

Improved Lp-Poincaré inequalities on the hyperbolic space

Original

Improved Lp-Poincaré inequalities on the hyperbolic space / Berchio, Elvise; D'Ambrosio, Lorenzo; Ganguly, Debdip; Grillo, Gabriele. - In: NONLINEAR ANALYSIS. - ISSN 0362-546X. - 157:(2017), pp. 146-166. [10.1016/j.na.2017.03.016]

Availability:

This version is available at: 11583/2675840 since: 2017-07-06T11:07:29Z

Publisher:

Elsevier Ltd

Published

DOI:10.1016/j.na.2017.03.016

Terms of use:

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

(Article begins on next page)

IMPROVED L^p -POINCARÉ INEQUALITIES ON THE HYPERBOLIC SPACE

ELVISE BERCHIO, LORENZO D'AMBROSIO, DEBDIP GANGULY, AND GABRIELE GRILLO

ABSTRACT. We investigate the possibility of improving the p -Poincaré inequality $\|\nabla_{\mathbb{H}^N} u\|_p^p \geq \Lambda_p \|u\|_p^p$ on the hyperbolic space, where $p > 1$ and $\Lambda_p := [(N-1)/p]^p$ is the best constant for which such inequality holds. We prove several different, and independent, improved inequalities, one of which is a Poincaré-Hardy inequality, namely an improvement of the best p -Poincaré inequality in terms of the Hardy weight r^{-p} , r being geodesic distance from a given pole. Certain Hardy-Maz'ya-type inequalities in the Euclidean half-space are also obtained.

1. INTRODUCTION

Let \mathbb{H}^N denote the hyperbolic space of dimension $N \geq 2$, $\nabla_{\mathbb{H}^N}$, $\Delta_{\mathbb{H}^N}$ and $dv_{\mathbb{H}^N}$ its Riemannian gradient, Laplacian and measure, respectively. It is well known that the L^2 spectrum of $-\Delta_{\mathbb{H}^N}$ is bounded away from zero. More precisely one has $\sigma(-\Delta_{\mathbb{H}^N}) = [(N-1)^2/4, +\infty)$. As a byproduct, the quadratic form inequality

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 dv_{\mathbb{H}^N} \geq \frac{(N-1)^2}{4} \int_{\mathbb{H}^N} u^2 dv_{\mathbb{H}^N}$$

holds for all $u \in C_c^\infty(\mathbb{H}^N)$. See e.g. [14] for an elementary proof. Besides, another inequality which one is very familiar within the Euclidean setting, namely *Hardy's inequality*, holds true as well on \mathbb{H}^N , so that one has, at least for $N \geq 3$,

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 dv_{\mathbb{H}^N} \geq \frac{(N-2)^2}{4} \int_{\mathbb{H}^N} \frac{u^2}{r^2} dv_{\mathbb{H}^N},$$

where $r := \rho(x, x_0)$ denotes geodesic distance from a fixed pole x_0 . In fact, such inequality holds on any Cartan-Hadamard manifold, where the latter are defined as those manifolds which are complete, simply connected and have nonpositive sectional curvatures. See [12] for details. Hardy-type inequalities have been the object of a large amount of research in the past decades, see for example, with no claim of completeness, [3, 4, 8, 9, 10, 11, 13, 15, 16, 18, 21, 22, 23, 25, 27, 30, 32].

A combination of these inequalities was given in [1] and then rediscovered by other methods in [6]. A simplified version of it reads

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 dv_{\mathbb{H}^N} - \frac{(N-1)^2}{4} \int_{\mathbb{H}^N} u^2 dv_{\mathbb{H}^N} \geq \frac{1}{4} \int_{\mathbb{H}^N} \frac{u^2}{r^2} dv_{\mathbb{H}^N} \quad (1.1)$$

for all $u \in C_c^\infty(\mathbb{H}^N)$, and the constants in (1.1) are sharp (the sharpness of the constant $(N-1)^2/4$ in the l.h.s. being obvious), see [6]. The sharpness of related inequalities in more general manifolds and similar improved inequalities of Rellich type, which are again sharp

Key words and phrases. p -Poincaré inequality, hyperbolic space, Poincaré-Hardy inequality .

23 in suitable senses, are also proved in [6]. See also [5] for related higher order Poincaré-Hardy
24 inequalities.

25 No L^p analogue of (1.1) is known for $p \neq 2$. It is our purpose here to initiate a study of
26 *improved p -Poincaré inequalities* on \mathbb{H}^N , where we take the attitude of looking for improve-
27 ments of the L^p -gap inequality

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p dv_{\mathbb{H}^N} \geq \left(\frac{N-1}{p} \right)^p \int_{\mathbb{H}^N} |u|^p dv_{\mathbb{H}^N}, \quad (1.2)$$

28 valid for all $u \in C_c^\infty(\mathbb{H}^N)$, where it is known that the constant $\left(\frac{N-1}{p} \right)^p$ is the best one for
29 such an inequality to hold, see [28] (a simpler proof of this fact will anyway be given below
30 in Lemma 2.1).

31 In fact, let $-\Delta_{p,\mathbb{H}^N}$ denote the p -Laplacian operator on \mathbb{H}^N , namely

$$\Delta_{p,\mathbb{H}^N} u := \operatorname{div}_{\mathbb{H}^N} (|\nabla_{\mathbb{H}^N} u|^{p-2} \nabla_{\mathbb{H}^N} u) \quad (1.3)$$

32 It is well-known that \mathbb{H}^N is a p -hyperbolic manifold, i.e., $-\Delta_{p,\mathbb{H}^N}$ admits a positive Green's
33 function by which the validity of a Hardy-type inequality follows. Less evident is the answer
34 to the following question:

35 *Problem.* Does there exist a nonnegative, not identically zero weight W such that the
36 following improved Poincaré inequality

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p dv_{\mathbb{H}^N} - \left(\frac{N-1}{p} \right)^p \int_{\mathbb{H}^N} |u|^p dv_{\mathbb{H}^N} \geq \int_{\mathbb{H}^N} W |u|^p dv_{\mathbb{H}^N} \quad (1.4)$$

37 holds for all $u \in C_c^\infty(\mathbb{H}^N)$?

38 A first affirmative answer to the above question was given in [7], see formula (5.25) there.
39 In fact, the authors prove the following result:

40 **Proposition 1.1** ([7]). *Let $p > 1$ and $N \geq 2$. Set $r := \varrho(x, x_0)$ with $x_0 \in \mathbb{H}^N$ fixed. There
41 exists a radial weight $0 < W = W(r)$ such that for all $u \in C_c^\infty(\mathbb{H}^N)$ there holds*

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p dv_{\mathbb{H}^N} - \left(\frac{N-1}{p} \right)^p \int_{\mathbb{H}^N} |u|^p dv_{\mathbb{H}^N} \geq \int_{\mathbb{H}^N} W |u|^p dv_{\mathbb{H}^N}.$$

42 *Furthermore,*

43 • *near x_0 there holds*

$$W(r) \underset{r \rightarrow 0}{\sim} \begin{cases} \left(\frac{N-p}{p} \right)^p \frac{1}{r^p} & \text{if } N > p, \\ \left(\frac{N-1}{N} \right)^N \frac{1}{r^{N(\log \frac{1}{r})^N}} & \text{if } N = p, \\ C \frac{1}{r^{\frac{p(N-1)}{p-1}}} & \text{if } N < p, \end{cases} \quad (1.5)$$

44 *where $C = C(p, N) := \left(\frac{p-1}{p} \right)^p \left(\int_0^\infty (\sinh s)^{-\frac{N-1}{p-1}} ds \right)^{-p}$ for $N < p$.*

• *Near infinity, there holds*

$$W(r) = \Lambda_p \frac{(N-1)p}{2(N-1+2(p-1))} \sinh(r)^{-2} + o(e^{-3r}) \quad \text{as } r \rightarrow \infty.$$

45 Hence, the given improvement of the Poincaré inequality is stated in terms of a weight
46 which is power-like near a given pole but exponentially decaying at infinity.

47 In the present paper we construct different examples of weights W for which inequality
48 (1.4) holds and that are slowly decaying at infinity. In any case, due to their asymptotic
49 behavior the weights provided are not globally comparable. For instance, we prove the
50 existence of a weight which is bounded but does not globally vanish at infinity. Finally,
51 in a suitable range of p we improve the Poincaré inequality via the Hardy weight $W =$
52 $\frac{C}{\varrho^p(x, x_0)}$, where $\varrho(x, x_0)$ is the geodesic distance from $x_0 \in \mathbb{H}^N$ fixed and $C = C(N, p)$ is a
53 positive constant. This choice seems to be the best compromise to capture the non euclidean
54 behavior of inequality (1.4) at infinity without losing too much information at the origin.
55 An uncertainty principle Lemma for the shifted Laplacian then follows immediately. The
56 techniques applied in the proofs are: hyperbolic symmetrization and p -convex inequalities
57 together with a suitable transformation which uncovers the Poincaré term. Furthermore,
58 super-solution technique and potential inequalities have been exploited.

59 The paper is organized as follows. In Section 2 we state our main results on \mathbb{H}^N , Theorems
60 2.2-2.5. Section 3 discusses a related result in the Euclidean half-space, which is the key
61 one to prove some of the results valid on \mathbb{H}^N but can have some independent interest,
62 see Theorem 3.2. Section 4 contains, for the convenience of the reader, a concise proof
63 of Proposition 1.1. Section 5 discusses the proofs of Theorem 3.2 and, consequently, of
64 Theorem 2.2, which is an improvement of the Poincaré inequality in terms of a weight having
65 different asymptotics in different “directions” and, in particular, not vanishing everywhere
66 at infinity. Theorem 2.3, which states a *Hardy-type improvement* of the Poincaré inequality
67 in the spirit of [1], [6], is proven in Section 6. Our final result, Theorem 2.5, deals with a
68 related weighted inequality on the whole \mathbb{H}^N . Even if it is not a direct improvement of the
69 Poincaré inequality for $p \neq 2$, it has an independent interest in itself due to the asymptotic
70 behavior of the involved weight. It is proved in Section 7, where as byproduct we obtain a
71 Poincaré type inequality on geodesic balls.

72 2. PRELIMINARIES AND RESULTS

73 We have mentioned before that inequality (1.2) holds, and that the constant

$$\Lambda_p := \left(\frac{N-1}{p} \right)^p \quad (2.1)$$

74 appearing there is optimal. This is in fact a particular case of the work given in [28], but
75 we provide a simple proof below for the convenience of the reader.

76 **Lemma 2.1.** *Let $N \geq 2$, $p > 1$ and set Λ_p as in (2.1). There holds*

$$\inf_{u \in W^{1,p}(\mathbb{H}^N) \setminus \{0\}} \frac{\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p \, dv_{\mathbb{H}^N}}{\int_{\mathbb{H}^N} |u|^p \, dv_{\mathbb{H}^N}} = \Lambda_p. \quad (2.2)$$

77 *Proof.* Considering the upper half space model for \mathbb{H}^N , namely $\mathbb{R}_+^N = \{(x, y) \in \mathbb{R}^{N-1} \times \mathbb{R}^+\}$
78 endowed with the Riemannian metric $g_{ij} = \frac{\delta_{ij}}{y^2}$ and using the expression of p -Laplacian (1.3)
79 in these coordinates we have

$$\Delta_{p, \mathbb{H}^N} u = y^N \partial_i (y^{p-N} |\nabla u|^{p-2} \partial_i u).$$

80 By computing $-\Delta_{p,\mathbb{H}^N}$ for the function $\rho(x, y) := y^\alpha \in W_{loc}^{1,p}(\mathbb{H}^N)$ where $\alpha := \frac{N-1}{p-1}$, one
81 has

$$-\Delta_{p,\mathbb{H}^N} \rho = \alpha^{p-2} \alpha (N-1 - \alpha(p-1)) y^{\alpha(p-1)} = 0.$$

82 Now we are in the position to apply Theorem 2.1 of [13], obtaining

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p \, dv_{\mathbb{H}^N} \geq \left(\frac{p-1}{p} \right)^p \int_{\mathbb{H}^N} |u|^p \frac{|\nabla_{\mathbb{H}^N} \rho|^p}{\rho^p} \, dv_{\mathbb{H}^N} = \Lambda_p \int_{\mathbb{H}^N} |u|^p \, dv_{\mathbb{H}^N}$$

83 for all $u \in C_c^\infty(\mathbb{H}^N)$ and hence, by density, for all $u \in W^{1,p}(\mathbb{H}^N)$.

On the other hand, for $\varepsilon > 0$, set

$$U_\varepsilon(x, y) = \left(\frac{y}{(1+y)^2 + |x|^2} \right)^{\frac{N-1+\varepsilon}{p}}.$$

Since in the coordinates (x, y) the volume element reads $dv_{\mathbb{H}^N} = \frac{dx \, dy}{y^N}$ and $\nabla_{\mathbb{H}^N} u = y^2 \nabla u$, we get

$$\int_{\mathbb{H}^N} |U_\varepsilon|^p \, dv_{\mathbb{H}^N} = \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \left(\frac{y}{(1+y)^2 + |x|^2} \right)^{N-1+\varepsilon} \frac{dx \, dy}{y^N}$$

and

$$\begin{aligned} & \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} U_\varepsilon|^p \, dv_{\mathbb{H}^N} \\ &= \left(\frac{N-1+\varepsilon}{p} \right)^p \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \left(\frac{(1-y^2 + |x|^2)^2 + 4|x|^2 y^2}{((1+y)^2 + |x|^2)^2} \right)^{p/2} \left(\frac{y}{(1+y)^2 + |x|^2} \right)^{N-1+\varepsilon} \frac{dx \, dy}{y^N} \\ &\leq \left(\frac{N-1+\varepsilon}{p} \right)^p \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \left(\frac{y}{(1+y)^2 + |x|^2} \right)^{N-1+\varepsilon} \frac{dx \, dy}{y^N} \end{aligned}$$

84 Hence, $U_\varepsilon(x, y) \in W^{1,p}(\mathbb{H}^N)$ for $\varepsilon > 0$ and $\frac{\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} U_\varepsilon|^p \, dv_{\mathbb{H}^N}}{\int_{\mathbb{H}^N} |U_\varepsilon|^p \, dv_{\mathbb{H}^N}} \leq \left(\frac{N-1+\varepsilon}{p} \right)^p$. By letting
85 $\varepsilon \rightarrow 0$, this argument completes the proof of the lemma. \square

86 Now we are in a situation to state our main results.

87 In first place, by exploiting the half-space model for \mathbb{H}^N and following the approach of
88 [31], here below we provide a weight that does not globally decay at infinity but which is
89 bounded near x_0 . Hence, this choice turns out to be best suited to capture the non euclidean
90 behaviour of \mathbb{H}^N which occurs at infinity. More precisely, we prove

91 **Theorem 2.2.** *Let $p > 1$, $N \geq 2$ and set Λ_p as in (2.1). There exists a bounded weight*
92 $0 < V \leq 1$ such that for all $u \in C_c^\infty(\mathbb{H}^N)$ there holds

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p \, dv_{\mathbb{H}^N} - \Lambda_p \int_{\mathbb{H}^N} |u|^p \, dv_{\mathbb{H}^N} \geq \left(\frac{N-1}{p} \right)^{p-2} C(N, p) \int_{\mathbb{H}^N} V |u|^p \, dv_{\mathbb{H}^N}, \quad (2.3)$$

93 where $C(N, p)$ is a positive constant that can be explicitly computed for which the following
 94 estimates hold

$$\begin{aligned}
 C(N, p) &\geq \frac{1}{4p'}, & \text{if } 1 < p \leq 4/3, \\
 C(N, p) &\geq \left(2(8 - 3p) + 2\sqrt{p'(8 - 3p)}\right)^{-1}, & \text{if } 4/3 < p \leq 2, \\
 C(N, p) &= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}p + 2\sqrt{p}}, & \text{if } 2 < p \leq 2(N - 1)^2, \\
 C(N, p) &= \left(\frac{p}{N - 1} + 2p + 2(N - 1)\right)^{-1}, & \text{if } p > 2(N - 1)^2,
 \end{aligned} \tag{2.4}$$

95 where $p' > 1$ denotes the conjugate exponent of p .

96 Furthermore, set $r := \varrho(x, x_0)$ with $x_0 \in \mathbb{H}^N$ fixed, we have

- 97 • for any $0 < \alpha \leq 1$ there exists an unbounded set $U_\alpha \subset \mathbb{H}^N$ such that $V|_{U_\alpha} \equiv \alpha$ and
 98 $U_\alpha \cap (B(x_0, 2r) \setminus B(x_0, r)) \neq \emptyset$ as $r \rightarrow +\infty$;
- 99 • for any $\beta > 0$ there exists an unbounded set $W_\beta \subset \mathbb{H}^N$ such that $V|_{W_\beta} \sim \sqrt{\frac{\beta}{2}} e^{-r/2}$
 100 as $r \rightarrow +\infty$.

101 It is worth noticing that the weight V can be written, in the half-space model, as
 102 $V(x_1, \dots, x_{N-1}, y) := \frac{y}{\sqrt{y^2 + x_1^2}}$, see Theorem 3.2 in Section 3 from which the above state-
 103 ments follow.

104 Even if both the inequalities provided by Proposition 1.1 and Theorem 2.2 are of the form
 105 (1.4) they seem to lose too much information, respectively, at infinity or near the origin. To
 106 this aim, a good compromise is represented by the following Poincaré-Hardy inequality

107 **Theorem 2.3.** Let $p \geq 2$ and $N \geq 1 + p(p - 1)$. Set Λ_p as in (2.1) and $r := \varrho(x, x_0)$ with
 108 $x_0 \in \mathbb{H}^N$ fixed. Then for $u \in C_c^\infty(\mathbb{H}^N)$ there holds

$$\begin{aligned}
 &\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p \, dv_{\mathbb{H}^N} - \Lambda_p \int_{\mathbb{H}^N} |u|^p \, dv_{\mathbb{H}^N} \\
 &\geq (p - 1) \left(\frac{N - 1}{p}\right)^{p-2} \left(\frac{p - 1}{p}\right)^2 \int_{\mathbb{H}^N} \frac{|u|^p}{r^p} \, dv_{\mathbb{H}^N}.
 \end{aligned} \tag{2.5}$$

109 **Remark 2.1.** From the above Theorem, we can easily infer that the best constant in the
 110 r.h.s. of (2.5), i.e.

$$c_p := \inf_{C_c^\infty(\mathbb{H}^N) \setminus \{0\}} \frac{\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p \, dv_{\mathbb{H}^N} - \Lambda_p \int_{\mathbb{H}^N} |u|^p \, dv_{\mathbb{H}^N}}{\int_{\mathbb{H}^N} \frac{|u|^p}{r^p} \, dv_{\mathbb{H}^N}},$$

111 blows up as $N \rightarrow \infty$ if $p > 2$. This does not happen in the linear case $p = 2$, where $c_2 = \frac{1}{4}$,
 112 see (1.1), where it is known that the constant c_2 is optimal. This issue was proved in [6]
 113 by providing an explicit super-solution for the corresponding Euler-equation, a construction
 114 that also allows to determine a remainder term for (1.1) of the type $\frac{1}{\sinh^2 r}$, see Remark 2.3.
 115 Unfortunately, this argument carries over to the case $p > 2$ only partially thereby allowing
 116 to prove Theorem 7.2 below on suitable geodesic balls.

117 As an immediate consequence of the previous result one gets the following *uncertainty*
 118 *principle* for the quadratic form of the shifted Laplacian. For a similar result, when $p = 2$,
 119 concerning the quadratic form of the Laplacian, see [23, Theorem 4.1].

120 **Corollary 2.4.** *Let $p \geq 2$ and $N \geq 1 + p(p-1)$. Set Λ_p as in (2.1) and $r := \varrho(x, x_0)$ with*
 121 *$x_0 \in \mathbb{H}^N$ fixed. Then for $u \in C_c^\infty(\mathbb{H}^N)$ there holds:*

$$\begin{aligned} & \left[\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p \, dv_{\mathbb{H}^N} - \Lambda_p \int_{\mathbb{H}^N} |u|^p \, dv_{\mathbb{H}^N} \right] \left[\int_{\mathbb{H}^N} |u|^p r^{p'} \, dv_{\mathbb{H}^N} \right]^{\frac{p}{p'}} \\ & \geq (p-1) \left(\frac{N-1}{p} \right)^{p-2} \left(\frac{p-1}{p} \right)^2 \left[\int_{\mathbb{H}^N} |u|^p \, dv_{\mathbb{H}^N} \right]^p, \end{aligned} \quad (2.6)$$

122 where $p' > 1$ denotes the conjugate exponent of p .

123 **Remark 2.2.** In Theorem 2.3, the restrictions $p \geq 2$ and $N \geq 1 + p(p-1)$ are technical. In
 124 particular, the latter only comes from the last step in the proof. Nevertheless, the very same
 125 assumption also appears in the Poincaré-Hardy inequality below where the constant Λ_p in
 126 (2.5) is replaced by a non-constant weight: $\Lambda_p H_p(r)$. Here, $H_p(r)$ is a positive function
 127 which is larger than one in $(0, r_p)$, smaller than one in $(r_p, +\infty)$, and that converges to one
 128 as $r \rightarrow +\infty$, see Figure 1 in Section 7. Since the proofs of the two theorems are completely
 129 different, we are led to believe that a deeper relation between the dimension restriction and
 130 the weight considered might exist.

131 **Theorem 2.5.** *Let $p \geq 2$ and $N \geq 1 + p(p-1)$. Set Λ_p as in (2.1) and $r := \varrho(x, x_0)$ with*
 132 *$x_0 \in \mathbb{H}^N$ fixed. Then for $u \in C_c^\infty(\mathbb{H}^N)$ there holds*

$$\begin{aligned} & \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p \, dv_{\mathbb{H}^N} - \Lambda_p \int_{\mathbb{H}^N} H_p(r) |u|^p \, dv_{\mathbb{H}^N} \geq \\ & \frac{(p-1)^{p-1} (N(p-2) + 1)}{p^p} \int_{\mathbb{H}^N} \frac{|u|^p}{r^p} \, dv_{\mathbb{H}^N} \\ & + \frac{(N-1)(N-1-p(p-1))(p-1)^{p-2}}{p^p} \int_{\mathbb{H}^N} \frac{|u|^p}{\sinh^p r} \, dv_{\mathbb{H}^N} \end{aligned} \quad (2.7)$$

133 where $H_p(r) = \left(\coth r - \left(\frac{p-1}{N-1} \right) \frac{1}{r} \right)^{p-2}$.

134 **Remark 2.3.** When $p = 2$, the statement of Theorem 2.5 includes that of Theorem 2.3
 135 providing a further remainder term. Unfortunately, the weight H_p is larger than one only for
 136 r small, hence (2.7) is not an improvement of the p -Poincaré inequality if $p \neq 2$. Nevertheless,
 137 for functions having support outside large balls the inequality becomes very "close" to the
 138 Poincaré one, see Lemma 7.1.

139 In Section 7, from Theorem 2.5, we deduce an inequality involving the same weight of
 140 (2.5) but holding on geodesic balls.

141 3. RELATED HARDY-MAZ'YA-TYPE INEQUALITIES ON HALF-SPACE

142 This section is devoted to the study of improved Hardy-Maz'ya-type inequalities on upper
 143 half space. There have been an extensive research on Hardy-Maz'ya inequality (see [17, 19,
 144 24, 26]). Our main goal here is to present some Hardy-Maz'ya inequalities strictly related
 145 to our Poincaré-Hardy inequalities on the hyperbolic space. We begin with the counterpart
 146 of Lemma 2.1:

147 **Lemma 3.1.** *Let $p > 1$, $N \geq 2$ and set Λ_p as in (2.1). Then for all $u \in C_c^\infty(\mathbb{R}_+^N)$ there*
 148 *holds*

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|\nabla u|^p}{y^{N-p}} dx dy \geq \Lambda_p \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|u|^p}{y^N} dx dy, \quad (3.1)$$

149 where ∇u denotes the euclidean gradient. Moreover the constant Λ_p appearing in (3.1) is
150 sharp.

151 *Proof.* The proof of Lemma 3.1 follows by noticing that in the upper half space model for
152 \mathbb{H}^N , see the proof of Lemma 2.1, (2.2) readily writes as the Hardy-Maz'ya-type inequality
153 (3.1). Hence, the statement of Lemma 3.1 comes as a corollary of Lemma 2.1. \square

154 Next we turn to the main result of this section. We improve (3.1) by providing a suitable
155 remainder term.

156 **Theorem 3.2.** *Let $p > 1$, $N \geq 2$ and set Λ_p as in (2.1). For all $u \in C_c^\infty(\mathbb{R}_+^N)$ there holds*

$$\begin{aligned} & \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|\nabla u|^p}{y^{N-p}} dx dy - \Lambda_p \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|u|^p}{y^N} dx dy \geq \\ & \left(\frac{N-1}{p}\right)^{p-2} C(N,p) \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|u|^p}{y^{N-1} \sqrt{y^2 + x_1^2}} dx dy. \end{aligned} \quad (3.2)$$

157 where $C(N,p)$ is a positive constant as in (2.4).

158 It's worth noting that Theorem 2.2 turns out to be a consequence of the above theorem.
159 We postpone the proofs of Theorem 3.2 and, hence, of Theorem 2.2 to Section 5.

160

4. PROOF OF PROPOSITION 1.1

161 We recall for the convenience of the reader the proof given in [7], only the asymptotics at
162 infinity not being explicitly given there. The proof relies on the well known classical Hardy
163 inequality with respect to the Green's function and exploiting its behavior on hyperbolic
164 space. More precisely, for $N \geq 2$ and $p > 1$, the following Hardy inequality holds (see [13],
165 [7]):

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p dv_{\mathbb{H}^N} \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^N} \left| \frac{\nabla G_p}{G_p} \right|^p |u|^p dv_{\mathbb{H}^N}, \quad (4.1)$$

166 for $u \in C_c^\infty(\mathbb{H}^N)$, where G_p is the Green's function of $-\Delta_{p,\mathbb{H}^N}$ which, up to a positive
167 multiplicative constant, is given by

$$G_p(r) := \int_r^\infty (\sinh s)^{-\frac{N-1}{p-1}} ds.$$

168 Indeed, if $p > N$, then $G_p \in W_{loc}^{1,p}(\mathbb{H}^N)$ and hence [13, Theorem 2.1] applies. For $1 < p \leq N$
169 the inequality (4.1) holds for functions $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$, and since $\{x_0\}$ is a compact set
170 of zero p -capacity, the claim follows from [13, Corollary 2.3].

171 The proof is then a calculus exercise involving the asymptotics of the function $G_p(r)$.
172 Indeed, Eq. (4.1) may be rewritten as

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p dv_{\mathbb{H}^N} - \Lambda_p \int_{\mathbb{H}^N} |u|^p dv_{\mathbb{H}^N} \geq \int_{\mathbb{H}^N} W |u|^p dv_{\mathbb{H}^N},$$

where

$$W(r) := \left(\frac{p-1}{p}\right)^p \left| \frac{G_p'(r)}{G_p(r)} \right|^p - \Lambda_p,$$

173 with Λ_p as in (2.1).

First we claim that $W > 0$. From the expression of the Green's function we have

$$\begin{aligned} G_p(r) &= \int_r^\infty (\sinh s)^{-\frac{N-1}{p-1}} ds = \int_r^\infty (\sinh s)^{-\frac{N-1}{p-1}-1} \sinh s ds \\ &< \int_r^\infty (\sinh s)^{-\frac{N-1}{p-1}-1} \cosh s ds = \int_{\sinh r}^\infty t^{-\frac{N-1}{p-1}-1} dt \\ &= \frac{p-1}{N-1} (\sinh r)^{-\frac{N-1}{p-1}}. \end{aligned}$$

174 Moreover, we also have $G'_p(r) = -(\sinh r)^{-\frac{N-1}{p-1}}$. Therefore,

$$\left| \frac{G'_p(r)}{G_p(r)} \right|^p > \left(\frac{N-1}{p-1} \right)^p,$$

175 and hence this proves $\left(\frac{p-1}{p} \right)^p \left| \frac{G'_p(r)}{G_p(r)} \right|^p > \Lambda_p$.

Let us turn to study the asymptotic behavior of W near the origin. First consider the case when $N \geq p$. Then, $G_p(r) \rightarrow \infty$ as $r \rightarrow 0$ and, using de L'Hôpital's rule, we obtain:

$$\lim_{r \rightarrow 0} \frac{r G'_p(r)}{G_p(r)} = \frac{p-N}{p-1} \quad \text{if } N > p$$

and

$$\lim_{r \rightarrow 0} \frac{r \log r G'_p(r)}{G_p(r)} = 1 \quad \text{if } N = p.$$

176 Whence, the stated asymptotics easily follows.

177 When $N < p$, in the second term above one has $\int_r^\infty (\sinh s)^{-\frac{N-1}{p-1}} ds < \infty$ as $r \rightarrow 0$.

178 Hence, (1.5) follows immediately by exploiting $\sinh r \sim r$ as $r \rightarrow 0$.

Finally, we study the asymptotics of W near infinity. For this we note that

$$\begin{aligned} G_p(r) &= \int_r^\infty (\sinh s)^{-\frac{N-1}{p-1}} ds = \int_{\sinh r}^\infty t^{-\frac{N-1}{p-1}} (1+t^2)^{-\frac{1}{2}} dt \\ &= \int_{\sinh r}^\infty t^{-\frac{N-1}{p-1}-1} \left[1 - \frac{1}{2t^2} + o\left(\frac{1}{t^3}\right) \right] dt, \quad r \rightarrow \infty \\ &= \frac{p-1}{N-1} (\sinh r)^{-\frac{N-1}{p-1}} - \left(2\frac{N-1}{p-1} + 4 \right)^{-1} (\sinh r)^{-\frac{N-1}{p-1}-2} + o\left((\sinh r)^{-\frac{N-1}{p-1}-3} \right), \end{aligned}$$

179 hence we have

$$\begin{aligned} \left| \frac{G'_p(r)}{G_p(r)} \right|^p &= \left| \frac{p-1}{N-1} - \left(2\frac{N-1}{p-1} + 4 \right)^{-1} (\sinh r)^{-2} + o\left((\sinh r)^{-3} \right) \right|^{-p} = \\ &= \left(\frac{N-1}{p-1} \right)^p \left(1 + \frac{p\frac{N-1}{p-1}}{2\left(\frac{N-1}{p-1} + 2\right)} (\sinh r)^{-2} + o\left((\sinh r)^{-3} \right) \right). \end{aligned}$$

180 This completes the proof.

5. PROOF OF THEOREM 3.2 AND THEOREM 2.2

181

182 **Proof of Theorem 3.2**

183 The key ingredients in the proof are the following Lemma 5.1 from [31] that we adapt
 184 to our situation with a suitable choice of the parameters, and the inequality (5.3) which
 185 represents an improvement of the analogous inequalities presented in [31].

Lemma 5.1. [31, Lemma 2.1] *Let Ω be a convex domain in \mathbb{R}^N and set $\delta(z) := \text{dist}(z, \partial\Omega)$ for any $z \in \Omega$. Let $d \in (-\infty, mp - 1)$ where $m \in \mathbb{N}_+$ and let $\mathbf{F} = (F_1, \dots, F_N)$ be a $C^1(\Omega)$ vector field in \mathbb{R}^N . Furthermore, let $w \in C^1(\Omega)$ be a nonnegative weight function and*

$$h_{p,m,d} := \left(\frac{mp - d - 1}{p} \right)^p.$$

186 Then, the following inequality holds

$$\begin{aligned} \int_{\Omega} \frac{|\nabla u|^p w}{\delta^{(m-1)p-d}} dz &\geq h_{p,m,d} \left(\int_{\Omega} \frac{|u|^p w}{\delta^{mp-d}} - \frac{p|u|^p \Delta \delta w}{(mp-d-1)\delta^{mp-d-1}} dz \right) \\ + h_{p,m,d} \int_{\Omega} \left[\frac{p \operatorname{div} \mathbf{F}}{mp-d-1} + \frac{p-1}{\delta^{mp-d}} \left(1 - |\nabla \delta - \delta^{mp-d-1} \mathbf{F}|^{\frac{p}{p-1}} \right) \right] |u|^p w dz &\quad (5.1) \\ + \left(\frac{mp-d-1}{p} \right)^{p-1} \int_{\Omega} \nabla w \cdot \left(\mathbf{F} - \frac{\nabla \delta}{\delta^{mp-d-1}} \right) |u|^p dz, \end{aligned}$$

187 for all $u \in C_c^\infty(\Omega)$.

188 We will apply Lemma 5.1 with $\Omega = \mathbb{R}_+^N$. Hence, $z = (x_1, \dots, x_{N-1}, y) = (x, y)$ with
 189 $x \in \mathbb{R}^{N-1}$, $y \in \mathbb{R}^+$, and $\delta(z) = y$. Furthermore, we fix $w = 1$, $m = 2$ and $d = mp - N$ so
 190 that $d < mp - 1$ for any $p \geq 1$ and $N > 1$ and we obtain $h_{p,m,d} = \Lambda_p$. Then, (5.1) reads as
 191 follows.

192 **Lemma 5.2.** *Let $p > 1$, $N \geq 2$ and set Λ_p as in (2.1). For any any $C^1(\mathbb{R}_+^N)$ vector field
 193 $\mathbf{F} = (F_1, \dots, F_N)$, the following inequality holds*

$$\begin{aligned} \int_{\mathbb{R}_+} \int_{\mathbb{R}^{N-1}} \frac{|\nabla u|^p}{y^{N-p}} dx dy - \Lambda_p \int_{\mathbb{R}_+} \int_{\mathbb{R}^{N-1}} \frac{|u|^p}{y^N} dx dy &\geq \\ \Lambda_p \int_{\mathbb{R}_+} \int_{\mathbb{R}^{N-1}} \left[\frac{p \operatorname{div} \mathbf{F}}{N-1} + \frac{p-1}{y^N} \left(1 - |(0, \dots, 0, 1) - y^{N-1} \mathbf{F}|^{\frac{p}{p-1}} \right) \right] |u|^p dx dy, &\quad (5.2) \end{aligned}$$

194 for all $u \in C_c^\infty(\mathbb{R}_+^N)$.

195 **Lemma 5.3.** *Let $b > 0$ and $s \in [0, 1]$ then*

$$1 - (1-s)^b \geq bs - q_b(b-1)s^2 \quad (5.3)$$

196 where

$$q_b := \begin{cases} 1 & \text{if } 1 \leq b \leq 2; \\ b/2 & \text{if } 0 < b < 1 \text{ or } 2 < b. \end{cases} \quad (5.4)$$

197 *Proof.* Taylor expansion of $(1-s)^b$ around 0 gives $(1-s)^b = 1 - bs + \frac{b}{2}(b-1)s^2 + R(s)$ where
 198 the reminder term $R(s)$ is given by $R(s) = -s^3 b(b-1)(b-2)(1-t)^{b-3}/6$ with a suitable
 199 $t \in [0, s]$. For $s \in [0, 1]$ and $b \geq 2$ or $0 < b \leq 1$, $R(s) \leq 0$ and the claim follows.

200 For the case $1 < b < 2$ the claim will follow by proving that the function $g(s) :=$
 201 $(1-s)^b - 1 + bs - (b-1)s^2$ is nonpositive on $[0, 1]$. To this end since $g''' > 0$ one deduces
 202 that g'' is negative on an interval $]0, s_0[$ and positive on $]s_0, 1[$, which in turn, with the

203 fact that $g'(0) = 0$ and $g'(1) > 0$, implies that g' has only a critical point on $]0, 1[$. Since
 204 $g(0) = g(1) = 0$ and $g'(1) > 0$ we obtain that the maximum of g is 0.

205 \square

For sake of brevity we introduce the following notation

$$I(u) := \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|\nabla u|^p}{y^{N-p}} \, dx \, dy - \Lambda_p \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{|u|^p}{y^N} \, dx \, dy,$$

and

$$w := \frac{y}{\sqrt{y^2 + x_1^2}}.$$

206 Next, in the spirit of [31, Theorem 4.1], for any $0 \leq a \leq 1$ we write (5.2) with $\mathbf{F}_1 :=$
 207 $(0, \dots, \frac{aw}{y^{N-1}})$. Since $0 \leq w \leq 1$ we get

$$\operatorname{div} \mathbf{F}_1 \geq (2 - N)a \frac{w}{y^N} - a \frac{w^2}{y^N}, \quad (5.5)$$

208 and, by using (5.3) with $b = p'$ and the fact that $0 \leq aw \leq 1$, we have

$$1 - |(0, \dots, 1) - y^{N-1} \mathbf{F}_1|^{p'} = 1 - (1 - aw)^{p'} \geq p'aw - q_{p'}(p' - 1)a^2w^2. \quad (5.6)$$

209 By using (5.5) and (5.6) in (5.2), the square bracket in right hand side can be estimated as

$$\begin{aligned} & \left[\frac{p \operatorname{div} \mathbf{F}_1}{N-1} + \frac{p-1}{y^N} \left(1 - |(0, \dots, 0, 1) - y^{N-1} \mathbf{F}_1|^{\frac{p}{p-1}} \right) \right] \\ & \geq a \frac{p}{N-1} \frac{w}{y^N} - a \left(\frac{p}{N-1} + q_{p'}a \right) \frac{w^2}{y^N} =: S_1 \end{aligned} \quad (5.7)$$

210 Therefore, from (5.2) we obtain

$$I(u) \geq \Lambda_p \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} S_1 |u|^p \, dx \, dy \quad (5.8)$$

211 for all $u \in C_c^\infty(\mathbb{R}_+^N)$.

212 Similarly, for any $0 \leq c \leq 1$, choosing $\mathbf{F}_2 = c \left(\frac{x_1 w^2}{y^N}, 0, \dots, 0, \frac{y w^2}{y^N} \right)$, by an explicit compu-
 213 tation we obtain

$$\operatorname{div} \mathbf{F}_2 = c(2 - N) \frac{w^2}{y^N} \quad (5.9)$$

214 and

$$|(0, \dots, 1) - y^{N-1} \mathbf{F}_2|^2 = 1 - c(2 - c)w^2. \quad (5.10)$$

215 Evaluating the square bracket in r.h.s. of (5.2), by using (5.3) with $b = p'/2$ and the fact
 216 $0 \leq c(2 - c)w^2 \leq 1$, we have

$$\begin{aligned} & \left[\frac{p \operatorname{div} \mathbf{F}_2}{N-1} + \frac{p-1}{y^N} \left(1 - |(0, \dots, 0, 1) - y^{N-1} \mathbf{F}_2|^{\frac{p}{p-1}} \right) \right] \\ & = \frac{p}{N-1} c(2 - N) \frac{w^2}{y^N} + \frac{p-1}{y^N} \left(1 - (1 - c(2 - c)w^2)^{p'/2} \right) \\ & \geq \frac{p}{N-1} c \left(1 - c \frac{N-1}{2} \right) \frac{w^2}{y^N} - (p-1)c^2(2 - c)^2 q_{p'/2} \left(\frac{p'}{2} - 1 \right) \frac{w^4}{y^N} =: S_2 \end{aligned} \quad (5.11)$$

217 \triangleright From Lemma 5.2 we deduce

$$I(u) \geq \Lambda_p \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} S_2 |u|^p \, dx \, dy \quad (5.12)$$

Case $1 < p \leq 2$. In this case since $0 \leq w \leq 1$ and $p'/2 - 1 = \frac{(2-p)}{2(p-1)} \geq 0$ we have

$$S_2 \geq \frac{w^2}{y^N} \frac{p}{N-1} f(c),$$

where

$$f(c) := c \left(1 - c \frac{N-1}{2} \right) - c^2 (2-c)^2 q_{p'/2} \frac{(2-p)(N-1)}{2p}.$$

Set $M := \max\{f(c), c \in [0, 1]\}$. Since $f(0) = 0$ and $f'(0) = 1 > 0$ we have that $M > 0$. Hence we have

$$S_2 \geq M \frac{p}{N-1} \frac{w^2}{y^N},$$

218 which in turns yields

$$I(u) \geq \Lambda_p \frac{p}{N-1} M \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{w^2}{y^N} |u|^p \, dx \, dy. \quad (5.13)$$

For $1 < p \leq 2$, since $q_{p'} = \frac{p}{2(p-1)}$, (5.8) reads as

$$\begin{aligned} I(u) &\geq \Lambda_p \frac{p}{N-1} a \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{w}{y^N} |u|^p \, dx \, dy \\ &\quad - \Lambda_p \frac{p}{N-1} a \left(1 + \frac{N-1}{2(p-1)} a \right) \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{w^2}{y^N} |u|^p \, dx \, dy. \end{aligned} \quad (5.14)$$

219 Multiplying (5.13) by $\frac{a}{M} \left(1 + \frac{N-1}{2(p-1)} a \right)$ and summing up to (5.14) we have

$$I(u) \geq \Lambda_p \frac{p}{N-1} \mu_1(a) \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{w}{y^N} |u|^p \, dx \, dy, \quad (5.15)$$

where

$$\mu_1(a) := \frac{a}{1 + \frac{a}{M} \left(1 + \frac{N-1}{2(p-1)} a \right)}.$$

220 Setting $C(N, p) := \frac{N-1}{p} \max\{\mu_1(a), a \in [0, 1]\}$ we get the claim.

Now we proceed to obtain an explicit estimate on $C(N, p)$. To this end we first look for some bounds on $M = \max\{f(c), c \in [0, 1]\}$. Since $c \geq 0$ and $(2-p) \geq 0$ from the chain of inequalities

$$f(c) \leq c \left(1 - c \frac{N-1}{2} \right) \leq \frac{1}{2(N-1)},$$

221 we deduce

$$M \leq \frac{1}{2}. \quad (5.16)$$

Next step is to estimate the maximum of μ_1 . The function $\mu_1(a)$ for $a \geq 0$ attains its maximum at $a_0 := \sqrt{\frac{2(p-1)}{N-1}} M$. From the bound $M \leq 1/2$, we immediately deduce that $0 < a_0 \leq 1$, and hence

$$C(N, p) = \frac{N-1}{p} \mu_1(a_0) = \frac{N-1}{p} \frac{M}{1 + \sqrt{\frac{2(N-1)M}{p-1}}} =: \gamma(M).$$

Since γ is increasing, a bound from below on M yields a bound from below on $C(N, p)$. Set $\beta := N - 1$ and $\delta := q_{p'/2} \frac{2-p}{p}$. For $0 \leq c \leq 1$, $f(c)$ can be estimated as

$$f(c) = c \left(1 - c \frac{\beta}{2} (1 + 4\delta) + 2\beta\delta c^2 \left(1 - \frac{1}{4}c \right) \right) \geq c \left(1 - c \frac{\beta}{2} (1 + 4\delta) \right).$$

That is, by choosing $c_0 := \frac{1}{\beta(1+4\delta)}$, we have

$$M \geq f(c_0) = \frac{1}{2\beta(1+4\delta)},$$

222 and hence

$$C(N, p) = \gamma(M) \geq \gamma \left(\frac{1}{2\beta(1+4\delta)} \right) = \frac{1}{2p(1+4\delta)} \frac{1}{1 + ((p-1)(1+4\delta))^{-1/2}}. \quad (5.17)$$

223 Now, taking into account that for $1 < p \leq 4/3$ one has $q_{p'/2} = \frac{p'}{4}$, while for $4/3 < p \leq 2$
224 one gets $q_{p'/2} = 1$, plugging $\delta = \frac{2-p}{p} q_{p'/2}$ in (5.17), we obtain the estimates.

225 **Case $p > 2$.** In this case we have for any $c \in [0, 1]$

$$S_2 \geq \frac{p}{N-1} c \left(1 - c \frac{N-1}{2} \right) \frac{w^2}{y^N} - (p-1) c^2 (2-c)^2 q_{p'/2} \frac{2-p}{2} \frac{w^4}{y^N} \quad (5.18)$$

$$\geq \frac{p}{N-1} c \left(1 - c \frac{N-1}{2} \right) \frac{w^2}{y^N}. \quad (5.19)$$

226 Choosing $c = 1/(N-1)$ we obtain

$$S_2 \geq \frac{p}{N-1} \frac{1}{2(N-1)} \frac{w^2}{y^N}, \quad (5.20)$$

227 and hence we have

$$I(u) \geq \Lambda_p \frac{p}{N-1} \frac{1}{2(N-1)} \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{w^2}{y^N} |u|^p \, dx \, dy \quad (5.21)$$

228 Since $1 < p' \leq 2$ we have that $q_{p'} = 1$ and (5.8) reads as

$$\begin{aligned} I(u) &\geq \Lambda_p \frac{p}{N-1} a \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{w}{y^N} |u|^p \, dx \, dy \\ &\quad - \Lambda_p \frac{p}{N-1} a \left(1 + \frac{N-1}{p} a \right) \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{w^2}{y^N} |u|^p \, dx \, dy \end{aligned} \quad (5.22)$$

229 Multiplying (5.21) by $2(N-1)a \left(1 + \frac{N-1}{p} a \right)$ and using (5.22) we have

$$I(u) \geq \Lambda_p \frac{p}{N-1} \mu_2(a) \int_{\mathbb{R}^+} \int_{\mathbb{R}^{N-1}} \frac{w}{y^N} |u|^p \, dx \, dy \quad (5.23)$$

where

$$\mu_2(a) := \frac{a}{1 + 2(N-1)a \left(1 + \frac{N-1}{p} a \right)}.$$

230 Setting $C(N, p) := \frac{N-1}{p} \max\{\mu_2(a), a \in [0, 1]\}$ we get the claim.

231 Now we proceed to compute $C(N, p)$. The maximum of μ_2 is achieved at $a_0 := \frac{1}{N-1} \sqrt{\frac{p}{2}}$
232 if $a_0 \leq 1$, at 1 else. That is,

233 - if $2 < p \leq 2(N-1)^2$ we have $C(N, p) = \frac{N-1}{p} \mu_2(a_0) = (\sqrt{2}(\sqrt{2}p + 2\sqrt{p}))^{-1}$;

234 - if $p > 2(N-1)^2$ we have $C(N, p) = \frac{N-1}{p} \mu_2(1) = \frac{N-1}{p} \left(1 + 2(N-1) + 2 \frac{(N-1)^2}{p}\right)^{-1}$.
 235 This concludes the proof of Theorem 3.2.

Remark 5.1. Let $1 < p < 2$. Here, we compute $C(2, p)$, that is when $N = 2$. In this case, with the same notation used in the proof of Theorem 3.2, the function f reads as

$$f(c) = c \left(1 - \frac{1}{2}c - \frac{1}{2}c(2-c)^2\delta\right).$$

Consider first the case $4/3 \leq p < 2$. In this case $\delta \in]0, 1/2]$ and the only critical point of f in $[0, 1]$ is at $c = 1$, therefore f attains its maximum at 1, that is $M = f(1) = (1-\delta)/2 < 1/2$. Therefore, by definition of $C(2, p)$ we have

$$C(2, p) = \frac{1}{p} \frac{(1-\delta)/2}{1 + \sqrt{\frac{1-\delta}{4(p-1)}}} = \frac{1}{p'} \frac{\sqrt{2}}{\sqrt{2p} + \sqrt{p}}.$$

Next we consider the case $1 < p < 4/3$. Now we have $\delta \in]1/2, +\infty[$ and the function f has in $[0, 1]$ two distinct critical value $c_0 = 1 - \sqrt{1 - \frac{1}{2\delta}}$ and $c_1 = 1$. Since $f''(1) = 2\delta - 1 > 0$, the maximum is attained at c_0 , that is $M = f(c_0) = \frac{1}{8\delta} (< 1/4)$. Therefore

$$C(2, p) = \frac{1}{p} \frac{(1/8\delta)}{1 + \sqrt{\frac{1/8\delta}{2(p-1)}}} = \frac{1}{p'} \frac{1}{2(2-p) + \sqrt{2-p}}.$$

236 Proof of Theorem 2.2

237 Letting $V(x_1, \dots, x_{N-1}, y) := \frac{y}{\sqrt{y^2 + x_1^2}}$, the proof of (2.3) follows at once from (3.2) by
 238 exploiting the half-space model for \mathbb{H}^N as explained in the proof of Lemma 2.1. Next, for any
 239 $\alpha \in (0, 1]$, set $U_\alpha := \{(x, y) \in \mathbb{R}_+^N : x_1 = ky \text{ with } k^2 = (1 - \alpha^2)/\alpha^2\}$. Clearly, $V|_{U_\alpha} \equiv \alpha$ and
 240 $V|_{U_\alpha} \rightarrow \alpha$ as $y \rightarrow +\infty$. Set $r := \varrho((x, y), (0, 1))$. Since $\cosh(r(x, y)) = \left(1 + \frac{(y-1)^2 + |x|^2}{2y}\right)$, we
 241 get that $r(x, y) \rightarrow +\infty$ as $y \rightarrow +\infty$ and the corresponding claim of Theorem 2.2 follows.

242 On the other hand, for any $\beta > 0$, take $W_\beta := \{(x_1, 0, \dots, 0, \beta) \in \mathbb{R}_+^N\}$. Then, for any
 243 $\beta > 0$, one has $V|_{W_\beta} \rightarrow 0$ as $x_1 \rightarrow +\infty$. Furthermore, $r|_{W_\beta} \rightarrow +\infty$ if and only if $x_1 \rightarrow +\infty$
 244 and $V|_{W_\beta} \sim \sqrt{\frac{\beta}{2}} e^{-r/2}$ as $r \rightarrow +\infty$.

245 6. PROOF OF THEOREM 2.3 AND COROLLARY 2.4

246 Before proving Theorem 2.3, we recall some known results related to the symmetrization
 247 on the hyperbolic space. For any $\Omega \subset \mathbb{H}^N$ and $x_0 \in \mathbb{H}^N$ fixed, denote with Ω^* the geodesic
 248 ball $B(x_0, r)$ having the same measure of Ω . For $u \in C_c^\infty(\Omega)$, the hyperbolic symmetrization
 249 of u is the unique nonnegative and decreasing function u^* defined in Ω^* such that the level
 250 sets $\{x \in \Omega^* : u^*(x) > t\}$ are concentric balls having the same measure of the level sets
 251 $\{x \in \Omega : |u(x)| > t\}$. See [2] for more details.

252 **Lemma 6.1.** *Let $p \geq 1$ and $N \geq 2$. For every $u, v \in C_c^\infty(\mathbb{H}^N)$, there holds*

$$\begin{aligned} \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p \, dv_{\mathbb{H}^N} &\geq \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u^*|^p \, dv_{\mathbb{H}^N}, \\ \int_{\mathbb{H}^N} |u|^p \, dv_{\mathbb{H}^N} &= \int_{\mathbb{H}^N} |u^*|^p \, dv_{\mathbb{H}^N}, \end{aligned}$$

253

254 and

$$\int_{\mathbb{H}^N} |uv| \, dv_{\mathbb{H}^N} \leq \int_{\mathbb{H}^N} u^* v^* \, dv_{\mathbb{H}^N},$$

255 where $*$ denotes the hyperbolic symmetrization.

256 Next we state a p -convexity lemma. The proof of the following lemma can be obtained
257 as an application of Taylor's formula, we refer to [20] for further details.

258 **Lemma 6.2.** *Let $p \geq 1$ and ξ, η be real numbers such that $\xi \geq 0$ and $\xi - \eta \geq 0$. Then*

$$(\xi - \eta)^p + p\xi^{p-1}\eta - \xi^p \geq \begin{cases} \max\{(p-1)\eta^2\xi^{p-2}, |\eta|^p\}, & \text{if } p \geq 2, \\ \frac{1}{2}p(p-1)\frac{\eta^2}{(\xi+|\eta|)^{2-p}}, & \text{if } 1 \leq p \leq 2. \end{cases}$$

259 Now we turn to prove an *optimal* inequality which is one of the key ingredient in proving
260 Theorem 2.3.

261 **Lemma 6.3.** *For all $v \in W^{1,p}(0, \infty)$ and $1 < l \leq p$, there holds*

$$\int_0^\infty |v(r)|^{p-l} (\coth r)^{p-l} |v'(r)|^l \, dr \geq \left(\frac{p-1}{p}\right)^l \int_0^\infty \frac{|v(r)|^p}{r^p} \, dr. \quad (6.1)$$

262 Furthermore, the constant $\left(\frac{p-1}{p}\right)^l$ in (6.1) is sharp.

Proof. We first prove the claim for $v \in C_c^\infty(0, \infty)$. Write

$$\begin{aligned} \int_0^\infty \frac{|v(r)|^p}{r^p} \, dr &= \frac{-1}{p-1} \int_0^\infty |v(r)|^p \frac{d}{dr} (r^{-(p-1)}) \, dr \\ &= \left(\frac{p}{p-1}\right) \int_0^\infty \frac{|v(r)|^{p-2} v(r) v'(r)}{r^{p-1}} \, dr \\ &\leq \left(\frac{p}{p-1}\right) \int_0^\infty \frac{|v(r)|^{p-1} |v'(r)|}{r^{p-1}} \, dr \\ &= \left(\frac{p}{p-1}\right) \int_0^\infty \frac{|v(r)|^{\frac{p(l-1)}{l}}}{r^{\frac{p(l-1)}{l}}} \frac{|v(r)|^{\frac{p-l}{l}} |v'(r)|}{r^{\frac{p-l}{l}}} \, dr \\ &\leq \left(\frac{p}{p-1}\right) \left(\int_0^\infty \frac{|v(r)|^p}{r^p} \, dr\right)^{\frac{l-1}{l}} \left(\int_0^\infty \frac{|v(r)|^{p-l} |v'(r)|^l}{r^{p-l}} \, dr\right)^{\frac{1}{l}}. \end{aligned}$$

263 Since $\coth r \geq \frac{1}{r}$ for all $r > 0$, we conclude

$$\int_0^\infty |v(r)|^{p-l} (\coth r)^{p-l} |v'(r)|^l \, dr \geq \left(\frac{p-1}{p}\right)^l \int_0^\infty \frac{|v(r)|^p}{r^p} \, dr.$$

Now, noticing that by using Young inequality and the classical Hardy inequality with exponent p , we have

$$\int_0^\infty |v(r)|^p \, dr + \int_0^\infty |v'(r)|^p \, dr \geq c \int_0^\infty |v(r)|^{p-l} (\coth r)^{p-l} |v'(r)|^l \, dr,$$

264 the claim follows by density argument.

265 Next we turn to the optimality issue. For $\varepsilon > 0$ and $\delta > 0$, consider

$$V_\varepsilon^\delta(r) := \begin{cases} r^{\frac{p-1+\delta}{p}}, & 0 < r < \varepsilon \\ \varepsilon^{\frac{p-1+\delta}{p}}, & \varepsilon \leq r < 1 \\ \varepsilon^{\frac{p-1+\delta}{p}}(2-r), & 1 \leq r < 2 \\ 0, & r \geq 2. \end{cases}$$

266 Clearly, $V_\varepsilon^\delta(r) \in W^{1,p}(0, \infty)$ for $\varepsilon > 0, \delta > 0$. Furthermore, we have

$$\int_0^\infty \frac{|V_\varepsilon^\delta(r)|^p}{r^p} dr \geq \int_0^\varepsilon \frac{r^{p-1+\delta}}{r^p} dr = \int_0^\varepsilon r^{\delta-1} dr.$$

267 On the other hand, using the fact $\sinh r \geq r$, we obtain

$$\begin{aligned} & \int_0^\infty |V_\varepsilon^\delta(r)|^{p-l} (\coth r)^{p-l} |(V_\varepsilon^\delta(r))'|^l dr = \\ & \left(\frac{p-1+\delta}{p}\right)^l \int_0^\varepsilon r^{\frac{(p-1+\delta)(p-l)}{p}} (\coth r)^{p-l} r^{\frac{(\delta-1)l}{p}} dr \\ & + \varepsilon^{p-1+\delta} \int_1^2 (2-r)^{p-l} (\coth r)^{p-l} dr \\ & = \left(\frac{p-1+\delta}{p}\right)^l \int_0^\varepsilon r^{p-1+\delta-l} (\coth r)^{p-l} dr + c\varepsilon^{p-1+\delta} \\ & \leq \left(\frac{p-1+\delta}{p}\right)^l (\cosh \varepsilon)^{p-l} \int_0^\varepsilon \frac{r^{p-1+\delta-l}}{(\sinh r)^{p-l}} dr + c\varepsilon^{p-1+\delta} \\ & \leq \left(\frac{p-1+\delta}{p}\right)^l (\cosh \varepsilon)^{p-l} \int_0^\varepsilon r^{\delta-1} dr + c\varepsilon^{p-1+\delta}. \end{aligned}$$

268 Hence,

$$Q := \inf_{v \in W^{1,p}(0, \infty) \setminus \{0\}} \frac{\int_0^\infty |v(r)|^{p-l} (\coth r)^{p-l} |v'(r)|^l dr}{\int_0^\infty \frac{|v(r)|^p}{r^p} dr} \leq \left(\frac{p-1+\delta}{p}\right)^l (\cosh \varepsilon)^{p-l} + c\delta\varepsilon^{p-1}.$$

269 First letting $\varepsilon \rightarrow 0$, and then with $\delta \rightarrow 0$, we conclude that

$$Q \leq \left(\frac{p-1}{p}\right)^l.$$

270 This proves the optimality and concludes the proof.

271

□

272 Proof of Theorem 2.3 and of Corollary 2.4

273 By hyperbolic symmetrization, i.e., in view of Lemma 6.1, we may assume $u \in C_c^\infty(\mathbb{H}^N)$
 274 nonnegative, radially symmetric and non increasing. Hence, to prove (2.5), it is enough to
 275 show the validity of the following inequality

$$\int_0^\infty |u'(r)|^p (\sinh r)^{N-1} dr - \left(\frac{N-1}{p}\right)^p \int_0^\infty (u(r))^p (\sinh r)^{N-1} dr$$

$$\geq (p-1) \left(\frac{N-1}{p}\right)^{p-2} \left(\frac{p-1}{p}\right)^2 \int_0^\infty \frac{(u(r))^p}{r^p} (\sinh r)^{N-1} dr. \quad (6.2)$$

276

277

278 Let us define a suitable transformation which allows to put the Poincaré term into evi-
279 dence:

$$v(r) := (\sinh r)^{\frac{N-1}{p}} u(r)$$

280 so that

$$v'(r) = (u'(r))(\sinh r)^{\frac{N-1}{p}} + \left(\frac{N-1}{p}(\sinh r)^{\frac{N-1}{p}} \coth r\right) u,$$

281 hence $v \in W^{1,p}(0, \infty)$, and

$$(u'(r))(\sinh r)^{\frac{N-1}{p}} = v'(r) - \left(\frac{N-1}{p}(\sinh r)^{\frac{N-1}{p}} \coth r\right) u.$$

At this point we apply the p -convexity Lemma 6.2. By taking

$$\xi = \left(\frac{N-1}{p}\right) (\sinh r)^{\frac{N-1}{p}} \coth r u > 0 \quad \text{and} \quad \eta = v'(r)$$

282 and using Lemma 6.2 for $p \geq 2$, we obtain

$$\begin{aligned} |u'(r)|^p (\sinh r)^{N-1} &\geq (p-1) \left(\frac{N-1}{p}\right)^{p-2} v^{p-2}(r) (\coth r)^{p-2} (v'(r))^2 \\ &\quad + \left(\frac{N-1}{p}\right)^p (\sinh r)^{N-1} (\coth r)^p u^p(r) \\ &\quad - p \left(\frac{N-1}{p}\right)^{p-1} (\sinh r)^{\frac{(N-1)(p-1)}{p}} (\coth r)^{p-1} u^{p-1}(r) v'(r) \\ &= (p-1) \left(\frac{N-1}{p}\right)^{p-2} v^{p-2}(r) (\coth r)^{p-2} (v'(r))^2 \\ &\quad + \left(\frac{N-1}{p}\right)^p (\sinh r)^{N-1} (\coth r)^p u^p(r) \\ &\quad - p \left(\frac{N-1}{p}\right)^{p-1} (\coth r)^{p-1} v^{p-1}(r) v'(r). \end{aligned}$$

283 Integrating both sides of above inequality and applying Lemma 6.3 with $l = 2$, we get

$$\begin{aligned} \int_0^\infty |u'(r)|^p (\sinh r)^{N-1} dr &\geq (p-1) \left(\frac{N-1}{p}\right)^{p-2} \int_0^\infty v^{p-2}(r) (\coth r)^{p-2} (v'(r))^2 dr \\ &\quad + \left(\frac{N-1}{p}\right)^p \int_0^\infty (\coth r)^p v^p(r) dr \\ &\quad - \left(\frac{N-1}{p}\right)^{p-1} \int_0^\infty (\coth r)^{p-1} \frac{d}{dr} (v(r))^p dr \end{aligned}$$

$$\begin{aligned} &\geq (p-1) \left(\frac{N-1}{p}\right)^{p-2} \left(\frac{p-1}{p}\right)^2 \int_0^\infty \frac{v^p(r)}{r^p} dr \\ &+ \left(\frac{N-1}{p}\right)^p \int_0^\infty F(r)(v(r))^p dr, \end{aligned}$$

284 where $F(r) := (\coth r)^p - \frac{p(p-1)}{N-1} \frac{(\coth r)^p}{\cosh^2 r}$ and in the integration by parts we have used the
285 definition of v and the fact that $N > p$. Then, (6.2) follows by showing that $F(r) \geq 1$ for
286 all $r > 0$ or equivalently that

$$\tilde{F}(r) := (N-1) \cosh^p r - (N-1) \sinh^p r - p(p-1) \cosh^{p-2} r \geq 0,$$

287 for all $r > 0$. By rewriting

$$\tilde{F}(r) = \cosh^{p-2} r (N-1 - p(p-1)) + (N-1) \sinh^2 r (\cosh^{p-2} r - \sinh^{p-2} r),$$

288 we immediately infer that $\tilde{F}(r)$ is non negative provided that $N \geq 1 + p(p-1)$, and also
289 the condition is necessary. This completes the proof of Theorem 2.3. \square

290 **Proof of Corollary 2.4.** It suffices to notice that, by Hölder inequality:

$$\begin{aligned} \int_{\mathbb{H}^N} |u|^p dv_{\mathbb{H}^N} &= \int_{\mathbb{H}^N} \frac{|u|}{r} |u|^{p-1} r dv_{\mathbb{H}^N} \\ &\leq \left(\int_{\mathbb{H}^N} \frac{|u|^p}{r^p} dv_{\mathbb{H}^N} \right)^{\frac{1}{p}} \left(\int_{\mathbb{H}^N} |u|^{p'} r^{p'} dv_{\mathbb{H}^N} \right)^{\frac{1}{p'}}. \end{aligned}$$

291 The conclusion follows by using inequality (2.5). \square

292

7. PROOF OF THEOREM 2.5

293 Before proving Theorem 2.5 we collect here below the main properties of the weight H_p .
294 This will clarify also the meaning of inequality (2.7), see also Figure 1.

295 **Lemma 7.1.** *Let $H_p : \mathbb{R}^+ \rightarrow \mathbb{R}$ be defined as in the statement of Theorem 2.5 with $p > 2$*
296 *and $N \geq 1 + p(p-1)$. Then, the following holds*

297

298 (a) *For all $r > 0$, $H_p(r) > 0$, $H_p(r) \sim \left(\frac{N-p}{N-1}\right)^{p-2} \frac{1}{r^{p-2}}$ as $r \rightarrow 0^+$, and $H_p(r) \rightarrow 1^-$ as*
299 *$r \rightarrow \infty$.*

300 (b) *There exists a unique $r_p \in (0, \infty)$ such that $H_p(r) \geq 1$ for $r \in (0, r_p]$ and $H_p(r) < 1$*
301 *for $r \in (r_p, \infty)$.*

302 *Proof.* We set

$$\tilde{H}_p(r) := \coth r - \left(\frac{p-1}{N-1}\right) \frac{1}{r}, \quad r > 0.$$

303 Then, the property of H_p can be readily deduced from that of \tilde{H}_p .

The sign and the asymptotics of \tilde{H}_p follows from fact that

$$\coth r > \frac{1}{r} \text{ in } (0, \infty), \quad \coth r \sim \frac{1}{r} \text{ as } r \rightarrow 0^+, \quad \text{and } \coth r \rightarrow 1 \text{ as } r \rightarrow \infty.$$

304 To prove assertion (b), we note that

$$\tilde{H}'_p(r) = (N-1)^{-1} \left(\frac{-(N-1)r^2 + (p-1) \sinh^2 r}{r^2 \sinh^2 r} \right) =: \frac{(N-1)^{-1}}{r^2 \sinh^2 r} h(r). \quad (7.1)$$

305 Since $h'''(r) = 8(p - 1) \cosh r \sinh r > 0$ for all $r > 0$, $h''(0) = -2(N - p)$, and $h'(0) =$
 306 $h(0) = 0$ one readily deduces the existence of a unique $r_0 > 0$ such that $h(r) < 0$ in $(0, r_0)$,
 307 $h(r_0) = 0$ and $h(r) > 0$ in (r_0, ∞) . Hence, $\tilde{H}'_p(r) < 0$ in $(0, r_0)$ and $\tilde{H}'_p(r) > 0$ in (r_0, ∞) .
 308 This fact and assertion (a) gives the existence of a unique $r_p \in (0, r_0)$ for which (b) holds
 309 where r_p clearly satisfies

$$\coth r_p - 1 - \frac{p - 1}{N - 1} \frac{1}{r_p} = 0. \tag{7.2}$$

310 □

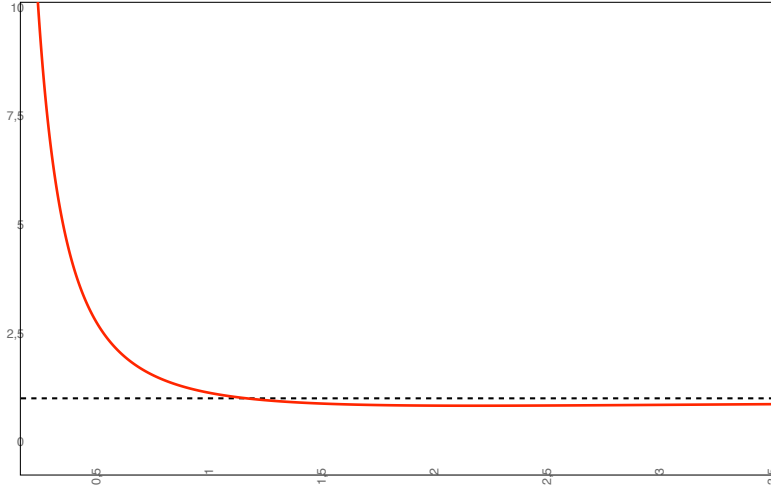


FIGURE 1. The plot of $y = H_p(r)$ for $p = 4$ and $N = 13$. The dotted line is $y = 1$ and the intersection point of the two curves is the point r_p as defined in Lemma 7.1-(b).

311

312 **Proof of Theorem 2.5**

313 The p -Laplacian operator in radial coordinates on the hyperbolic space writes

$$\begin{aligned} \Delta_{p, \mathbb{H}^N} u(r) &:= \Delta_p u(r) = (p - 1)|u'(r)|^{p-2}u''(r) + (N - 1) \coth r |u'(r)|^{p-2}u'(r) \\ &:= |u'(r)|^{p-2}L_p u(r), \end{aligned} \tag{7.3}$$

314 where $L_p u(r) = (p - 1)u''(r) + (N - 1) \coth r u'(r)$.

315 Set $g(r) = \left(\frac{r}{\sinh r}\right)^{\frac{(N-1)}{p}}$ and $f(r) = r^{\frac{p-N}{p}}$, some straightforward computations give

$$\begin{aligned} L_p g(r) &= \frac{-(N - 1)}{p} \left[\frac{(N - 1) - p(p - 1)}{p} \frac{1}{\sinh^2 r} + \left(\frac{N - 1}{p}\right) \right. \\ &\quad \left. + \frac{(p - 1)(p - (N - 1))}{p} \frac{1}{r^2} + \frac{(N - 1)(p - 2) \coth r}{p} \frac{1}{r} \right] g(r) \end{aligned} \tag{7.4}$$

316 and

$$L_p f(r) = \left[\frac{N(N-p)(p-1)}{p^2} \frac{1}{r^2} - (N-1) \coth r \frac{N-p}{p} \frac{1}{r} \right] f(r) \quad (7.5)$$

317 Using (7.4) and (7.5), we deduce for $\tilde{g}(r) = g(r)f(r)$,

$$\begin{aligned} L_p \tilde{g}(r) &= (L_p g(r))f(r) + (L_p f(r))g(r) \\ &+ 2(p-1) \left(\frac{-(N-1)}{p} \coth r + \frac{N-1}{p} \frac{1}{r} \right) g(r)f'(r) \\ &= - \left[\left(\frac{N-1}{p} \right)^2 \tilde{g} + \frac{(p-1)^2}{p^2} \frac{1}{r^2} \tilde{g} + \frac{(p-1)(p-2)(N-1)}{p^2} \left(\frac{\coth r}{r} \right) \tilde{g} \right. \\ &\quad \left. + \frac{(N-1)(N-1-p(p-1))}{p^2} \frac{1}{\sinh^2 r} \tilde{g} \right]. \end{aligned} \quad (7.6)$$

318 In view of Eq. (7.3) and Eq. (7.6) we obtain

$$\begin{aligned} -\Delta_p \tilde{g} - \left(\frac{N-1}{p} \right)^2 |\tilde{g}'|^{p-2} \tilde{g} &= \\ \frac{(p-1)^2}{p^2} \frac{1}{r^2} |\tilde{g}'|^{p-2} \tilde{g} + \frac{(p-1)(p-2)(N-1)}{p^2} \left(\frac{\coth r}{r} \right) |\tilde{g}'|^{p-2} \tilde{g} & \\ + \frac{(N-1)(N-1-p(p-1))}{p^2} \frac{1}{\sinh^2 r} |\tilde{g}'|^{p-2} \tilde{g}. & \end{aligned} \quad (7.7)$$

319 Furthermore, we have

$$\begin{aligned} \tilde{g}'(r) &= (g'(r))f(r) + (f'(r))g(r) \\ &= -\frac{1}{p} \left((N-1) \coth r - (p-1) \frac{1}{r} \right) \tilde{g}(r). \end{aligned} \quad (7.8)$$

320 Namely,

$$|\tilde{g}'(r)|^{p-2} = \left(\frac{N-1}{p} \right)^{p-2} H_p(r) \tilde{g}^{p-2}(r),$$

321 with $H_p(r)$ as defined in the statement of Theorem 7.2. On the other hand, a further
322 computation using (7.8) and the fact $\coth r > \frac{1}{r}$, gives

$$\begin{aligned} |\tilde{g}'(r)|^{p-2} &= \frac{(p-1)^{p-2}}{p^{p-2} r^{p-2}} \left(\frac{N-1}{p-1} r \coth r - 1 \right)^{p-2} \tilde{g}^{p-2}(r) \\ &\geq \frac{(p-1)^{p-2}}{p^{p-2}} \frac{\tilde{g}^{p-2}(r)}{r^{p-2}}. \end{aligned} \quad (7.9)$$

323 Substituting (7.9) in (7.7) we conclude

$$\begin{aligned}
-\Delta_p \tilde{g} - \left(\frac{N-1}{p}\right)^p H_p(r) \tilde{g}^{p-1} &\geq \frac{(p-1)^p}{p^p} \frac{1}{r^p} \tilde{g}^{p-1} \\
&+ \frac{(p-1)^{p-1}(p-2)(N-1)}{p^p} \left(\frac{\coth r}{r}\right) \frac{1}{r^{p-2}} \tilde{g}^{p-1} \\
&+ \frac{(N-1)(N-1-p(p-1))}{p^2} \frac{1}{\sinh^2 r} \tilde{g}^{p-1} \\
&\geq \frac{(p-1)^{p-1}(N(p-2)+1)}{p^p} \frac{1}{r^p} \tilde{g}^{p-1} \\
&+ \frac{(N-1)(N-1-p(p-1))}{p^2} \frac{1}{\sinh^2 r} \tilde{g}^{p-1}.
\end{aligned}$$

324 This proves that $\tilde{g}(r) = \left(\frac{r}{\sinh r}\right)^{\frac{N-1}{p}} r^{\frac{p-N}{p}}$ is a super-solution of the equation corresponding to
325 (2.7). Hence, by Allegretto-Piepenbrink theorem for p -Laplacian setting, (for detail see [29,
326 Theorem 2.3]) inequality (2.7) follows immediately for functions in $C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$. To extend the inequality for functions belonging to $C_c^\infty(\mathbb{H}^N)$ one argues as in the proof of Proposition 1.1. Namely, since $N > p$, the set $\{x_0\}$ is compact and has zero p -capacity, therefore the
327 completion of $C_c^\infty(\mathbb{H}^N)$ and $C_c^\infty(\mathbb{H}^N \setminus \{x_0\})$ with respect to the norm $(\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^p dv_{\mathbb{H}^N})^{1/p}$
328 coincides (see [13, Proposition A.1]). This concludes the proof.

331 As a consequence of Theorem 2.5 we have the following

332 **Theorem 7.2.** *Let $p \geq 2$ and $N \geq 1 + p(p-1)$. Let Λ_p be as in (2.1) and $r := \varrho(x, x_0)$ with
333 $x_0 \in \mathbb{H}^N$ fixed. Then for $u \in C_c^\infty(B(x_0, r_p))$ there holds*

$$\begin{aligned}
&\int_{B(x_0, r_p)} |\nabla_{\mathbb{H}^N} u|^p dv_{\mathbb{H}^N} - \Lambda_p \int_{B(x_0, r_p)} |u|^p dv_{\mathbb{H}^N} \\
&\geq \frac{(p-1)^{p-1}(N(p-2)+1)}{p^p} \int_{B(x_0, r_p)} \frac{|u|^p}{r^p} dv_{\mathbb{H}^N} \\
&+ \frac{(N-1)(N-1-p(p-1))(p-1)^{p-2}}{p^p} \int_{B(x_0, r_p)} \frac{|u|^p}{\sinh^p r} dv_{\mathbb{H}^N}
\end{aligned} \tag{7.10}$$

334 where $B(x_0, r_p)$ is the geodesic ball of radius r_p centered at x_0 and where we let, for $p > 2$,
335 $r_p = r_p(N)$ be the unique positive solution to the equation

$$\coth r_p - 1 - \frac{p-1}{N-1} \frac{1}{r_p} = 0,$$

336 whereas $r_2 := +\infty$ (namely $B(x_0, r_2) = \mathbb{H}^N$).

337 In particular, for every $p > 2$ the map $N \mapsto r_p(N)$ is strictly increasing in $[1 + p(p-1), +\infty)$ and $\lim_{N \rightarrow +\infty} r_p(N) = +\infty$ while, for every $N > 3$ the map $p \mapsto r_p$ is strictly
338 decreasing in $(2, \frac{1+\sqrt{4N-3}}{2}]$.

Proof. The proof readily follows by combining the statements of Theorem 2.5 and Lemma 7.1. In particular equation (7.2) implicitly defines a map $N \mapsto r_p(N)$. By differentiating in (7.2) one gets

$$\frac{d}{dN}(r_p(N)) = -\frac{(p-1)r_p \sinh^2 r_p}{(N-1)h(r_p)},$$

where the function h is as defined in (7.1). Since from the proof of Lemma 7.1-(b) we know that $h(r_p) < 0$, we conclude that the map $N \mapsto r_p(N)$ is strictly increasing. On the other hand, equation (7.2) also implicitly defines a map $p \mapsto r_p$. In this case we get

$$\frac{d}{dp}(r_p) = \frac{r_p \sinh^2 r_p}{(N-1)h(r_p)} < 0.$$

340 Hence, the map $p \mapsto r_p(N)$ is strictly decreasing. \square

341 **Acknowledgments.** The first author is partially supported by the Research Project FIR
 342 (Futuro in Ricerca) 2013 “Geometrical and qualitative aspects of PDE’s” (Italy). The second
 343 author is partially supported by the PRIN project n. 201274FYK7_004 “Aspetti Variazionali
 344 e Perturbativi nei Problemi Differenziali non Lineari”. The third author is partially sup-
 345 ported at the Technion by a fellowship of the Israel Council for Higher Education. The fourth
 346 author is partially supported by the PRIN project “Equazioni alle derivate parziali di tipo
 347 ellittico e parabolico: aspetti geometrici, disuguaglianze collegate, e applicazioni” (Italy).
 348 The first, second and fourth authors are members of the Gruppo Nazionale per l’Analisi
 349 Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di
 350 Alta Matematica (INdAM).

351 REFERENCES

- 352 [1] K. Akutagawa, H. Kumura, *Geometric relative Hardy inequalities and the discrete spectrum of*
 353 *Schrodinger operators on manifolds*, Calc. Var. Part. Diff. Eq. 48 (2013), 67-88.
- 354 [2] A. Baernstein II, *A unified approach to symmetrisation, Partial differential equations of elliptic type*
 355 (Cortona, 1992), 47-91, Sympos. Math., XXXV, Cambridge Univ. Press, Cambridge, 1994.
- 356 [3] G. Barbatis, S. Filippas, A. Tertikas, *Series expansion for L^p Hardy inequalities*, Indiana Univ. Math.
 357 J. 52 (2003), 171-190.
- 358 [4] G. Barbatis, S. Filippas, A. Tertikas, *A unified approach to improved L^p Hardy inequalities with best*
 359 *constants*, Trans. Amer. Soc. 356 (2004), 2169-2196.
- 360 [5] E. Berchio, D. Ganguly, *Improved higher order Poincaré inequalities on the hyperbolic space via Hardy-*
 361 *type remainder terms*, Commun. Pure Appl. Anal. 15 (2016), no. 5, 1871-1892.
- 362 [6] E. Berchio, D. Ganguly, G. Grillo, *Sharp Poincaré-Hardy and Poincaré-Rellich inequalities on the hy-*
 363 *perbolic space*, to appear in J. Funct. Anal., Arxiv:1507.02550.
- 364 [7] B. Bianchini, L. Mari, M. Rigoli, *Yamabe type equations with a sign-changing nonlinearity, and the*
 365 *prescribed curvature problem*, J. Diff. Eq. 260 (2016), no. 10, 7416-7497.
- 366 [8] Y. Bozhkov, E. Mitidieri, *Conformal Killing vector fields and Rellich type identities on Riemannian*
 367 *manifolds*, I Lecture Notes of Seminario Interdisciplinare di Matematica 7 (2008), 65-80.
- 368 [9] Y. Bozhkov, E. Mitidieri, *Conformal Killing vector fields and Rellich type identities on Riemannian*
 369 *manifolds*, II. Mediterr. J. Math. 9 (2012), no. 1, 1-20.
- 370 [10] H. Brezis, M. Marcus, *Hardy’s inequalities revisited*, Ann. Scuola Norm. Sup. Cl. Sci. (4) 25 (1997),
 371 217-237.
- 372 [11] H. Brezis, J. L. Vazquez, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Com-
 373 plut. Madrid 10 (1997), 443-469.
- 374 [12] G. Carron, *Inegalites de Hardy sur les varietes Riemanniennes non-compactes*, J. Math. Pures Appl.
 375 (9) 76 (1997), 883-891.
- 376 [13] L. D’Ambrosio, S. Dipierro, *Hardy inequalities on Riemannian manifolds and applications*, Ann. Inst.
 377 H. Poincaré Anal. Non Linéaire 31 (2014), 449-475.
- 378 [14] E. B. Davies, *Heat kernel and Spectral Theory*, Cambridge University Press, 1989.
- 379 [15] B. Devyver, M. Fraas, Y. Pinchover, *Optimal Hardy weight for second-order elliptic operator: an answer*
 380 *to a problem of Agmon*, J. Funct. Anal. 266 (2014), 4422-4489.
- 381 [16] B. Devyver, Y. Pinchover, *Optimal L^p Hardy-type inequalities*, Ann. Inst. H. Poincaré Anal. Non Linéaire
 382 33 (2016), 93-118.

- 383 [17] S. Filippas, L. Moschini, A. Tertikas, *Sharp trace Hardy-Sobolev-Maz'ya inequalities and the fractional*
 384 *Laplacian*, Arch. Ration. Mech. Anal. 208 (2013), no. 1, 109-161.
- 385 [18] S. Filippas, A. Tertikas, *Optimizing improved Hardy inequalities*, J. Funct. Anal. 192 (2002), 186-233.
- 386 [19] S. Filippas, A. Tertikas, J. Tidblom, *On the structure of Hardy-Sobolev-Maz'ya inequalities*, J. Eur.
 387 Math. Soc. (JEMS) 11 (2009), no. 6, 1165-1185.
- 388 [20] F. Gazzola, H. Grunau, E. Mitidieri, *Hardy inequalities with optimal constants and remainder terms*,
 389 Trans. Amer. Math. Soc. 356 (2004), 2149-2168.
- 390 [21] N. Ghoussoub, A. Moradifam, *Bessel pairs and optimal Hardy and Hardy-Rellich inequalities*, Math.
 391 Ann. 349 (2011), 1-57.
- 392 [22] I. Kombe, M. Ozaydin, *Improved Hardy and Rellich inequalities on Riemannian manifolds*, Trans. Amer.
 393 Math. Soc. 361 (2009), no. 12, 6191-6203.
- 394 [23] I. Kombe, M. Ozaydin, *Rellich and uncertainty principle inequalities on Riemannian manifolds*, Trans.
 395 Amer. Math. Soc. 365 (2013), no. 10, 5035-5050.
- 396 [24] G. Mancini, K. Sandeep, *On a semilinear equation in \mathbb{H}^n* . Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2008),
 397 5 vol. VII, 635-671.
- 398 [25] M. Marcus, V. J. Mizel, Y. Pinchover, *On the best constant for Hardy's inequality in \mathbb{R}^n* , Trans. Am.
 399 Math. Soc. 350 (1998), 3237-3255.
- 400 [26] V. G. Maz'ya, *Sobolev spaces*, Springer-Verlag, Berlin, 1985.
- 401 [27] E. Mitidieri, *A simple approach to Hardy inequalities*, Mat. Zametki 67 (2000), 563-572.
- 402 [28] Q. A. Ngo, V. H. Nguyen, *Sharp constant for Poincaré-type inequalities in the hyperbolic space*, preprint
 403 2016, arXiv:1607.00154.
- 404 [29] Y. Pinchover, K. Tintarev, *Ground state alternative for p -Laplacian with potential term*, Calc. Var.
 405 Part. Diff. Equ. 28 (2007), no. 2, 179-201.
- 406 [30] A. Tertikas, N. B. Zographopoulos, *Best constants in the Hardy-Rellich inequalities and related improve-*
 407 *ments*, Adv. Math. 209 (2007), 407-459.
- 408 [31] J. Tidblom, *A Hardy inequality in the half-space*, J. Funct. Anal. 221 (2005), 482-495.
- 409 [32] Q. Yang, D. Su, Y. Kong, *Hardy inequalities on Riemannian manifolds with negative curvature*, Com-
 410 mun. Contemp. Math. 16 (2014), no. 2, 1350043.

DIPARTIMENTO DI SCIENZE MATEMATICHE,
 POLITECNICO DI TORINO,
 411 CORSO DUCA DEGLI ABRUZZI 24, 10129 TORINO, ITALY.
E-mail address: `elvise.berchio@polito.it`

DIPARTIMENTO DI MATEMATICA,
 412 UNIVERSITA' DEGLI STUDI DI BARI ,
 VIA E. ORABONA 4, I-70125 BARI, ITALY.
E-mail address: `lorenzo.dambrosio@uniba.it`

DEPARTMENT OF MATHEMATICS,
 413 TECHNION, ISRAEL INSTITUTE OF TECHNOLOGY,
 HAIFA 32000, ISRAEL.
E-mail address: `gdeb dip@technion.ac.il`

DIPARTIMENTO DI MATEMATICA,
 414 POLITECNICO DI MILANO,
 PIAZZA LEONARDO DA VINCI 32, 20133 MILANO, ITALY.
E-mail addresses: `gabriele.grillo@polimi.it`