Diffusion of innovation in large scale graphs

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**Abstract**—Will a new smartphone application diffuse deeply in the population or will it sink into oblivion soon? To predict this, we argue that common models of spread of innovations based on cascade dynamics or epidemics may not be fully adequate. In this paper, we model the spread of a new technological item in a population through a novel network dynamics where diffusion is based on the word-of-mouth and the persuasion strength increases the more the product is diffused. In this paper we carry on an analysis on large scale graphs to show off how the parameters of the model, the topology of the graph and, possibly, the initial diffusion of the item, determine whether the spread of the item is successful or not. The paper is completed by a number of numerical simulations corroborating the analytical results.

**Index Terms**—Network dynamics, innovation diffusion, epidemics, phase transitions

## 1 Introduction

In this paper we consider a novel network dynamics that models the diffusion of a new technological item (e.g. a smartphone application) in a population. The scenario is that of a set of agents (individuals) that are connected through a network and modify their state (1 or 0 indicating whether they have the new item or not) at random times, through an interaction with their neighbors. Specifically, the state update is due to two different mechanisms: a gossip persuasion that pushes adoption of the new item (state changes from 0 to 1) by contacting a neighbor already possessing it, and a spontaneous regression where an agent autonomously drops the new item (state changes from 1 to 0) .

The original feature of this model, with respect to classical epidemic models ([1], [2]), lays in the fact that the strength of the gossip persuasion depends on the global diffusion already reached by the new technological item in the population. Instead of a diffusion channel, the gossip mechanism plays here the role of a learning channel. It is the media through which agents gets aware of the existence of this new item, while its attraction for a potential new adopter in the end depends on the size of the diffusion of such an item not just among the neighbors, but in the whole population. This is a phenomenon known in economics as a "positive externalities" effect [3].

In many papers ([4], [5], [6], [7]), the spread of innovations and of new behaviors is modeled and studied under a network game scenario. In this setting each agent tends to maximize its own payoff by coordinating its choice with that of its neighbors. This hypothesis seems to be realistic when dealing with what we might call "big choices", such as the terms of economic contracts [6] or the choice of an operating system [7]. In such cases, a wrong decision can be very costly for the one who took it, therefore it is reasonable that an individual contacts many of his or her friends/colleagues (i.e. the neighbors) before taking the "big choice". In this work we instead focus on those we might call "light choices", (e.g. downloading an application for smartphone or joining an online community/social network). In such cases, negative consequences of the choice are usually mild, hence we can assume that individuals take their own choices after a pairwise interaction with one of their neighbors (a recent survey [8] supports our hypothesis highlighting the centrality of the world-of-mouth in the spread of assets), instead of involving their whole neighborhood in the choice.

The presence of the spontaneous dropping mechanism has various motivations. It may model a tendency to abandon technologies that have a maintenance cost or, also, a limitation of the time during which an agent can influence their neighbors. Mathematically, the case when only the persuasion mechanism is present is not particularly interesting as in this case the item, if originally present, will eventually diffuse to the whole population, independently on the strength of the mechanism and on the topology of the network. Our analysis, however, covers also, as a limit case, the situation when the dropping mechanism is not present.

Our model was first introduced in [9] and there studied on large scale graphs under a mean field assumption (i.e. assuming that each individual is connected to all the others). The mean field assumption is clearly quite restrictive for application to real networks. In this paper, we extend the analysis to more general interaction networks encompassing classical random graphs (e.g. configuration model) typically used to model social networks. This paves the way for the application of our model to network topologies reconstructed from real-world data. We will address this issue further on in this paper.

Formally, our model is a jump Markov process [10] on the space of state configurations of all agents. As for the Susceptible-Infected-Susceptible (SIS) model [11], the regression mechanism induces an absorbing state that is the configuration where every agent is in the 0 state. Standard probabilistic arguments (such as Borel-Cantelli lemma) allow to conclude that with probability 1, the system is

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absorbed in that configuration in a finite time. The key point is thus to analyze the behavior of the dynamics (maximum level of diffusion and its persistence in time) before the absorbing event.

In the SIS model, we witness the presence of two different regimes determined by a threshold value that is function of connectivity features of the graph. If the strength of the contagion mechanism with respect to the rate of regression is below this threshold, the epidemic quickly decades to the absorbing state. Whereas if it is above this threshold, the epidemic expands and remains persistent in the population for a time exponentially large in the size of the population (see [12]). In our model, besides these two regimes, we witness in many cases the presence of a third intermediate regime where the behavior strictly depends on the initial condition, namely the original fraction of agents in state 1 in the population. In this third regime, if the initial fraction of 1's in the population is below a certain level, the item will not be able to spread. Whereas if it is sufficiently large, the persuasion mechanism will be able to push towards a wide and persistent diffusion. This is the main novelty of our model with respect to standard SIS, where the initial condition instead (as long as the fraction of infected agents is initially non zero) never influences qualitatively the behavior of the system. In [9] we proved the existence of this intermediate regime for some mean field network topologies. A different mean field model with the same driving mechanism was considered in [13] (Chapter 17) and an analogous dependence on the initial fraction of adopters was found.

The main result of this paper is Theorem 17. It generalizes the main result of [9] to general graphs, relating the presence of the intermediate regime, where the behavior strongly depends on the initial condition, to the topology of the interaction network (measured through the degrees, the spectral radius of the adjacency matrix and the bottleneck ratio). Even if results in Theorem 17 are not exhaustive as in the mean field case, nevertheless, they are sufficient to prove the existence of the intermediate regime for a number of relevant families of graphs including random Erdős-Rényi graphs and random graphs with prescribed (bounded) degree distribution.

In the remaining part of this Section, we present a formal description of the proposed model along with all notation and concepts used throughout the paper. Section 2 is devoted to a brief recall of the mean field results, whose analysis is in [9], and Section 3 to the analysis on general graphs. Finally, Section 4 discusses the application to a number of specific families of graphs. In this last Section, we also present a number of numerical results corroborating the analytical results. These simulations actually show the presence of phase transitions among the different regimes also in less connected families of graphs, for which our theoretical results are not sufficient to give conclusive results in this sense.

### 1.1 Description of the model

Let $G = (V, E)$ be a directed graph with a finite set of nodes $V$, called *agents*, and a set of (directed) edges $E \subseteq V \times V$. Put $N = |V|$. The presence of an edge $(v, w)$ has to be interpreted in the sense that agent $v$ is influenced by agent $w$. Let $\mathcal{N}_v$ be the set of the (out-)neighbors of $v$, namely

$$\mathcal{N}_v = \{w \in V : (v, w) \in E\} ,$$

that is the set of agents who influence $v$. The (out-)degree of $v$ is denoted by $d_v = |\mathcal{N}_v|$. Agents are described by their *state*, precisely

$$X_v(t) = \begin{cases} 1 \text{ if } v \text{ has the item at time } t \\ 0 \text{ if } v \text{ has not the item at time } t. \end{cases}$$

States are assembled in a vector $X(t) \in \{0, 1\}^V$, called *configuration* of the system at time $t$. $\delta_v$ denotes the configuration where agent $v$ has state 1, whereas all the other agents have state 0. $\mathbb{1}$ denotes the *pure configuration* where all nodes have state 1 and, consequently, $0\mathbb{1}$ denotes the *pure configuration* where all nodes have state 0. Given a configuration $\mathbf{y} \in \{0, 1\}^V$, whose $v$-th component is denoted by $y_v$, we define $z(\mathbf{y}) := N^{-1}\mathbb{1}^*\mathbf{y} = N^{-1}\sum_v y_v \in [0, 1]$, which is the fraction of agents in state 1 in configuration $\mathbf{y}$.

Dynamics is defined as follows: nodes and edges are equipped with independent Poisson clocks. For the sake of simplicity, agents activate at rate 1, whereas edges activate at rate $\beta\bar{d}^{-1}$, where $\bar{d}$ is the average degree of the graph (this rescaling with respect to $\bar{d}$ is useful in presenting our large scale results). When agent $v$ or edge $(v, w)$ activates, agent $v$ has the possibility to revise its state according to the following rules.

- **Persuasion by gossip**: If
  - edge $(v, w)$ activates at time $t$,
  - $X_v(t) = 0$ and $X_w(t) = 1$,
  - $z(X(t)) = z$,

  then, agent $v$ updates its state to 1 with probability $\phi(z)$ where $\phi : [0, 1] \to [0, 1]$ is a function, called *persuasion strength*, whose properties are described below.
- **Spontaneous regression**: If
  - agent $v$ activates at time $t$,
  - $X_v(t) = 1$,

  then, agent $v$ updates its state to 0.

Formally, $X(t)$ is a jump Markov process on $\{0, 1\}^V$ whose non-zero transition rates from $X(t) = \mathbf{y}$ are:

$$\begin{cases} \lambda_{\mathbf{y},\mathbf{y}+\delta_v} = \beta\bar{d}^{-1}(1 - y_v) \sum\limits_{w \in N_v} y_w \phi(z(\mathbf{y})) \\ \lambda_{\mathbf{y},\mathbf{y}-\delta_v} = y_v. \end{cases} \tag{1}$$

Notice that when $\phi$ is constant, this model reduces to the standard SIS model [11]. The main feature and novelty of this model is the fact that, when the function $\phi$ is instead not constant, the gossip dynamics is affected by the global distribution of the state in the population of agents. In this model agents influence each other through two "information channels": the one determined by the graph edges and the another one due to the pressure of the global population state. These two channels are coupled through the persuasion mechanism described above.

In this paper we assume that $\phi$ is a $C^2$ function possessing the following properties:

(A1) $\phi$ is non-decreasing: $\phi'(z) \geq 0, \forall\, z$;

(A2) $\phi$ is concave: $\phi''(z) \leq 0, \forall z$;
(A3) $\phi'(0) > \phi(0)$.

From this moment on, we refer to a $\phi$ satisfying properties (A1) to (A3) as to an *admissible persuasion strength*.

**Remark 1.** *We briefly motivate these properties. Property (A1) is a natural consequence of the "positive externalities effect" [3] cited before: the more the new item is diffused, the higher is its persuasion strength. Regarding property (A2), a sub-linear growth of the persuasion strength with respect to the diffusion of the new item can be inferred from real world observations: trivially, an increase of a single new agent adopting the item has a bigger impact when adopters are still few than when they are more numerous. Finally, (A3) is assumed for the sake of simplicity: the case $\phi'(0) < \phi(0)$ (studied in [9] for complete graphs) leads to a theory with essentially no difference with respect to the SIS model.*

As for the SIS dynamics [11], the pure configuration $0\mathbb{1}$ is the only absorbing state of $X(t)$ and from every configuration there is a non zero probability of reaching it in finite time. Consequently, $X(t)$ enters the absorbing state $0\mathbb{1}$ in a finite time with probability 1. Our aim is to study the behavior of the system in the transient phase, i.e. before the occurrence of the absorbing event. The analysis will be carried on by considering the evolution of the fraction of 1's in the population: $Z(t) := z(X(t)) = N^{-1} \sum_v X_v(t)$, taking values in $\mathcal{S}_N = \{0, 1/N, \ldots, 1\}$.

In this paper, the admissible persuasion strength $\phi$ is assumed to be fixed once and for all. Our goal is to analyze the transient behavior of the process $Z(t)$ in dependence of the parameter $\beta$ as well of the graph topology. Typically we will consider sequences of graphs with increasing size, i.e $N \to +\infty$.

Our main results, Theorem 17 and Corollary 19, show that, for expansive families of graphs (e.g. complete graphs, Erdős-Rényi graphs, configuration model with bounded degrees), a double bifurcation phenomenon occurs, with probability converging to 1 as $N \to \infty$. The first bifurcation takes place with respect to the parameter $\beta$. It is governed by four threshold values $\beta_1 < \beta_2 < \beta_3 < \beta_4$ that depend on topological properties of the graphs family. If $\beta < \beta_1$, the process $Z(t)$ enters forever into an $\epsilon$-neighborhood of 0 in a time independent on the size of the graph. If $\beta > \beta_4$, $Z(t)$ reaches a level $z_s(\beta)$ in a time that does not depend on $N$ and remains above that level for an exponentially (in $N$) long time. In the regime $\beta_2 < \beta < \beta_3$, we witness a further bifurcation with respect to the initial condition $Z(0)$, governed by thresholds $z'_u(\beta)$ and $z''_u(\beta)$. In this intermediate regime, if $Z(0) < z'_u(\beta)$ we have the analogous fast extinction phenomenon that takes place for $\beta < \beta_1$, whereas if $Z(0) > z''_u(\beta)$, we have the analogous slow extinction phenomenon as for $\beta > \beta_4$. This results are not exhaustive, since we are not able to tackle the regimes $\beta_1 < \beta < \beta_2$ and $\beta_3 < \beta < \beta_4$. However, the extensive simulations carried on in Section 4, suggest that, in the expansive graph families of interest, no other behavior shows up and that transitions among the various regimes are indeed sharp.

Coherently with to our interpretation of the model, from now on the two behaviors described above, namely the fraction of 1's that quickly becomes smaller than $\epsilon$, or instead that remains large for a long time are respectively called, a *failure* and a *success*. As already pointed out, the main novelty of our model is the presence of the intermediate regime where failure and success depend on the initial condition.

In order to apply this model to real world cases, a crucial point is the estimation of the various thresholds, in particular $z''_u(\beta)$ for its role in possible marketing strategies. Indeed, this value represents the fraction of agents needed to possess the new technological item at the beginning (possibly as consequence of initial launch offers and free trials) to guarantee that a large and persistent diffusion will take place. Concretely, to obtain an estimation of these thresholds, two steps must be taken. Starting from the real data, first the structure of the interaction network must be inferred. Then the parameter $\beta$ and the function $\phi$ must be tuned. Many efforts have been made to tackle the first step, such as in [14], [15], [16], [17], [18]. Instead, the tuning step remains an open problem. At the end of this paper, in a paragraph devoted to this issue, we will give some suggestions of possible strategies to tackle it.

The main difficulty in the analysis of $Z(t)$ relies in the fact that, when $G$ is not a complete graph, the process is not Markovian. This is because the distribution of 1's in a neighborhood of a node is in general different from the global distribution of 1's in the population.

To study $Z(t)$ on a general graph, we introduce the idea of *active edges*. An edge $(v, w)$ is called active at time $t$ if $X_v(t) = 0$ and $X_w(t) = 1$. If we now denote by $\xi(t) = \xi(X(t))$ the fraction of active edges at time $t$ (namely $\xi(t)$ is the ratio between the number of active edges at time $t$ and $|E|$), the process $Z(t)$ is Markovian when conditioned to $\xi(t)$. Notably, when conditioned to $\xi(t) = \xi$, $Z(t)$ is a birth and death jump Markov process whose transition rates from the state $z$ to $z + 1/N$ and $z - 1/N$ are, respectively

$$\begin{cases} \lambda^+(z, \xi) = |E|\bar{d}^{-1}\beta\xi\phi(z) = N\beta\xi\phi(z) \\ \lambda^-(z, \xi) = Nz. \end{cases} \tag{2}$$

Of course, the difficulty arises from the fact that the process $\xi(t)$ is not explicitly known. The next section is devoted to recall the main results in the mean field case, analyzed in [9]. This is essentially the only case in which $\xi(t)$ can be expressed as a deterministic function of $Z(t)$, so that $Z(t)$ is Markovian itself. The general case will be taken up in Section 3.

Throughout the paper, we use the notation $\mathbb{P}_z[\cdot]$ to denote the probability conditioned to the initial condition $Z(0) = z$. In all our results concerning the behavior of the process $Z(t)$ positive constants show up to denote times and rate of probability decay and they typically depend on the model (e.g. the parameter $\beta$ and the graph topology) through a single scalar variable, say $x$. We use the following jargon: we say that $A = A(x)$ is bounded ($A$ is bounded away from 0) when $x$ is bounded away from a set of point $S$ if for every $\delta > 0$, there exists $\tilde{\delta} > 0$ such that $A < \tilde{\delta}$ ($A > \tilde{\delta}$) if $|x - s| > \delta$ for each $s \in S$.

## 2 The Mean Field Case

In this section we assume $G$ to be a complete graph with each node equipped with a self-loop (this last assumption only simplifies the notation and it has no effect in the large scale analysis). Under this assumption, the fraction of active edges is a deterministic function of $Z(t)$, in particular $\xi(t) = Z(t)(1-Z(t))$. This immediately implies that $Z(t)$ is a birth and death jump Markov process whose transition rates can be determined from (2):

$$\begin{cases} \lambda^+(z) = N\beta z(1-z)\phi(z) \\ \lambda^-(z) = Nz. \end{cases} \tag{3}$$

For such processes a quite complete analysis is available and it has been developed in [9]. Of course, in this case the local gossip interaction and the global influence are somehow mixed together, but some key interesting phenomena can already be observed here. The main result of [9], which is the object of our extension in Section 3, is the following theorem.

**Theorem 2.** *Consider the birth and death jump Markov process $Z(t)$ whose transition rates are given in (3) where $\beta > 0$ and $\phi$ is an admissible persuasion strength (i.e. properties (A1), (A2), and (A3) are satisfied). Put*

$$\beta^* = \left[\max_{z\in[0,1]}(1-z)\phi(z)\right]^{-1}, \quad A = 14(1+\beta)^2 \tag{4}$$

*Then, for every $\varepsilon > 0$ we can find $C_\epsilon > 0$ and $T_\varepsilon > 0$ for which the following holds true for every $N > 0$.*

1) *if $\beta < \beta^*$, then $\forall z$,*

$$\mathbb{P}_z\left(\sup_{t\geq T_\varepsilon} Z(t) > \varepsilon\right) \leq 5Ne^{-C_\epsilon N};$$

2) *if $\beta^* < \beta < \phi(0)^{-1}$, then*
*$\forall z < z_u(\beta) - \varepsilon$,*

$$\mathbb{P}_z\left(\sup_{t\geq T_\varepsilon} Z(t) > \varepsilon\right) \leq 5Ne^{-C_\varepsilon N},$$

*$\forall z > z_u(\beta) + \varepsilon$,*

$$\mathbb{P}_z\left(\inf_{t\in[T_\varepsilon, T_\varepsilon + e^{C_\varepsilon N}]} Z(t) < z_s(\beta) - \varepsilon\right) \leq AN^2e^{-C_\varepsilon N};$$

3) *if $\beta > \phi(0)^{-1}$, then $\forall z > \varepsilon$*

$$\mathbb{P}_z\left(\inf_{t\in[T_\varepsilon, T_\varepsilon + e^{C_\varepsilon N}]} Z(t) < z_s(\beta) - \varepsilon\right) \leq AN^2e^{-C_\varepsilon N}.$$

*With the understanding that if $\phi(0) = 0$ case 3) does not show up. Points $z_s(\beta)$ and $z_u(\beta)$, when they exist, can be characterized as follows*

$$\begin{aligned} z_u(\beta) &= \min\{z > 0 \mid \beta(1-z)\phi(z) - 1 = 0\}, \\ z_s(\beta) &= \min\{z > z_u(\beta) \mid \beta(1-z)\phi(z) - 1 = 0\}. \end{aligned} \tag{5}$$

**Remark 3.** *The constants $C_\varepsilon$ and $T_\varepsilon$ appearing in the statement of Theorem 2 depend on the choice of the parameter $\beta$. They can be chosen in such a way that, for every $\epsilon > 0$, $C_\varepsilon$ is bounded away from $0$ and $T_\varepsilon$ is bounded when $\beta$ is bounded away from $\beta^*$ and $\phi(0)^{-1}$.*

**Remark 4.** *Choosing $\phi(z) = z$, which is the case considered in the simulations of Section 4, we have that $\beta^* = 4$ and, for $\beta \geq 4$, an explicit computation shows that:*

$$z_u(\beta) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4}{\beta}}, \quad z_s(\beta) = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4}{\beta}}. \tag{6}$$

In [9], Theorem 2 has actually a slightly different formulation. There, the expression bounding the probabilities are all pure exponentials with no polynomial factors in $N$. This simplification has been obtained at the price of a validity that is only for sufficiently large $N$. Here, we prefer this more precise formulation for the use we will make of it in Section 3. The proof of this theorem, whose details are in [9], is based on the application of Kurtz's Theorem [19], along with a couple of technical lemmas. Below we recall a simple version of Kurtz's theorem and one of these lemmas as they will be directly used later on in the proofs of Section 3.

Kurtz's theorem allows to approximate a birth and death jump Markov process with the solution of an associated ODE.

**Theorem 5. (Kurtz)** *Let $Z(t)$ be a birth and death process on the state-space $S_N$ with transitions rates, respectively, $\lambda^+(z) = Nf^+(z)$ and $\lambda^-(z) = Nf^-(z)$ where $f^+$ and $f^-$ are Lipschitz continuous functions of $z$. Suppose that $Z(0) = z_0$ deterministically. Consider the Cauchy problem:*

$$\begin{cases} z'(t) = f^+(z) - f^-(z) \\ z(0) = z_0. \end{cases} \tag{7}$$

*Then, for every $T > 0$ $\exists$ $C > 0$, such that the following exponential decay holds:*

$$\mathbb{P}\left(\sup_{0\leq t\leq T}|Z(t) - z(t)| > \epsilon\right) \leq 4\exp\left(-CN\epsilon^2\right). \tag{8}$$

*Moreover, $C$ only depends on $T$ and on the sup norm $||f^+ - f^-||_\infty$ and is bounded away from $0$ when $T$ and $||f^+ - f^-||_\infty$ are bounded.*

Finally, we have the following result.

**Lemma 6.** *Let $Z(t)$ be a birth and death process on the state-space $S_N$ with transitions rates, respectively, $\lambda^+(z)$ and $\lambda^-(z)$ satisfying the following properties:*

- *$\lambda^+(0) = \lambda^-(0) = 0$;*
- *there exist $\delta > 0$ such that*

$$\lambda^-(z) \geq (1+\delta)\lambda^+(z), \quad \forall z > 0. \tag{9}$$

*Then, called $C = \ln(1+\delta)$, for any $\varepsilon > 0$ and $z < \varepsilon$, it holds*

$$\mathbb{P}_z\left(\exists t \geq 0 \mid Z(t) > 2\varepsilon\right) \leq \varepsilon Ne^{-C\varepsilon N}. \tag{10}$$

*Proof.* First, for any $k \in \{0, \ldots, \lceil 2\delta N\rceil\}$, put $e_k = \mathbb{P}_{k/N}\left(\exists t \geq 0 \mid Z(t) \geq \lceil 2\varepsilon N\rceil/N\right)$. A straightforward argument based on conditioning on the first transition shows that the values $e_k$s satisfy the Laplace equation:

$$e_k = \frac{\lambda^+(k/N)e_{k+1} + \lambda^-(k/N)e_{k-1}}{\lambda^+(k/N) + \lambda^-(k/N)}. \tag{11}$$

This, along with the boundary condition $e_0 = 0$, gives

$$(e_{k+1} - e_k) = \prod_{j=1}^{k}\frac{\lambda^-(j/N)}{\lambda^+(j/N)}e_1. \tag{12}$$

Since $e_{\lceil\varepsilon N\rceil} \leq 1$, we obtain

$$e_1 \leq \left(\sum_{k=0}^{\lceil 2\varepsilon N\rceil - 1}\prod_{j=1}^{k}\frac{\lambda^-(\frac{j}{N})}{\lambda^+(\frac{j}{N})}\right)^{-1} \leq \left(\prod_{j=1}^{\lceil 2\varepsilon N\rceil - 1}\frac{\lambda^-(\frac{j}{N})}{\lambda^+(\frac{j}{N})}\right)^{-1} \tag{13}$$

Combining (12) and (13) we finally obtain

$$\begin{aligned} e_{\lfloor \varepsilon N \rfloor} = & \sum_{k=0}^{\lfloor \varepsilon N \rfloor -1} \prod_{j=1}^{k} \frac{\lambda^-(j/N)}{\lambda^+(j/N)} e_1 \leq \lfloor \varepsilon N \rfloor \frac{\prod_{j=1}^{\lfloor \varepsilon N \rfloor -1} \frac{\lambda^-(j/N)}{\lambda^+(j/N)}}{\prod_{j=1}^{\lceil 2\varepsilon N \rceil -1} \frac{\lambda^-(\frac{j}{N})}{\lambda^+(\frac{j}{N})}} \\ \leq & \lfloor \varepsilon N \rfloor \prod_{j=\lfloor \varepsilon N \rfloor}^{\lfloor 2\varepsilon N \rfloor -1} \frac{\lambda^+(j/N)}{\lambda^-(j/N)} \leq \lfloor \varepsilon N \rfloor (1+\delta)^{-\varepsilon N}, \end{aligned}$$

which yields the thesis. ■

## 3 Analysis on general graphs

In this section, we partially extend Theorem 2, proving the existence of the three regimes (in particular of the intermediate one), for a large family of expander graphs.

For the epidemic SIS model, an estimation of the mean absorbing time has been carried on general graphs [11], [12], [20]. Notably, fast extinction results have been obtained [12] by upper bounding the original process with another one whose transition rates depend linearly on the state variable $\mathbf{x}$ and for which, consequently, the moment analysis turns out to be particularly simple. The key graph parameter in this estimation is the spectral radius of the corresponding adjacency matrix. On the other hand, slow extinction has been analyzed by essentially estimating the fraction of active edges in terms of bottleneck ratios of the graph and then lower bounding the process with a simple birth and death process.

However, the techniques developed in [12] can not be straightforwardly applied to our model. Indeed, the presence of the term $\phi(z)$ poses a number of technical issues that are absent in the SIS model. This will be particularly evident in the analysis of the intermediate regime where success or failure depends on the initial condition.

In the next three subsections we determine a series of lower and upper bounds of process $Z(t)$ using standard coupling techniques, inspired by the results in [11], [12], [20]. Specifically, in 3.1 we propose a lower bound based on bottleneck estimation that allows to prove the existence of the success regimes - thereby extending the second part of item 2) and item 3) of Theorem 2. In 3.2 instead, we propose a quite simple upper bound sufficient to prove the existence of the failure regime - extending item 1) of Theorem 2. Finally, in 3.3 we develop a stronger upper bound using a linearization technique and a second moment analysis in order to prove the existence of the failure regime depending on the initial condition - extending the first part of item 2) of Theorem 2. This last case is the most original technical part of this paper. Though inspired by [11], [12], [20], our technique is based on a detailed second order analysis of the bounding process, which is not needed in the analysis of the SIS model. All these partial results are assembled in 3.4 where the main result is finally stated and proved.

Fixed a strongly connected graph $G=(V,E)$, we denote by $A \in \{0,1\}^V$ its adjacency matrix ($A_{uv}=1 \iff (u,v) \in E$) and by $\rho_A$ its spectral radius. The Cheeger constant [21] of $G$ is defined as

$$\gamma = \gamma_G = \inf_{U \subset V} \frac{|\{(u,v) | u \in U,\, v \in V \setminus U\}|}{\min\{|U|, |V \setminus U|\}}. \tag{14}$$

We recall that $X(t)$ is a jump Markov process on the state-space $\{0,1\}^V$ having transition rates given by (1). $Z(t) = z(X(t))$ denotes the total fraction of 1's in the population and $\xi(t) = \xi(X(t))$ is the fraction of active edges.

### 3.1 A bottleneck-based lower bound

The following result, inspired by an argument used in [22], yields a lower bound of the process $Z(t)$ in terms of a birth and death jump Markov process.

**Proposition 7.** *There exists a coupling of the process $X(t)$ with a birth and death jump Markov process $\tilde{Z}(t)$ over $\mathcal{S}_N$ having transition rates*

$$\begin{cases} \tilde{\lambda}^+(z) = N\beta \bar{d}^{-1}\gamma z(1-z)\phi(z) \\ \tilde{\lambda}^-(z) = Nz, \end{cases} \tag{15}$$

*in such a way that $Z(t) \geq \tilde{Z}(t)$ for all $t$.*

*Proof.* From (14), choosing as $U$ the set of all agents with state equal to 0, we obtain

$$\begin{aligned} \gamma \leq & \frac{\xi|E|}{\min\{z|V|, (1-z)|V|\}} \leq \frac{\xi|E|}{z(1-z)|V|} = \\ & = \frac{\xi \bar{d}|V|}{z(1-z)|V|} = \frac{\xi \bar{d}}{z(1-z)}. \end{aligned}$$

This implies that the fraction of active edges satisfies the inequality $\xi \geq \gamma \bar{d}^{-1} z(1-z)$. This yields, using (2),

$$\lambda^+(z,\xi) \geq \tilde{\lambda}^+(z) \qquad \text{and} \qquad \lambda^-(z,\xi) = \tilde{\lambda}^-(z).$$

The proof is now concluded by applying a simple coupling argument, similarly to the one used to prove Theorem 8.8 in [11]. ■

The following Corollary proves the existence of the two success regimes, extending the second part of item 2) and item 3) of Theorem 2 to general graphs.

**Corollary 8.** *Put $z_u = z_u(\beta \bar{d}^{-1}\gamma)$ and $z_s = z_s(\beta \bar{d}^{-1}\gamma)$ as defined in (5). For every $\varepsilon > 0$ we can find $C_\varepsilon > 0$ and $T_\varepsilon > 0$ for which the following hold true*

1) *if $\bar{d}\gamma^{-1}\beta^* < \beta$, then, $\forall z > z_u + \varepsilon$,*

$$\mathbb{P}_z\left(\inf_{t \in [T_\varepsilon, T_\varepsilon + e^{C_\varepsilon N}]} Z(t) < z_s - \varepsilon\right) \leq AN^2 e^{-C_\varepsilon N};$$

2) *if, moreover, $\bar{d}\gamma^{-1}\phi(0)^{-1} < \beta$, then, $\forall z > \varepsilon$,*

$$\mathbb{P}_z\left(\inf_{t \in [T_\varepsilon, T_\varepsilon + e^{C_\varepsilon N}]} Z(t) < z_s - \varepsilon\right) \leq AN^2 e^{-C_\varepsilon N},$$

*where $A = 14(1+\beta)^2$. The constants $C_\varepsilon$ and $T_\varepsilon$ only depend on the quantity $\bar{d}^{-1}\gamma\beta$ and are, respectively, bounded away from 0 and bounded, when this quantity is bounded away from $\beta^*$ and $\phi(0)^{-1}$.*

*Proof.* Using Proposition 7, process $Z(t)$ can be lower bounded by a birth and death jump Markov process $\tilde{Z}(t)$ having transition rates (15). Hence $\mathbb{P}_z[Z(t) < \bar{z}] \leq \mathbb{P}_z[\tilde{Z}(t) < \bar{z}]$, $\forall z, \bar{z}$. A comparison with (3) shows that the transition rates of $\tilde{Z}(t)$ coincide with the ones of the mean field model (3) with $\beta$ replaced by $\beta \bar{d}^{-1}\gamma$. Results then follow from items 2) and 3) of Theorem 2. ■

### 3.2 A degree-based upper bound

In this subsection we provide a simple upper bound that depends only on the degrees of the nodes in the graph. Let $\Delta$ be the maximum in-degree in $G$, then the following proposition holds.

**Proposition 9.** *There exists a coupling of the process $X(t)$ with a birth and death jump Markov process $\tilde{Z}(t)$ over $\mathcal{S}_N$ having transition rates*

$$\begin{cases} \tilde{\lambda}^+(z) = N\Delta\bar{d}^{-1}\beta z\phi(z) \\ \tilde{\lambda}^-(z) = Nz, \end{cases} \tag{16}$$

*in such a way that $Z(t) \leq \tilde{Z}(t)$ for all $t$.*

*Proof.* This simply follows from the estimation

$$\xi = \frac{|\{(u,v)|X_u = 0, X_v = 1\}|}{|E|} \leq \frac{\Delta z n}{\bar{d}n} = \Delta\bar{d}^{-1}z.$$

■

The following Corollary proves the existence of the failure regime, extending item 1) of Theorem 2 to general graphs.

**Corollary 10.** *If $\beta < \bar{d}\Delta^{-1}\phi(1)^{-1}$, then, for every $\varepsilon > 0$ we can find $C_\varepsilon > 0$ and $T_\varepsilon > 0$ for which*

$$\mathbb{P}_z\left(\sup_{t\geq T_\varepsilon} Z(t) > \varepsilon\right) \leq 5Ne^{-C_\varepsilon N}, \quad \forall\, z.$$

*For every $\varepsilon > 0$, the constants $C_\varepsilon$ and $T_\varepsilon$ only depend on the quantity $\beta\bar{d}^{-1}\Delta$ and are, respectively, bounded away from 0 and bounded, when this quantity is bounded away from $\phi(1)^{-1}$.*

*Proof.* From Proposition 9, process $Z(t)$ can be upper bounded by a birth and death jump Markov process $\tilde{Z}(t)$ having transition rates (16). Hence $\mathbb{P}_z[Z(t) > \bar{z}] \leq \mathbb{P}_z[\tilde{Z}(t) > \bar{z}]$, $\forall z, \bar{z}$. We now apply Theorem 5 to the process $\tilde{Z}(t)$. The assumption $\beta < \bar{d}\Delta^{-1}\phi(1)^{-1}$ implies that 0 is asymptotically stable in the ODE (7). More precisely, the solution $z(t)$ of (7) with $z(0) = 1$ converges to 0 when $t \to +\infty$ and this clearly implies that, for every fixed $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that $z(t) < \varepsilon/2$ for every $t \geq T_\varepsilon$ and for every initial condition $z(0) = z$. Moreover, $T_\varepsilon$ only depends monotonically on the quantity $\bar{d}^{-1}\beta$ and blows up when this quantity approaches $\phi(1)^{-1}$. Using (5) with $T = T_\varepsilon$ we thus obtain that there exists $C'_\varepsilon > 0$ such that

$$\mathbb{P}_z\left(\tilde{Z}(T_\varepsilon) > \varepsilon/2\right) \leq 4e^{-C'_\epsilon N} \quad \forall z. \tag{17}$$

It follows from Theorem 5 and from the considerations above on $T_\varepsilon$ that $C'_\varepsilon$ only depends on $\bar{d}^{-1}\beta$ and it is bounded away from 0, when this quantity is bounded away from $\phi(1)^{-1}$. If we apply Lemma 6 to $\tilde{Z}(t)$ with $1 + \delta = \bar{d}\Delta^{-1}\beta^{-1}\phi(1)^{-1} > 1$, we obtain that, for every $z < \varepsilon/2$,

$$\mathbb{P}_z\left(\exists t \geq 0 \,|\, \tilde{Z}(t) > \varepsilon\right) \leq \frac{\varepsilon}{2}Ne^{-[\ln(1+\delta)]\varepsilon/2N}. \tag{18}$$

From (17) and (18) we finally obtain that, $\forall\, z$

$$\mathbb{P}_z\left(\sup_{t\geq T_\varepsilon} Z(t) > \varepsilon\right) \leq \mathbb{P}_z\left(\sup_{t\geq T_\varepsilon} \tilde{Z}(t) > \varepsilon\right) \leq 5Ne^{-C_\varepsilon N},$$

where $C_\varepsilon = \min\{C'_\varepsilon, [\ln(\bar{d}\Delta^{-1}\beta^{-1}\phi(1)^{-1})]\varepsilon/2\}$. In consideration of the properties already discussed for the quantity $C'_\varepsilon$, the result is now proven. ■

### 3.3 An upper bound based on linearization

What remains to be shown is the existence, for general graphs, of the failure regime as described in the first part of 2) of Theorem 2. This is fundamental in order to prove the existence of the bifurcation with respect to the initial condition. In this subsection we tackle this issue by upper bounding the process $Z(t)$ with another jump Markov process whose transition rates depend linearly on the configuration vector. In [12] a similar idea was used to analyze the SIS model. However, while in [12] it was sufficient to carry on a first moment analysis of the linearized process to prove the existence of the fast extinction regime, here a much more complex analysis is needed. In fact, differently from the SIS model, in our model the fraction of 1s influences the persuasion strength and, ultimately, the behavior of the system. In the study of the bifurcation in dependence on the initial condition, to make our bounding technique effective, we must make sure that the fraction of 1s always remains below a certain threshold. For this, a first moment analysis of the linearized process is no longer sufficient. It must be coupled with a concentration result based on a second moment analysis.

Consider the jump Markov process $Y(t)$ over $\Theta = \mathbb{N}^V$:

$$\begin{cases} \bar{\lambda}_{\mathbf{y},\mathbf{y}+\delta_v} &= \mu \sum\limits_{w\in\mathcal{N}_v} y_w \\ \bar{\lambda}_{\mathbf{y},\mathbf{y}-\delta_v} &= y_v, \end{cases} \tag{19}$$

where $\mu = \beta\bar{d}^{-1}\phi(1)$.

Notice that the original process $Z(t)$, taking values in $\{0,1\}^V$, can be trivially extended to $\Theta$ by simply putting $\lambda_{\mathbf{y},\mathbf{y}+\delta_v} = 0$ if $y_v > 0$ and using the same expression for $\lambda_{\mathbf{y},\mathbf{y}-\delta_v} = y_v$. In the case when $\beta < \bar{d}\rho_A^{-1}\phi(1)^{-1}$, it follows that $\bar{\lambda}_{\mathbf{y},\mathbf{y}+\delta_v} \geq \lambda_{\mathbf{y},\mathbf{y}+\delta_v}$ for all $\mathbf{y}$ and for all $v$. We now consider any coupling between $X(t)$ and $Y(t)$ such that $X(0) = Y(0)$ and $X(t) \leq Y(t)$ (entry-wise) for all $t$. Clearly, it holds $Z(t) \leq Z_Y(t) := z(Y(t))$ for all $t$.

In the remaining part of this section, we study the behavior of $Z_Y(t)$. In this way, we will later derive the fast extinction result for $Z(t)$ and we will finally complete the analysis of the bifurcation phenomena.

The analysis of $Z_Y(t)$ proceeds as follows. In Lemma 11, we provide an upper bound for its first moment. Then, an analysis on its second moment and a bound on its variance is provided in Lemmas 13 and 14. Finally, we combine these results in Lemma 15 that analyzes the asymptotic behavior of $Z_Y(t)$.

**Lemma 11.**

$$\mathbb{E}[Z_Y(t)] \leq \exp((\mu\rho_A - 1)t)Z(0)^{1/2}. \tag{20}$$

*Proof.* Let us denote the first moment of the process $Y(t)$ by $M^{(1)}(t) = \mathbb{E}(Y(t))$. The distribution $p(t) \in [0,1]^\Theta$ of $Y(t)$ satisfies the forward Kolmogorov equation $\dot{p} = -pL(\bar{\lambda})$ where $L(\bar{\lambda})$ is the Laplacian of the process (i.e. $L(\bar{\lambda})_{xy} = \sum_{y'} \bar{\lambda}_{xy'} - \bar{\lambda}_{xy}$). Therefore, $M^{(1)}(t)$ satisfies the ODE

$$\dot{M}^{(1)} = (\mu A - I)M^{(1)}. \tag{21}$$

We can thus estimate

$$||M^{(1)}(t)|| \le \exp((\mu\rho_A - 1)t)||Y(0)||, \tag{22}$$

where $\rho_A$ is the spectral radius of $A$. This yields

$$\begin{aligned}\mathbb{E}[Z_Y(t)] \quad \le \quad & n^{-1}n^{1/2}\exp((\mu\rho_A - 1)t)||X(0)|| = \\ & = \exp((\mu\rho_A - 1)t)Z(0)^{1/2}.\end{aligned} \tag{23}$$

Proof is now completed. ■

**Remark 12.** *Since $\mu = \beta\bar{d}^{-1}\phi(1)$, it holds*

$$\mu\rho_A - 1 = \beta\bar{d}^{-1}\rho_A\phi(1) - 1.$$

*Hence, $\beta < \bar{d}\rho_A^{-1}\phi(1)^{-1}$ implies $\mu\rho_A - 1 < 0$, yielding an exponential decay of $\mathbb{E}[Z(t)]$ to 0 . However, as already pointed out above, this is not yet sufficient to generalize item 1) of Theorem 2.*

We now undertake a second order analysis of the process $Y(t)$. To this aim, put $M^{(2)} = \mathbb{E}(Y(t)Y(t)^*)$ and $\Omega = M^{(2)} - M^{(1)}M^{(1)*}$.

**Lemma 13.** *$\Omega$ satisfies the ODE*

$$\dot{\Omega} = \mu(A\Omega + \Omega A) - 2\Omega + \mu\mathrm{diag}(AM^{(1)}) + \mathrm{diag}(M^{(1)}), \tag{24}$$

*with $\Omega(0) = 0$.*

*Proof.* Using the Kolmogorov equation it follows that

$$\begin{aligned}\dot{M}^{(2)} \quad = \quad & \sum_{x\in\Theta} \dot{p}_x xx^* \\ = \quad & \sum_{x\in\Theta} \mu \sum_{v\in V} p_{x-\delta_v}(A(x-\delta_v))_v xx^* + \\ & + \sum_{x\in\Theta}\sum_{v\in V} p_{x+\delta_v}(x_v+1)xx^* + \\ & - \sum_{x\in\Theta} \mu \sum_{v\in V} p_x(Ax)_v xx^* + \\ & - \sum_{x\in\Theta}\sum_{v\in V} p_x x_v xx^*.\end{aligned} \tag{25}$$

We rearrange the first two terms of (25) by adding and subtracting $\delta_v$ to both $x$ and $x^*$, expanding the products and, finally, changing the indexes. We thus obtain

$$\begin{aligned}& \sum_{x\in\Theta} \mu \sum_{v\in V} p_{x-\delta_v}(A(x-\delta_v))_v xx^* = \\ & \quad = \mu \sum_{v\in V}\sum_{x\in\Theta} p_x(Ax)_v xx^* + \\ & \quad + \mu(AM^{(2)} + M^{(2)}A) + \mu\mathrm{diag}(AM^{(1)}),\end{aligned} \tag{26}$$

and

$$\begin{aligned}& \sum_{x\in\Theta}\sum_{v\in V} p_{x+\delta_v}(x_v+1)xx^* = \\ & \quad = \sum_{v\in V}\sum_{x\in\Theta} p_x x_v xx^* - 2M^{(2)} + \mathrm{diag}(M^{(1)}).\end{aligned} \tag{27}$$

Substituting (26) and (27) into (25), we finally obtain

$$\begin{aligned}\dot{M}^{(2)} \quad = \quad & \mu(AM^{(2)} + M^{(2)}A) - 2M^{(2)} + \\ & + \mu\mathrm{diag}(AM^{(1)}) + \mathrm{diag}(M^{(1)}).\end{aligned} \tag{28}$$

Thesis now follows by differentiating the expression $M^{(1)}M^{(1)*}$ with the use of (21) and then subtracting it from (28). ■

We can now bound $\mathrm{Var}(Z_Y(t)) = N^{-2}\mathbb{1}^*\Omega\mathbb{1}$ through the following Lemma.

**Lemma 14.**

$$\mathrm{Var}(Z_Y(t)) \le N^{-1/2}\frac{1+\mu\rho_A}{1-\mu\rho_A}e^{(\mu\rho_A - 1)t}Z_Y(0)^{1/2}. \tag{29}$$

*Proof.* Let $\mathcal{S}(V)$ be the set of symmetric matrices over $V$ and let $\mathcal{L} : \mathcal{S}(V) \to \mathcal{S}(V)$ be the linear operator given by $\mathcal{L}(M) = \mu(AM + MA) - 2M$. Then, using (24), we can represent the centered second moment as

$$\Omega(t) = \int_0^t \exp((t-s)\mathcal{L})U(s)\,\mathrm{d}s,$$

where

$$U(t) = \mu\mathrm{diag}(AM^{(1)}(t)) + \mathrm{diag}(M^{(1)}(t)). \tag{30}$$

Hence,

$$\mathrm{Var}(Z_Y(t)) = N^{-2}\mathbb{1}^*\int_0^t \mathbb{1}^*[\exp((t-s)\mathcal{L})U(s)]\,\mathrm{d}s\,\mathbb{1}. \tag{31}$$

Using the representation

$$\exp(t\mathcal{L})M = \exp(t(\mu A - I))M\exp(t(\mu A - I)),$$

we can estimate the variance as follows:

$$\mathrm{Var}(Z_Y(t)) \le N^{-2}N^{1/2}\int_0^t e^{(2(t-s)(\mu\rho_A - 1)}||U(s)||\,\mathrm{d}s\,N^{1/2},$$

where $||U(s)||$ is the induced 2-norm of $U(s)$. From (30) we can estimate this norm as

$$\begin{aligned}||U(s)|| \quad \le \quad & \mu\max_v|\delta_v^*AM^{(1)}| + \max_v|\delta_v^*M^{(1)}| \\ \le \quad & (\mu\rho_A + 1)\exp((\mu\rho_A - 1)s)||Y(0)||.\end{aligned}$$

Combining this estimation with the previous inequality, we obtain the thesis. ■

We are now ready to analyze the convergence behavior of the process $Z_Y(t)$ in the case when $\mu\rho_A < 1$.

**Lemma 15.** *Assume that $\mu\rho_A < 1$. For every $\epsilon > 0$ there exists a time $T_\varepsilon > 0$ and a constant $K_\varepsilon > 0$ such that*

*1) if $Z_Y(0) \le a^2$, it holds*

$$\mathbb{P}(\exists t \ge 0\,|\,Z_Y(t) > a + \varepsilon) \le K_\varepsilon N^{-1/2}; \tag{32}$$

*2) for every $Z_Y(0)$, it holds*

$$\mathbb{P}(\exists t \ge T_\epsilon\,|\,Z_Y(t) > \epsilon) \le K_\epsilon N^{-1/2}. \tag{33}$$

*Moreover, for every $\varepsilon > 0$, the constants $K_\varepsilon$ and $T_\varepsilon$ only depend on the quantity $\mu\rho_A$ and are bounded when this quantity is bounded away from 1.*

*Proof.* Consider the underlying discrete time Markov chain $\tilde{Y}(k)$ for $k = \{0, 1, \dots\}$ and the corresponding $Z_{\tilde{Y}}(k) = z(\tilde{Y}(k))$. The Poisson process $\Lambda(t)$ governing the jumps of $Y(t)$ has intensity $\nu = (\beta + 1)N$. Hence it holds

$$\begin{aligned}\mathrm{Var}(Z_Y(t)) \quad = \quad & \sum_{k=0}^{+\infty}\mathrm{Var}(Z_{\tilde{Y}}(k))\mathbb{P}(\Lambda(t) = k) \\ \ge \quad & \mathrm{Var}(Z_{\tilde{Y}}(\lfloor\nu t\rfloor))\mathbb{P}(\Lambda(t) = \lfloor\nu t\rfloor).\end{aligned} \tag{34}$$

The last multiplicative term of (34) can be lower bounded using Stirling's approximation:

$$\begin{aligned}\mathbb{P}(\Lambda(t) = \lfloor\nu t\rfloor) = \frac{(\nu t)^{\lfloor\nu t\rfloor}}{\lfloor\nu t\rfloor!}e^{-\nu t} \ge \\ \ge \frac{(\nu t)^{\lfloor\nu t\rfloor}}{\lfloor\nu t\rfloor^{\lfloor\nu t\rfloor}}\frac{e^{\lfloor\nu t\rfloor}}{\sqrt{2\pi\lfloor\nu t\rfloor}}e^{-\nu t} \ge \frac{1}{9\lfloor\nu t\rfloor}.\end{aligned} \tag{35}$$

From (34) and (35) we obtain that

$$\mathrm{Var}(Z_{\tilde{Y}}(k)) \leq 9k\mathrm{Var}(Z_Y(k/\nu)), \quad \forall\, k = \{0, 1, \dots\}. \quad (36)$$

Combining the assumption $Z_Y(0) \leq a^2$ with the estimation in Lemma 11, we bound

$$\begin{aligned}
&\mathbb{P}(\exists t \geq 0 \,|\, Z_Y(t) > a + \varepsilon) \leq \\
&\quad \leq \mathbb{P}(\exists k \geq 0 \,|\, Z_{\tilde{Y}}(k\lfloor \varepsilon N/2 \rfloor) > a + \varepsilon/2) \\
&\quad \leq \sum_{k\geq 0} \mathbb{P}(|Z_Y(k\lfloor \varepsilon N/2 \rfloor) - \mathbb{E}(Z_Y(k\lfloor \varepsilon N/2 \rfloor))| \geq \varepsilon/2) \\
&\quad \leq \frac{4}{\varepsilon^2} \sum_{k \geq 0} \mathrm{Var}(Z_{\tilde{Y}}(k\lfloor \varepsilon N/2 \rfloor)).
\end{aligned}$$

Using now estimation (36) and Lemma 14, we obtain item 1).

Item 2) can be proven in a similar fashion. First, we notice that Lemma 11 implies that there exists $T_\epsilon > 0$ such that $\mathbb{E}(Z_Y(t)) \leq \epsilon/2$ for all $t \geq T_\epsilon$. We then conclude using again the variance estimation in (14). ■

We are now ready to go back to our original process $Z(t)$. The following Corollary proves the existence of two failure regimes, one that does not depend on the initial condition and the second one that does depend on it, extending items 1) and the first part of item 2) of Theorem 2 to general graphs, respectively. Notice that a different extension of item 1) was already obtained in Corollary 10. Later on we will comment on the relation between these two estimations.

**Corollary 16.** *For every $\varepsilon > 0$, there exist $K_\varepsilon > 0$, $K'_\varepsilon > 0$, $T_\varepsilon > 0$, and $T'_\varepsilon > 0$ s.t.*

1) *if $\beta < \bar{d}\rho_A^{-1}\phi(1)^{-1}$, then for every $z$,*

$$\mathbb{P}_z(\exists t \geq T_\varepsilon \,|\, Z(t) > \epsilon) \leq K_\varepsilon N^{-1/2}; \quad (37)$$

2) *if $\bar{d}\rho_A^{-1}\phi(1)^{-1} < \beta < \bar{d}\rho_A^{-1}\phi(0)^{-1}$, let $z^*$ be the unique solution of the equation $\phi(z^*) = \beta^{-1}\bar{d}\rho_A^{-1}$ and assume that $\varepsilon < z^*/2$. Then, for every $z \leq (z^* - 2\varepsilon)^2$, it holds*

$$\mathbb{P}_z(\exists t \geq T'_\varepsilon \,|\, Z(t) > \varepsilon) \leq K'_\varepsilon N^{-1/2}. \quad (38)$$

*Moreover, the constants $K_\varepsilon$ and $T_\varepsilon$ only depend on the quantity $\beta\bar{d}^{-1}\rho_A$ and are bounded when this quantity is bounded away from $\phi(1)^{-1}$. The constants $K'_\varepsilon$ and $T'_\epsilon$ only depend on $\varepsilon$.*

*Proof.* Item 1) is a straightforward consequence of the stochastic domination between $Z(t)$ and $Z_Y(t)$ and of item 2) is a consequence of Lemma 15 with $\mu = \beta\bar{d}^{-1}\phi(1)$.

Regarding item 2), notice first of all that because of the assumptions on $\phi$, we have that $|\{z|\phi(z) = w\}| = 1$ for every $w \in [\phi(0), \phi(1))$. This implies the uniqueness of $z^*$. At this stage, we consider the jump Markov process $Y(t)$ with transition rates given by (19) and $\mu = \beta\bar{d}^{-1}\phi(z^* - \varepsilon)$. We notice that

$$\mu\rho_A = \beta\bar{d}^{-1}\phi(z^* - \varepsilon)\rho_A = \frac{\phi(z^* - \varepsilon)}{\phi(z^*)} < 1, \quad (39)$$

and that $\bar{\lambda}_{\mathbf{y},\mathbf{y}+\delta_\mathbf{v}} \geq \lambda_{\mathbf{y},\mathbf{y}+\delta_\mathbf{v}}$ as long as $\mathbf{y}$ is such that $z(\mathbf{y}) \leq \mathbf{z}^* - \varepsilon$. Put

$$\bar{T} = \inf\{t \,|\, Y(t) > z^* - \varepsilon\}.$$

We can establish a coupling between $X(t)$ and $Y(t)$ such that $X(0) = Y(0)$ and $X(t) \leq Y(t)$ for all $t < \bar{T}$. Choose now $T_\varepsilon$ that satisfies item 2) of Proposition 15. It holds

$$\begin{aligned}
&\mathbb{P}(\exists t \geq T_\varepsilon \,|\, Z(t) > \epsilon) = \\
&\quad = \mathbb{P}(\exists t \geq T_\varepsilon \,|\, Z(t) > \varepsilon\,, \bar{T} = +\infty) + \\
&\qquad + \mathbb{P}(\exists t \geq T_\varepsilon \,|\, Z(t) > \epsilon\,, \bar{T} < +\infty) \\
&\quad \leq \mathbb{P}(\exists t \geq T_\epsilon \,|\, Z_Y(t) > \varepsilon) + \mathbb{P}(\exists t \geq 0 \,|\, Z_Y(t) > z^* - \varepsilon).
\end{aligned}$$

Result now follows from Proposition 15 with $a = z^* - 2\varepsilon$ and $\delta = \varepsilon$. The fact that we get constants $K'_\varepsilon$ and $T'_\epsilon$ that only depend on $\varepsilon$ is due to the fact that for every $\varepsilon > 0$ the quantity $\mu\rho_A$ in (39) is uniformly bounded away from 1 when $\beta$ varies in the specified interval. ■

### 3.4 The core result

The main result of this paper can be finally obtained by combining Corollaries 8, 10 and 16. For the sake of readability, we recall here the standing assumptions.

Let $G = (V, E)$ be a fixed graph having average degree $\bar{d}$, maximum degree $\Delta$, Cheeger constant $\gamma$ and spectral radius of the adjacency matrix $\rho_A$. We recall that $X(t)$ is a jump Markov process on $\{0, 1\}^V$ governed by the transition rates (1) and $Z(t) = z(X(t))$ is a process counting the fraction of nodes having state 1. The persuasion strength $\phi$ is assumed to be admissible, namely it satisfies properties (A1), (A2), and (A3). The following result holds true.

**Theorem 17.** *Let $z'_u \leq z''_u < z_s$ be points in $[0, 1]$ defined by*

$$\begin{aligned}
\phi(\sqrt{z'_u}) &= \beta^{-1}\bar{d}\rho_A^{-1}, \\
z''_u &= z_u(\Delta\bar{d}^{-1}\beta), \\
z_s &= z_s(\Delta\bar{d}^{-1}\beta),
\end{aligned} \quad (40)$$

*where $z_u(\cdot)$ and $z_s(\cdot)$ have been defined in (5). Depending on the conditions of the various parameters each of this point may exist or not. Below, whenever we write them, we are implicitly affirming their existence. Let $A = 14(1+\beta)^2$.*

*For every $\varepsilon > 0$ we can find $C^i_\varepsilon > 0$, $T^i_\varepsilon > 0$ for $i = 1, 2, 3$, $K_\varepsilon > 0$, and $S_\varepsilon > 0$ such that*

1) *if $\beta < \bar{d}\Delta^{-1}\phi(1)^{-1}$, then $\forall z$,*

$$\mathbb{P}_z\left(\sup_{t \geq T^1_\varepsilon} Z(t) > \varepsilon\right) \leq 5Ne^{-C^1_\varepsilon N};$$

2) *if $\bar{d}\gamma^{-1}\beta^* < \beta < \bar{d}\rho_A^{-1}\phi(0)^{-1}$, then*
   - $\forall z < z'_u - 4\varepsilon$,

$$\mathbb{P}_z\left(\sup_{t \geq S_\varepsilon} Z(t) > \varepsilon\right) \leq K_\epsilon N^{-1/2},$$

   - $\forall z > z''_u + \varepsilon$,

$$\mathbb{P}_z\left(\inf_{t \in [T^2_\varepsilon, T^2_\varepsilon + e^{C^2_\varepsilon N}]} Z(t) < z_s - \varepsilon\right) \leq AN^2e^{-C^2_\varepsilon N};$$

3) *if $\bar{d}\gamma^{-1}\phi(0)^{-1} < \beta$, then $\forall z > \varepsilon$,*

$$\mathbb{P}_z\left(\inf_{t \in [T^2_\varepsilon, T^2_\varepsilon + e^{C^2_\varepsilon N}]} Z(t) < z_s - \varepsilon\right) \leq AN^2e^{-C^2_\varepsilon N}.$$

*(with the understanding that if $\phi(0) = 0$, then case 3) does not show up). Moreover, for every $\varepsilon > 0$, the various constants exhibit*

*the following dependence on the parameters. $C^1_\varepsilon$ and $T^1_\varepsilon$ only depend on the quantity $\beta\bar{d}^{-1}\Delta$ and are, respectively, bounded away from 0 and bounded, when this quantity is bounded away from $\phi(1)^{-1}$. $C^2_\varepsilon$ and $T^2_\varepsilon$ only depend on the quantity $\beta\bar{d}^{-1}\gamma$ and are, respectively, bounded away from 0 and bounded, when this quantity is bounded away from $\beta^*$ and $\phi(0)^{-1}$. Finally, $K_\varepsilon$ and $S_\varepsilon$ only depend on $\varepsilon$.*

*Proof.* Item 1) is a consequence of Corollary 10. Regarding item 2), notice first that the following inequalities hold

$$\gamma \le \rho_A \le \bar{d} \le \Delta \quad \text{and} \quad \phi(1)^{-1} \le \beta^* \le \phi(0)^{-1}. \tag{41}$$

The first and third inequalities of the first expression are trivial, while the second one comes form [23]. The second expression is a direct consequence of the monotonicity of $\phi(z)$. This yields, in particular, $\gamma^{-1}\beta^* \ge \rho_A^{-1}\phi(1)^{-1}$. Item 2) now follows from item 2) of Corollary 16 and from item 1) of Corollary 8. Finally, item 3) comes from item 2) of Corollary 8. ■

**Remark 18.** *Using item 1) of Corollary 16, we can obtain the following variant of item 1) of Theorem 17:*

*1 bis) if $\beta < \bar{d}\rho_A^{-1}\phi(1)^{-1}$, then $\forall\, z$,*

$$\mathbb{P}_z\left(\sup_{t\ge T'_\varepsilon} Z(t) > \varepsilon\right) \le C'_\varepsilon N^{-1/2},$$

*where the constants $C'_\varepsilon$ and $T'_\varepsilon$ only depend on the quantity $\beta\bar{d}^{-1}\rho_A$ and are bounded when this quantity is bounded away from $\phi(1)^{-1}$. On the one hand, 1 bis) improves the result by widening the interval for $\beta$, as $\rho_A \le \Delta$ and the gap between the two quantities may actually be large. On the other hand, it weakens the result in terms of probability decay. Bound 1 bis) can be useful when dealing with large-scale random graphs, where results on the concentration of $\rho_A$ and $\Delta$ have already a slow decay in probability, so that the exponential decay would be lost in any case.*

Theorem 17 is a result that holds true for any possible graph. Of course, its most interesting use is for sequences of graphs having size $N \to +\infty$. Since the various thresholds and constants involved in the statement depend on graph properties (and thus ultimately on $N$), suitable assumptions on the sequence of graphs are needed in order for the three regimes to be observed in the large scale limit, similarly to the mean field case. Notably, in order to ensure the existence of the intermediate regime with the bifurcation with respect to the initial condition, we need to consider graphs where the Cheeger constant (14) and the average degree have the same asymptotic behavior, as the population size $N$ grows. In the next section we will show some very interesting topologies with this property. To sum up, in Theorem 17 we notice two important differences with respect to the results established in Theorem 2. First, the cases considered in Theorem 17 are not exhaustive as the various intervals considered for the parameter $\beta$ do not cover the whole positive line. Second, exponential decay of probabilities is not always insured.

## 4 Analytical and Numerical Results on Specific Topologies

In this section, we discuss the application of Theorem 17 to specific sequences of graphs with increasing size $N$. For the sake of simplicity we stick to the case $\phi(z) = z$. This choice of the persuasion strength yields $\beta^* = 4$ and to the occurrence of only the cases depicted in items 1) and 2) of Theorem 17.

First we introduce the notion of a regularly expansive sequence of graphs that includes popular random graphs examples like Erdős-Rényi graphs and random configuration models. For such graphs, we show that Theorem 17 guarantees the existence of the two regimes: the first one where failure always occurs and the second one where both failure and success may occur, depending on the initial condition. Finally we present some numerical simulations on Erdős-Rényi graphs and random configuration models corroborating our analytical results. We conclude with some simulations on graphs for which Theorem 17 does not give any information. Such simulations suggest that these bifurcation phenomena should hold under less stringent assumptions than those assumed in results.

We recall below the graph parameters that need to be computed (or at least estimated) in order to use Theorem 17:

- $\bar{d}$ and $\Delta$ are, respectively, the average and the largest degree of the graph;
- $\gamma$ is the Cheeger constant of the graph, defined in (14);
- $\rho_A$ is the spectral radius of the adjacency matrix.

A sequence of graphs $G_N$ with increasing number of nodes $N$ is called $(a, e_1, e_2)$-*regularly expansive* if, for every $N$,

$$\bar{d}\Delta^{-1} \ge a \qquad \text{and} \qquad e_1 \le \bar{d}\rho_A^{-1} \le \bar{d}\gamma^{-1} \le e_2.$$

Notice that, because of (41), we can always choose $e_1 \ge a$.

For such graph sequences, Theorem 17 can be reformulated as follows.

**Corollary 19.** *Assume that $\phi(z) = z$ and that $G_N$ is a $(a, e_1, e_2)$-regularly expansive sequence of graphs. Let $z'_u \le z''_u < z_s$ be points defined by*

$$\begin{aligned} z'_u &= \beta^{-2}e_1^2, \\ z''_u &= \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4e_2}{\beta}}, \\ z_s &= \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4e_1}{\beta}}. \end{aligned} \tag{42}$$

*Let $A = 14(1+\beta)^2$. For every $\beta > 0$ and for every $\varepsilon > 0$ we can find $C_\varepsilon > 0$, $K_\varepsilon > 0$, and $T_\varepsilon > 0$ for which the following holds true for every $N$:*

*1) if $\beta < a$, then $\forall z$,*

$$\mathbb{P}_z\left(\sup_{t\ge T_\varepsilon} Z(t) > \varepsilon\right) \le 5Ne^{-C_\varepsilon N};$$

*2) if $\beta > 4e_2$, then $\forall z < z'_u - 4\varepsilon$,*

$$\mathbb{P}_z(\exists t \ge T_\epsilon \mid Z(t) > \epsilon) \le K_\epsilon N^{-1/2};$$

*and $\forall z > z''_u + \varepsilon$,*

$$\mathbb{P}_z\left(\inf_{t\in[T_\varepsilon, T_\varepsilon + e^{C_\varepsilon N}]} Z(t) < z_s - \varepsilon\right) \le AN^2 e^{-C_\varepsilon N}.$$

*Proof.* It is an immediate consequence of Theorem 17, of the explicit formulas (6), and the inequalities (41). ∎

**Remark 20.** *We stress the fact that the quantities $C_\varepsilon$, $K_\varepsilon$ and $T_\varepsilon$ appearing in the statement of Corollary 19, only depend on the graph parameters $a, e, 1, e_2$ and on the choice of $\beta$, but not on the size $N$.*

We notice that, the condition $a > 0$ ensures that there exists a transition with respect to the parameter $\beta$ from the failure regime to the intermediate regime. Instead, the condition $e_1 > 0$ ensures the presence of the transition with respect to the initial condition in the second regime.

Below we present two fundamental examples of random graph ensembles which yield, under specific assumptions, regularly expansive graphs sequences.

**Example 1. (Erdős-Rényi graphs)** *The Erdős-Rényi model $G(N,p)$ is the first random graph model, introduced in 1959 [24]. $G(N,p)$ is a random undirected graph with $N$ nodes where each edge $\{u,v\}$ is independently present with a probability $p \in (0,1)$. The degree of each node is thus a realization of a binomial random variable with parameters $N-1$ and $p$, which means that the expected average degree is $(N-1)p$. Standard concentration results [11] show that with high probability (w.h.p.)[1] as $N \to \infty$,*

$$\bar{d} \asymp \Delta \asymp Np.$$

*Here we restrict our analysis to the case above connectivity threshold, i.e. when $\frac{\ln N}{Np} \to 0$. In this regime, w.h.p. $G(N,p)$ is connected [11] and, moreover,*

$$\bar{d}\gamma^{-1} = 2 + o(1)\,[11] \qquad \text{and} \qquad \bar{d}\rho_A^{-1} = 1 + o(1)\,[25].$$

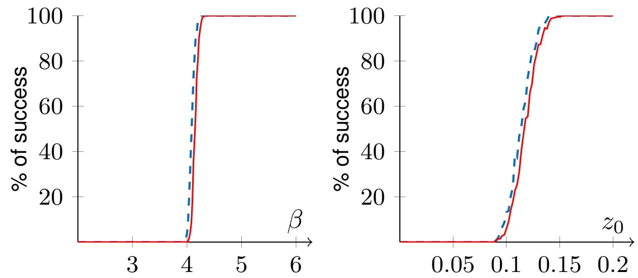
*This implies that in the connectivity regime, $G(N,p)$ is w.h.p. $(1-\delta, 1-\delta, 2+\delta)$-regularly expansive for any $\delta > 0$.*

**Example 2. (Configuration model)** *Consider a probability distribution $q_d$ over $\{3, \dots, d_{max}\}$. The configuration model $G(N, q_d)$ is a random undirected graphs with $N$ nodes whose degrees are independent random variables distributed according to $q_d$ and where edges are created through a random permutation (see [26] for details). Notice that, by construction, $3 \le \bar{d} \le \Delta \le d_{max}$. Moreover, $\exists\, \alpha > 0$ such that $\gamma \ge \alpha$ for all finite $N$ and w.h.p. as $N \to \infty$ [26]. Consequently, $G(N, q_d)$ is w.h.p. $(3/d_{max}, 3/d_{max}, d_{max}/\alpha)$-regularly expansive.*

Numerical simulations with Erdős-Rényi graphs and with regular configuration models are shown in Figs. 1 and 2. In particular, Fig. 1 shows the bifurcation with respect to $\beta$ and the bifurcation with respect to the initial condition in the intermediate regime for $\beta = 10$. Simulations seem to show that such bifurcations are sharp as it was happening in the mean field case (notice also that the bifurcation with respect to $\beta$ is placed in the same position $\beta^* = 4$). Fig. 2 deepens the analysis of the bifurcation with respect to the initial condition that seems to become sharper for larger and larger values of $N$.

In this final paragraph we consider some examples of graph sequences that are not regularly expansive and for which, consequently, Theorem 17 can not infer the presence of the phase transitions. Nevertheless, we show through numerical simulations that such phenomena (or at least some of them) do take place.

1. A family of events $E_n$ occur with high probability (w.h.p.) if $\mathbb{P}[E_n] \ge 1 - CN^{-\alpha}$, for some $\alpha > 0$.




(a) Success varying $\beta$, $z_0 = 1$. (b) Success varying $z_0$, $\beta = 10$.

Fig. 1: Simulations on random graphs with $N = 1000$. In Erdős-Rényi graphs $p = 0.05$ (blue dashed), in the regular configuration model $\bar{d} = 20$ (red solid).

**Example 3. (Barábasi-Albert model)** *The Barábasi-Albert model is a random graph model introduced in 1999 to represent social networks [27]. Starting from an initial connected graph, at each time step a node is added to the graph and it is connected to $m$ existing nodes with a probability proportional to their degrees, until there are $N$ nodes (see [27] for details). This algorithm constructs a graph whose degree distribution follows asymptotically a power-law [27] (in particular $\mathbb{P}[d_v = k] \propto k^{-3}$). As $N \to \infty$ it is immediate to verify that $\bar{d} = m + o(1)$ (due to construction). On the other hand, from [25], $\exists\, \alpha > 0$ such that w.h.p.*

$$\Delta = \sqrt{n}(1+o(1)), \quad \rho_A = \sqrt[4]{n}(1+o(1)) \quad \text{and} \quad \gamma \ge \alpha.$$

*Therefore, Barábasi-Albert graphs are only $(0, 0, m/\alpha + \delta)$-regularly expansive, for any $\delta > 0$.*

**Example 4. (Toroidal graphs)** *A 1-torus is a cyclic graph $C_n$. A $k$-torus can be defined as the cartesian product between $k$ 1-tori with $\sqrt[k]{N}$ nodes each [28]. For a $k$-torus we have that $\gamma \asymp N^{-k/2}$ and $\bar{d} = 2k$ is constant. Therefore, $\bar{d}/\gamma$ always diverges.*

Therefore, in these two examples Corollary 19 can not be applied to prove any phase transitions neither on $\beta$, nor on

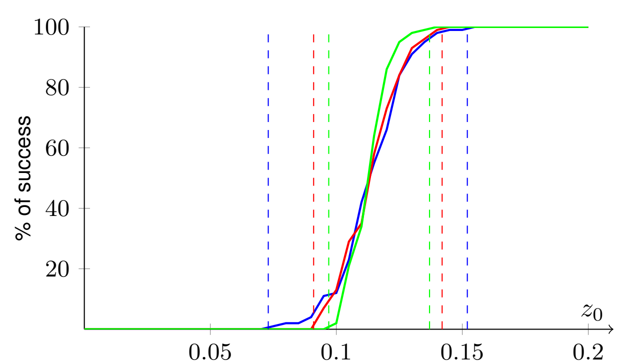
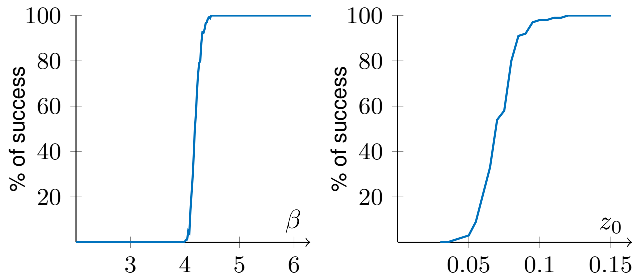
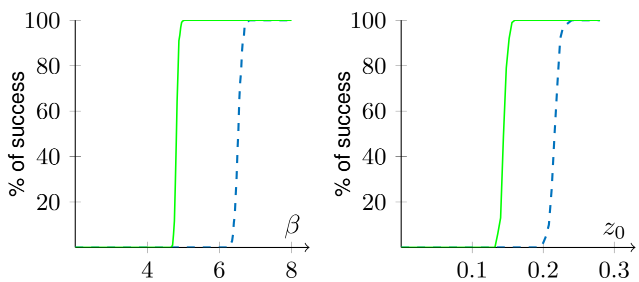



Fig. 2: Simulations on Erdős-Rényi graphs with $\beta = 10$ and $n = 800$ (blue), $n = 1200$ (red) and $n = 1600$ (green). The vertical dotted lines are the estimated thresholds $z_u'$ and $z_u''$. As $N$ increases the transition seems to be sharper. Notice that the analytical thresholds from Corollary 19 are $z_u' = 0.01$ and $z_u'' \simeq 0.2764$.




(a) Success varying on $\beta$, $z_0 = 1$. (b) Success varying $z_0$, $\beta = 10$.

Fig. 3: Simulations of the dynamics on Barábasi-Albert graphs with $N = 1000$ and $m = 6$.




(a) Success varying $\beta$, $z_0 = 1$. (b) Success varying $z_0$, $\beta = 10$.

Fig. 4: Simulations of the dynamics on $k$-tori (dashed in blue $k = 1$, solid in green $k = 2$) with $N = 1024$.

the initial condition. Nevertheless, in the case of Barábasi-Albert graphs, simulations presented in Figs. 3a and 3b show the existence of the two different regimes: the failure regime, and a regime where success and failure are both possible depending on the initial condition. However, from our simulations, the phase transition in this intermediate regime seems to be smooth, even increasing $N$ (see Fig. 3b).

Even in the case of $k$-tori, simulations (see Fig. 4) seem to show the existence of the intermediate regime. In this case the transition with respect to the initial condition seems to be sharp, as one can see in Fig. 4b. Finally, the simulations in Fig. 4 also suggest that the various thresholds for a $k$-torus are monotonically decreasing in the dimension $k$. This is intuitive for the role played by the connectivity features of the graph in the diffusion dynamics.

## 5 Conclusions and future research

In this work we have proposed a novel network dynamics modeling the diffusion of the adoption of a new technological item, such as a smart-phone application or a PC program. For the spread of such "light choices", we have proposed a novel gossip diffusion mechanism whose strength depends on the global diffusion of the item in the community, coupled with a spontaneous regression drift. This model can also be interpreted as a generalization of an SIS epidemic model with a non fixed infection probability. We have proven that, for important classes of random interaction graphs (e.g. Erdős-Rényi graphs and random configuration models with fixed and bounded degree distribution), depending on the strength of the gossip mechanism and the connectivity of the graph, three possible regimes show up: one where the diffusion of the item fails and it quickly disappears, one where the diffusion succeed since the item diffuses to a consistent part of the population and persists for an exponentially long time (with respect to the size of the system), and, finally, one intermediate regime where both scenarios can appear, depending on the initial fraction of agents possessing the item. This intermediate regime is the main novelty of our model with respect to a standard SIS epidemic model, where no dependence on the initial condition shows up. Simulations seem to suggest that even for more general graphs (e.g. preferential attachment graphs, regular grids) such phase transitions phenomena do take place and a theoretical analysis of such cases is left for future research.

The main research lines arising from this work go in two different directions. On the one hand, in view of the results of the simulations in Section 4, as already pointed out above, we are interested in extending our analytical results to some classes of non-expander graphs used to model social networks (e.g. scale-free networks). On the other hand, we want to test our model in real world case studies. Our aim is to acquire temporal data describing the spread of a technological issue on social networks (e.g. the use of a service or the downloads of an application for smartphone) and, against them, to test our model also in comparison with classical epidemic models and with standard model used for "big choices". Of course a key point is the tuning of the parameters appearing in the model, specifically $\beta$ and the function $\phi$. As the persuasion strength function $\phi$ is considered, the first step consists in choosing a parametrized family of functions $\phi(z)$ consistent with the properties $(A1)$, $(A2)$ and $(A3)$, one simple possibility being $\phi(z; a) = (a-1)^{-1}(a-z)z$, with $a > 2$. Thereafter, the new parameter $a$ and $\beta$ can be estimated from real world data using parameter identification methods similar to the one used for epidemic models [29]. Parameterizations with more degrees of freedom can also be considered, being careful of course not to over-fit the data.

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