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# Mixed Virtual Elements for discrete fracture network simulations<sup>☆</sup>

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## Abstract

The present work deals with the simulation of the flow in Discrete Fracture Networks (DFN), using the mixed formulation of the Virtual Element Method (VEM) on polygonal conforming meshes. The flexibility of the VEM in handling polygonal meshes is used to easily generate a conforming mesh even in the case of intricate DFNs. Mixed virtual elements of arbitrary polynomial accuracy are then used for the discretization of the velocity field. The well posedness of the resulting discrete problem is shown. Numerical results on simple problems are proposed to show convergence properties of the method with respect to known analytic solutions, whereas some tests on fairly complex networks are also reported showing its applicability and effectiveness.

*Keywords:* Mixed Virtual Elements, Discrete Fracture Networks, Mixed formulation, Fracture flows, Darcy flows

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## 1. Introduction

Effective flow simulations in underground fractured media are strategic in several practical contexts: protection of water resources, geothermal applications, Oil&Gas enhanced production and geological waste storage. All these applications share two possibly conflicting common characteristics: a high accuracy and reliability is required, whereas the uncertainty on the geometry and

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on the data demands for a huge number of simulations in order to provide probability distributions of the target quantities.

This work considers the problem of simulating the hydraulic head distribution in the subsoil, modeled as a Discrete Fracture Network (DFN) [1, 2, 3, 4, 5, 6], which is a randomly generated set of intersecting planar polygons resembling the fractures in a surrounding porous medium. DFNs are usually characterized by enormous geometrical complexities and by the presence of a large number of fractures forming an intricate network of intersections. Many novel numerical approaches have been recently developed, in order to circumvent problems arising in efficient flow simulations in realistic DFNs. One of the main difficulties consists in the meshing process, since conventional approaches rely on the conformity of the mesh at fracture intersections in order to enforce suitable matching conditions. The generation of a mesh conforming to fracture intersections might have a high computational cost, or even fail, as a consequence of the number of geometrical constraints, and could result in poor quality triangulations for the presence of distorted elements. Furthermore, as already mentioned, input data for DFN simulations are derived from probability distribution of soil properties, thus requiring a large number of costly simulation to derive reliable statistics on the quantity of interest.

Recently, a novel code for the simulation of the flow in DFNs with stochastic input data was proposed in [7, 8, 9]. In [10, 11] the complexity of DFN flow simulations is tackled resorting to dimensional reduction of the problem, removing the unknowns in the interior of the fractures and rewriting the problems at the interfaces. In [12, 13] the authors use the eXtended Finite Element Method (XFEM) in order to allow for the presence of interfaces in the domain not conforming to the mesh. The XFEM is also used in [14, 15]. In [16, 17, 18, 19, 20] the authors suggest the use of an optimization-based approach on non-conforming meshes to avoid any problem related to the generation of the mesh. The proposed optimization approach also provides a scalable resolution algorithm [21], and is used in conjunction with different discretization choices, ranging from standard finite elements, to the XFEM, [22, 23], or to the new virtual element method [24]. Recently, techniques as the Mimetic Finite Difference method (MFD, [25, 26]) have been used for flow simulations in DFNs, by [27, 28], as an example, and also the new Virtual Element Method (VEM, [29, 30, 31, 32, 33, 34]) was proposed, in addition to the already mentioned reference [24], also in [35, 36]. In these last two works, in particular, the authors take advantage of the flexibility of virtual elements to easily generate a polygonal mesh of the fracture network that satisfies certain conformity requirements with fracture intersections.

The use of mixed formulation in DFN simulations is a widely common choice, for the possibility of a direct computation of the Darcy velocity; see among others [12, 27, 28, 37, 38, 39, 40, 41, 42]. This improves the accuracy for simulations in which the velocity is to be used as the transport field of an advection-diffusion process of a passive scalar, as in the case of the evolution of the concentration of a pollutant in the subsoil.

In the present work, the framework proposed in [35] is extended to the use

of mixed virtual elements, thus combining the reliable meshing process used therein to the mentioned advantages of the mixed formulation. The continuous advection-diffusion-reaction problem in a DFN is presented in mixed form, introducing suitable matching conditions at fracture intersections for the pressure and velocity fields. The discrete formulation with mixed virtual elements of arbitrary polynomial accuracy is then derived, and a proof of well posedness is also provided. Numerical results on simple DFN configurations are first proposed, showing convergence rates of the numerical solution to the known exact solutions. Polynomial accuracy values ranging from  $k = 0$  to  $k = 5$  are considered. Afterwards, other numerical tests are shown on increasingly complex networks, in order to highlight the viability and the effectiveness of the method in dealing with realistic DFN configurations.

The presentation follows this outline: in Section 2 we describe the domain of interest, establish some notations and write the continuous model that describes the hydraulic head distribution within the DFN. In Section 3 the discrete formulation of the problem based on the mixed VEMs on each fracture is discussed and suitable coupling conditions at intersections are introduced. Well posedness of the discrete problem is shown. Some notes on the implementation are given in Section 4. Finally, in Section 5 validation tests are shown on advection-diffusion-reaction problems written on simple domains, together with an analysis of the performances of the method in solving pure diffusion problems on realistic DFNs.

We use the notation  $\|\cdot\|_{k,\omega}$  to indicate the  $H^k(\omega)$ -norm of vectors or scalar functions, on some set  $\omega \subset \mathbb{R}^2$ . In the case of a vector  $\mathbf{v} = (v_1, v_2)$ , we intend, e.g.,  $\|\mathbf{v}\|_{0,\omega}^2 = \int_{\omega} (v_1(x, y)^2 + v_2(x, y)^2) dx dy$ . Moreover, the symbol  $[[\mathbf{v} \cdot \mathbf{n}_{\sigma}]]_{\sigma}$  denotes the jump  $(\mathbf{v} \cdot \mathbf{n}_{\sigma}^+) - (\mathbf{v} \cdot \mathbf{n}_{\sigma}^-)$  across a segment  $\sigma$ , being  $\mathbf{n}_{\sigma}^+$ ,  $\mathbf{n}_{\sigma}^-$  the unit normal vectors to  $\sigma$  with opposite directions. We have that  $\mathbf{n}_{\sigma}$  is the unit normal vector to  $\sigma$  with one fixed orientation, and we observe that the definition of the jump is independent from the choice of  $\mathbf{n}_{\sigma}$ .

## 2. The continuous problem

The geometrical setting for the problem of interest is a *Discrete Fracture Network*  $\Omega$ , that is a finite set of planar polygonal *fractures* intersecting in the 3D space. Each fracture in  $\Omega$  is denoted by  $F_i$ , for some index  $i = \{1, \dots, N\} = \mathcal{I}$ , whereas intersections between fractures are called *traces* and indicated by  $\Gamma_{\ell}$ , for  $\ell = \{1, \dots, L\} = \mathcal{L}$ . We assume, for simplicity, that each intersection occurs between exactly two fractures, and we define, for each  $\ell \in \mathcal{L}$ ,  $\mathcal{I}_{\ell} = (i, j)$ , with  $i < j$ , as the ordered couple of indices of those fractures meeting at  $\Gamma_{\ell}$ , i.e.  $\Gamma_{\ell} = \bar{F}_i \cap \bar{F}_j$ . For each fracture  $F_i$ ,  $\mathcal{L}_i$  is the set of indices of those traces that  $F_i$  shares with other fractures.

The boundary of  $\Omega$ ,  $\partial\Omega$  is split in a Dirichlet part  $\Gamma_D \neq \emptyset$  and a Neumann part  $\Gamma_N$  with  $\partial\Omega = \Gamma_D \cup \Gamma_N$  and  $\Gamma_D \cap \Gamma_N = \emptyset$ . Let us denote by  $h$  the hydraulic head in  $\Omega$  and by  $h_i$  its restriction to  $F_i$  for  $i \in \mathcal{I}$ . Let further  $F_i$  be subdivided in a set of sub-domains  $F_{i,j}$ ,  $j \in \{1, \dots, N_i\}$ , such that the traces lying on  $F_i$  are

now part of the boundary of some of these sub-domains. Then, the hydraulic head  $h$  in  $\Omega$  is the solution of the following system of equations, which, for  $i \in \mathcal{I}$  and  $j \in \{1, \dots, N_i\}$  reads as:

$$\begin{cases} \operatorname{div}(-\mathbf{K}_i \nabla h_i + \mathbf{b}_i h_i) + \gamma_i h_i = f_i & \text{in } F_{i,j}, \\ h_i = h_{Di} & \text{on } \Gamma_{Di} \cap \partial F_{i,j}, \\ (-\mathbf{K}_i \nabla h_i + \mathbf{b}_i h_i) \cdot \mathbf{n}_{\Gamma_{Ni}} = h_{Ni} & \text{on } \Gamma_{Ni} \cap \partial F_{i,j}, \end{cases} \quad (1)$$

where  $\mathbf{K}_i$  is a uniformly positive definite tensor expressing the transmissivity of fracture  $F_i$ , whereas  $\partial F_{i,j}$  is the boundary of  $F_{i,j}$ , and  $\partial F_i$  is the boundary of  $F_i$  which is split in a Dirichlet part  $\Gamma_{Di} = \Gamma_D \cap \partial F_i$  on which the value  $h_{Di}$  is prescribed and a Neumann part  $\Gamma_{Ni} = \Gamma_N \cap \partial F_i$ . Across  $\Gamma_{Ni}$  a total (diffusive and advective) flux is imposed equal to  $h_{Ni}$ . Finally  $\mathbf{n}_{\Gamma_{Ni}}$  is the outward unit normal vector to the Neumann boundary.

Problems on the fractures are coupled together by natural matching conditions expressing the continuity of  $h$  at traces and the balance of fluxes: for all  $\ell \in \mathcal{L}$ , if  $\mathcal{I}_\ell = (i, j)$ ,

$$h_i|_{\Gamma_\ell} - h_j|_{\Gamma_\ell} = 0, \quad (2)$$

$$\frac{\partial h_i}{\partial \mathbf{n}_{\Gamma_\ell}^i} + \frac{\partial h_j}{\partial \mathbf{n}_{\Gamma_\ell}^j} = 0. \quad (3)$$

where  $\mathbf{n}_{\Gamma_\ell}^i$  is the unit normal vector to  $\Gamma_\ell$  with a fixed orientation on  $F_i$ .

In order to introduce the variational formulation of problem (1), let us set the following functional spaces: for  $i \in \mathcal{I}$  and  $j = 1, \dots, N_i$ ,

$$\begin{aligned} \mathbf{H}(\operatorname{div}, F_{i,j}) &:= \left\{ \mathbf{v} \in [\mathbf{L}^2(F_{i,j})]^2 : \operatorname{div}(\mathbf{v}) \in \mathbf{L}^2(F_{i,j}) \right\}, \\ \mathbf{H}_0(\operatorname{div}, F_{i,j}) &:= \left\{ \mathbf{v} \in \mathbf{H}(\operatorname{div}, F_{i,j}) : (\mathbf{v} \cdot \mathbf{n}_{\Gamma_{Ni}})|_{\partial F_{i,j}} = 0 \right\}. \end{aligned}$$

We define

$$\begin{aligned} \mathbb{V}_{i,0} &:= \{ \mathbf{v}_i := (\mathbf{v}_{i,j})_{j=1,\dots,N_i} : \mathbf{v}_{i,j} \in \mathbf{H}_0(\operatorname{div}, F_{i,j}) \quad \forall j \in \{1, \dots, N_i\} \}, \\ \mathbb{V}_i &:= \{ \mathbf{v}_i := (\mathbf{v}_{i,j})_{j=1,\dots,N_i} : \mathbf{v}_{i,j} \in \mathbf{H}(\operatorname{div}, F_{i,j}) \quad \forall j \in \{1, \dots, N_i\} \}, \\ \mathbb{V} &:= \{ \mathbf{v} := (\mathbf{v}_i)_{i=1,\dots,N} : \mathbf{v}_i \in \mathbb{V}_{i,0} \quad \forall i \in \mathcal{I} \}, \\ \mathbb{Q} &:= \{ q = (q_i)_{i=1,\dots,N} : q_i \in \mathbf{L}^2(F_i) \quad \forall i \in \mathcal{I} \}, \\ \mathbb{G} &:= \left\{ \mu = (\mu_\ell)_{\ell=1,\dots,M} : \mu_\ell \in \mathbf{H}^{\frac{1}{2}}(\Gamma_\ell) \quad \forall \ell \in \mathcal{L} \right\}, \end{aligned}$$

endowed with the following natural norms:

$$\begin{aligned}
\|\mathbf{v}\|_{\mathbb{V}_i} &:= \left( \|\mathbf{v}\|_{0,F_i}^2 + \sum_{j=1}^{N_i} \|\operatorname{div}(\mathbf{v})\|_{0,F_{i,j}}^2 \right)^{\frac{1}{2}} & \forall i \in \mathcal{I}, \mathbf{v} \in \mathbb{V}, \\
\|\mathbf{v}\|_{\mathbb{V}} &:= \left( \sum_{i \in \mathcal{I}} \|\mathbf{v}\|_{\mathbb{V}_i}^2 \right)^{\frac{1}{2}} & \forall \mathbf{v} \in \mathbb{V}, \\
\|q\|_{\mathbb{Q}} &:= \left( \sum_{i \in \mathcal{I}} \|q\|_{0,F_i}^2 \right)^{\frac{1}{2}} & \forall q \in \mathbb{Q}, \\
\|\mu\|_{\mathbb{G}} &:= \left( \sum_{\ell \in \mathcal{L}} \|\mu_\ell\|_{\frac{1}{2},\Gamma_\ell}^2 \right)^{\frac{1}{2}} & \forall \mu \in \mathbb{G},
\end{aligned}$$

in which  $\|v\|_{\ell,\omega}$  denotes, as usual, the norm of function  $v$  in  $\mathbf{H}^\ell(\omega)$ .

By defining  $\nu_i := \mathbf{K}_i^{-1}$ ,  $\beta_i := \mathbf{K}_i^{-1}\mathbf{b}_i$ ,  $\forall i \in \mathcal{I}$ , and introducing, on each fracture  $F_i$ ,  $i \in \mathcal{I}$  the new variables  $\mathbf{u}_i := -\mathbf{K}_i \nabla h_i + \mathbf{b}_i h_i$  and for each  $\ell \in \mathcal{L}$  formally defining  $\lambda_\ell = h|_{\Gamma_\ell}$ , we can recast (1) in the following dual variational form:

Find  $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_N$ , with  $\mathbf{u}_0 \in \mathbb{V}$ ,  $h \in \mathbb{Q}$  and  $\lambda \in \mathbb{G}$  such that

$$\begin{cases}
a_i(\mathbf{u}_0, \mathbf{v}) - d_i(\mathbf{v}, h) - b_i(h, \mathbf{v}) + g_i(\lambda, \mathbf{v}) = \\
\quad = -\langle \mathbf{v}_i \cdot \mathbf{n}, h_{D_i} \rangle_{\pm \frac{1}{2}, \Gamma_{D_i}} - a_i(\mathbf{u}_N, \mathbf{v}) & \forall \mathbf{v} \in \mathbb{V}, \\
d_i(\mathbf{u}_0, q) + c_i(h, q) = (f, q)_{F_i} - d_i(\mathbf{u}_N, q) & \forall q \in \mathbb{Q},
\end{cases} \quad (4)$$

where  $\mathbf{u}_N$  is any function such that, being  $\mathbf{u}_{i,N} = \mathbf{u}_N|_{F_i} \in \mathbb{V}_i$ , it is  $\mathbf{u}_{i,N} \cdot \mathbf{n}_{\Gamma_{N_i}} = h_{N_i}$  and we have defined

$$\begin{aligned}
a_i(\mathbf{w}, \mathbf{v}) &:= (\nu \mathbf{w}_i, \mathbf{v}_i)_{F_i} & \forall \mathbf{w}, \mathbf{v} \in \mathbb{V}, \\
b_i(q, \mathbf{v}) &:= (q, \beta_i \cdot \mathbf{v}_i)_{F_i} & \forall q \in \mathbb{Q}, \mathbf{v} \in \mathbb{V}, \\
c_i(p, q) &:= (\gamma_i p_i, q_i)_{F_i} & \forall p, q \in \mathbb{Q}, \\
d_i(\mathbf{v}, q) &:= (\operatorname{div}(\mathbf{v}_i), q_i)_{F_i} & \forall \mathbf{v} \in \mathbb{V}, q \in \mathbb{Q}, \\
g_i(\mu, \mathbf{v}) &:= \sum_{\ell \in \mathcal{L}_i} \left\langle \llbracket \mathbf{v}_i \cdot \mathbf{n}_{\Gamma_\ell}^i \rrbracket_{\Gamma_\ell}, \mu_\ell \right\rangle_{\pm \frac{1}{2}, \Gamma_\ell} & \forall \mu \in \mathbb{G}, \mathbf{v} \in \mathbb{V}.
\end{aligned} \quad (5)$$

Problems (4) on the fractures are coupled by the following variational equations, which follows from (3):

$$\left\langle \llbracket \mathbf{u} \rrbracket_{\Gamma_\ell}, \lambda_\ell \right\rangle_{\pm \frac{1}{2}, \Gamma_\ell} = 0 \quad \forall \lambda_\ell \in \mathbf{H}^{\frac{1}{2}}(\Gamma_\ell), \forall \ell \in \mathcal{L}, \quad (7)$$

where, if  $\mathcal{I}_\ell = (i, j)$ ,  $\{\{\mathbf{u}\}\}_{\Gamma_\ell} := \llbracket \mathbf{u}_i \cdot \mathbf{n}_{\Gamma_\ell}^i \rrbracket_{\Gamma_\ell} + \llbracket \mathbf{u}_j \cdot \mathbf{n}_{\Gamma_\ell}^j \rrbracket_{\Gamma_\ell}$ . After denoting by:

$$\begin{aligned}
a(\mathbf{w}, \mathbf{v}) &:= \sum_{i \in \mathcal{I}} a_i(\mathbf{w}, \mathbf{v}) & \forall \mathbf{w}, \mathbf{v} \in \mathbb{V}, \\
b(q, \mathbf{v}) &:= \sum_{i \in \mathcal{I}} b_i(q, \mathbf{v}) & \forall q \in \mathbb{Q}, \mathbf{v} \in \mathbb{V}, \\
c(p, q) &:= \sum_{i \in \mathcal{I}} c_i(p, q) & \forall p, q \in \mathbb{Q}, \\
d(\mathbf{v}, q) &:= \sum_{i \in \mathcal{I}} d_i(\mathbf{v}, q) & \forall \mathbf{v} \in \mathbb{V}, q \in \mathbb{Q}, \\
g(\mu, \mathbf{v}) &:= \sum_{i \in \mathcal{I}} g_i(\mu, \mathbf{v}) & \forall \mu \in \mathbb{G}, \mathbf{v} \in \mathbb{V},
\end{aligned}$$

problem (4)-(7) can be re-stated as the following system of equations on the whole network: find  $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_N$ , with  $\mathbf{u}_0 \in \mathbb{V}$ ,  $h \in \mathbb{Q}$  and  $\lambda \in \mathbb{G}$  such that

$$\begin{cases}
a(\mathbf{u}_0, \mathbf{v}) - b(h, \mathbf{v}) - d(\mathbf{v}, h) + g(\lambda, \mathbf{v}) = \\
\quad = -(h_D, \mathbf{v} \cdot \mathbf{n})_{\Gamma_D} - a(\mathbf{u}_N, \mathbf{v}) & \forall \mathbf{v} \in \mathbb{V}, \\
d(\mathbf{u}_0, q) + c(h, q) = (f, q) - d(\mathbf{u}_N, q) & \forall q \in \mathbb{Q}, \\
g(\mu, \mathbf{u}_0) = -g(\mu, \mathbf{u}_N) & \forall \mu \in \mathbb{G}.
\end{cases} \quad (8)$$

Notice that in the above problem,  $h$  is in general discontinuous at fracture intersections, and  $\lambda$  is playing the role of its trace. By classical arguments and assumptions on the regularity of the data, (see e.g. [43]), it can be proven that, if problem (1)-(3) is well posed, (8) has a unique solution, which coincides with the one of (1)-(3) [32, 43].

### 3. The discrete problem with the Mixed Virtual Element Method

The mixed formulation for the Virtual Element Method (VEM) has been recently presented in [30], with a followup work generalizing the method in [32], and, due to its recent introduction, the only work regarding its application, at the time of writing, is [44], which deals with Stokes flow. Here, we use a Mixed VEM space to discretize the velocity space  $\mathbb{V}$ , while the pressure and trace spaces  $\mathbb{Q}$  and  $\mathbb{G}$  are discretized in a standard way.

After a brief presentation of the mesh generation process proposed for the present work, a description of the mixed Virtual Element in the context fo DFN flow simulation is provided. The reader is, however, referred to [30, 32] for most of the theoretical results concerning the VEM.

#### 3.1. Meshing process

Let  $i \in \mathcal{I}$  be fixed and let us consider a generic fracture  $F_i$  in  $\Omega$  with traces  $\Gamma_\ell$ ,  $\ell \in \mathcal{L}_i$ . The local mesh on the fracture is obtained starting from a regular

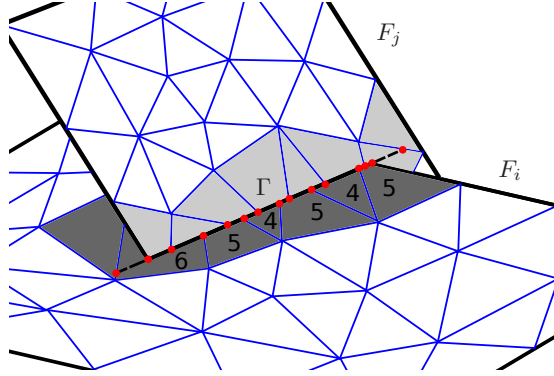


Figure 1: Detail of the VEM conforming mesh at two intersecting fractures

triangular mesh, that is not required to be geometrically conforming with the traces. The triangular elements of this mesh that intersect the traces are then cut into polygons that do not cross the traces. This process is performed independently on each of the fractures in  $\Omega$ , thus providing a polygonal mesh on each fracture that is locally conforming to the traces, i.e. with elements that do not cross any of the traces  $\Gamma_\ell$ ,  $\ell \in \mathcal{L}_i$  of  $F_i$ , but instead cover them with edges of the triangulation. A globally conforming mesh of the whole network is finally obtained adding, on each trace  $\Gamma_\ell$  on fracture  $F_i$  the vertices on  $\Gamma_\ell$  of the mesh of fractures  $F_j$ , with  $i, j \in \mathcal{I}$ : this increases the original number of edges of the mesh elements neighboring  $\Gamma_\ell$ , forming polygons with flat angles at some vertices. This discretization technique is the same proposed in [35] and exploits the flexibility of virtual elements in handling arbitrary polygonal elements. An example is proposed in Figure 1, where two intersecting fractures are shown. The shadowed regions on both fractures represent the original triangles that are subsequently cut into polygons. The trace segment is prolonged up to meeting the first edge in the mesh and new polygons are created not crossing the prolonged segment. The numbers in some of the new elements on  $F_i$  represent the number of edges of the new polygonal mesh elements, whereas the red dots indicate mesh vertices on the trace, shared by the two fractures. It can be noticed that in most cases a red dot corresponds to a polygon vertex with a flat angle.

After the generation of the globally conforming mesh, each trace  $\Gamma_\ell$ ,  $\ell \in \mathcal{L}$  is split by mesh vertices into  $n_e^{\Gamma_\ell}$  segments, each denoted by  $\sigma_t^\ell$ ,  $t = 1, \dots, n_e^{\Gamma_\ell}$ . In order to ease the presentation of the method it is convenient to assume that to each segment  $\sigma_t^\ell$  correspond two identical mesh edges on each fracture, thus having, in the global conforming mesh, a total of four copies of each trace edge, with their own outward unit normal vector. An example is shown in Figure 2, where a generic fracture  $F_i$  in  $\Omega$  is shown, with a single trace  $\Gamma_1$  perpendicular to the horizontal boundary edges, and passing through their midpoints. As shown, the original trace segment  $\sigma_1^1 \equiv \Gamma_1$  generates two mesh edges,  $e_1^{1,i,+}$  and

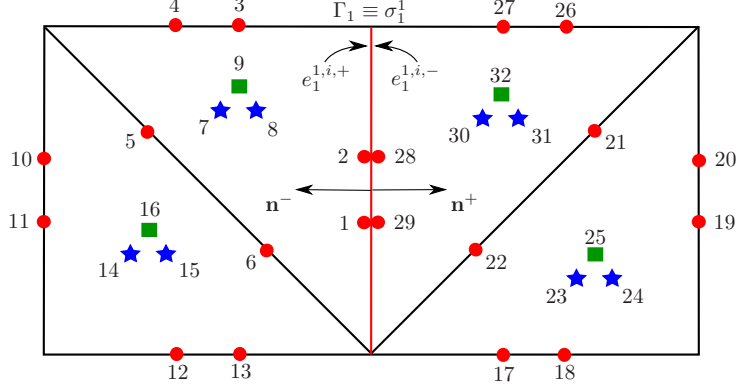


Figure 2: Schematic of the mesh on a selected fracture with the degrees of freedom of the discrete spaces for Virtual Elements of order  $k = 1$

$e_1^{1,i,-}$  each endowed with an outward unit normal vector,  $\mathbf{n}^+$ ,  $\mathbf{n}^-$ , with opposite direction.

Let us denote by  $\mathcal{T}_{\delta,i}$  the mesh obtained in such a way on each fracture,  $\mathcal{E}_{\delta,i}$  being the set of its edges, and we also set  $\mathcal{T}_{\delta} = \cup_{i \in \mathcal{I}} \mathcal{T}_{\delta,i}$  and  $\mathcal{E}_{\delta} = \cup_{i \in \mathcal{I}} \mathcal{E}_{\delta,i}$ . For each mesh element  $E \in \mathcal{T}_{\delta}$ , we set  $(x_E, y_E)$  to be the barycenter of  $E$ , and  $h_E$  to be the diameter of  $E$ .

### 3.2. Virtual spaces

Let  $i \in \mathcal{I}$  and  $E \in \mathcal{T}_{\delta,i}$ . The local VEM space for the velocity variable on each element  $E$  is

$$\mathbb{V}_{\delta,i}^E = \{ \mathbf{v}_{\delta} \in \mathbf{H}(\text{div}, E) : \mathbf{v}_{\delta} \cdot \mathbf{n}_e \in \mathbb{P}_k(e) \quad \forall e \in \partial E, \\ \text{div}(\mathbf{v}_{\delta}) \in \mathbb{P}_k(E), \text{ and } \text{rot}(\mathbf{v}_{\delta}) \in \mathbb{P}_{k-1}(E) \}, \quad (9)$$

in which  $\mathbb{P}_k(\omega)$  is the space of polynomials of maximum order  $k$ , and  $e \subset \partial E$  is one of the edges of the boundary of  $E$ , and  $\mathbf{n}_e$  is the unit outward normal vector to edge  $e$ . The condition on the rotor in (9) is required to fix the dimension of  $\mathbb{V}_{\delta,i}^E$ . The global discrete space on each element  $E$  is then:

$$\mathbb{V}_{\delta,i} := \{ \mathbf{v}_{\delta} \in \mathbb{V}_i : \mathbf{v}_{\delta}|_E \in \mathbb{V}_{\delta,i}^E \quad \forall E \in \mathcal{T}_{\delta,i} \}. \quad (10)$$

We remark that functions in  $\mathbb{V}_{\delta,i}$  have a continuous normal component in subfractures, whereas we have in general  $[\mathbf{v}_{\delta,i} \cdot \mathbf{n}_{\Gamma_{\ell}^i}] \neq 0, \forall \ell \in \mathcal{L}_i$ . There are many choices for the degrees of freedom (DOFs) of functions in  $\mathbb{V}_{\delta,i}$ , see, e.g. [30, 32, 45]. In order to describe the DOFs used for the present work, let us introduce the following set of scaled monomials on each mesh element  $E$ , that span the space  $\mathbb{P}_k(E)$  of dimension  $n_k = (k+1)(k+2)/2$ :

$$\mathbb{M}_k(E) = \left\{ m_{\alpha}(x, y) = \frac{(x - x_E)^{\alpha_1} (y - y_E)^{\alpha_2}}{h_E^{\alpha_1 + \alpha_2}}, \quad |\alpha| = \alpha_1 + \alpha_2 \leq k \right\}.$$

Then let us introduce also the space

$$\nabla\mathbb{P}_{k+1}(E) := \left\{ \mathbf{p} \in [\mathbb{P}_k(E)]^2 : \mathbf{p} = \nabla\hat{p} \text{ for some } \hat{p} \in \mathbb{P}_{k+1}(E) \right\}.$$

This space has dimension  $n_{k+1} - 1$ . Moreover, let  $(\nabla\mathbb{P}_{k+1}(E))^\perp$  be the  $L^2(E)$  orthogonal complement of  $\nabla\mathbb{P}_{k+1}(E)$  in  $(\mathbb{P}_k(E))^2$ , so that

$$(\mathbb{P}_k(E))^2 = (\nabla\mathbb{P}_{k+1}(E)) \oplus (\nabla\mathbb{P}_{k+1}(E))^\perp.$$

We remark that, in what follows, the space  $(\nabla\mathbb{P}_{k+1}(E))^\perp$  could be replaced by any other space  $(\nabla\mathbb{P}_{k+1}(E))^*$  satisfying  $(\nabla\mathbb{P}_{k+1}(E)) \oplus (\nabla\mathbb{P}_{k+1}(E))^* = (\mathbb{P}_k(E))^2$ . The space  $(\nabla\mathbb{P}_{k+1}(E))^\perp$  is here used only for presentation purposes, whereas for the implementation of the method the choice described in [46] is used, which is computationally less expensive.

The dimension of the space  $(\nabla\mathbb{P}_{k+1}(E))^\perp$  is

$$\begin{aligned} \dim \left[ (\nabla\mathbb{P}_{k+1}(E))^\perp \right] &= \dim \left[ (\mathbb{P}_k(E))^2 \right] - \dim [\nabla\mathbb{P}_{k+1}(E)] = \\ &= 2n_k - n_{k+1} + 1 = 2 \frac{(k+1)(k+2)}{2} - \frac{(k+2)(k+3)}{2} = \frac{(k+2)(k-1)}{2} + 1, \end{aligned}$$

from which we deduce that  $(\nabla\mathbb{P}_{k+1}(E))^\perp$  is not empty when  $k \geq 1$ . The space  $\nabla\mathbb{P}_{k+1}(E)$  is spanned by

$$\nabla\mathbb{M}_{k+1}(E) := \left\{ \mathbf{m} \in [\mathbb{M}_k(E)]^2 : \mathbf{m} = \nabla\hat{m} \text{ for some } \hat{m} \in \mathbb{M}_{k+1}(E), \hat{m} \neq 1 \right\},$$

whose orthogonal component in  $[\mathbb{P}_k(E)]^2$  is denoted by  $(\nabla\mathbb{M}_{k+1}(E))^\perp$ .

As degrees of freedom on  $\mathbb{V}_{\delta,i}^E$  we choose,  $\forall \mathbf{v}_\delta \in \mathbb{V}_{\delta,i}$ ,

- the value of  $\mathbf{v}_\delta \cdot \mathbf{n}_e$  at  $k+1$  internal points on  $e$ , for each edge  $e \subset \partial E$ ;
- if  $k \geq 1$ , the  $n_k - 1$  products  $(\mathbf{v}_\delta, \mathbf{m})_E$  for all  $\mathbf{m} \in \nabla\mathbb{M}_k(E)$ ;
- if  $k \geq 1$ , the  $(k+2)(k-1)/2 + 1$  products  $(\mathbf{v}_\delta, \mathbf{m}^\perp)_E$  for all  $\mathbf{m} \in (\nabla\mathbb{M}_{k+1}(E))^\perp$ ;

thus leading to the following number of degrees of freedom,  $n_{\text{dof}}^E$  for a polygonal element  $E \in \mathcal{T}_\delta$  with  $n_e^E$  vertices:

- if  $k = 0$ ,  $n_{\text{dof}}^E = n_e^E$ ;
- if  $k \geq 1$ ,  $n_{\text{dof}}^E = (k+1)n_e^E + k(k+2)$ .

A schematic of the degrees of freedom for the discrete VEM space is reported in Figure 2, in the case of a polynomial accuracy  $k = 1$ . According to the selected polynomial accuracy, two degrees of freedom are associated to each mesh edge and are marked with red dots and a consecutive numbering (the location of the DOFs in the picture is only indicative). Two DOFs are associated to each

copy of the edge lying on the trace and we remark that this allows for a velocity solution with discontinuous normal component across the traces of each fracture. The set of DOFs for the velocity is completed by the two moments with respect to the monomials in  $\nabla\mathbb{M}_1(E)$ , denoted with a star shaped blue marker inside the element, and the moment with respect to  $(\nabla\mathbb{M}_2(E))^\perp$ , marked with green squares inside the element.

When building the global velocity space  $\mathbb{V}_{\delta,i}$  on each fracture  $F_i$ , continuity of the normal component of  $\mathbf{v}_\delta \in \mathbb{V}_{\delta,i}$  is enforced across each mesh element edge that does not lie on any of the traces  $\Gamma_\ell$ ,  $\ell \in \mathcal{L}_i$ , thus allowing jumps of the normal component of the velocity across fracture intersections both on each fracture and across intersecting fractures.

As in the case of the primal VEM formulation, the shape functions in the local VEM space are not explicitly known in general except for their DOFs, and thus computability of operators on the VEM functions is one of the key aspects in the VEM framework. Let us observe that the divergence of each VEM function can be exactly computed, by means of the set of DOFs. Indeed, since  $\text{div}(\mathbf{v}_\delta)|_E \in \mathbb{P}_k(E)$ , we can exactly compute its components in the basis  $\mathbb{M}_k(E)$  since, by Green's formula we have:

$$(\text{div}(\mathbf{v}_\delta), m)_E = -(\mathbf{v}_\delta, \nabla m)_E + (\mathbf{v}_\delta \cdot \mathbf{n}_e, m)_{\partial E} \quad \forall m \in \mathbb{M}_k(E), \quad (11)$$

and the right-hand side is computable using the first two sets of degrees of freedom. Having computed  $\text{div}(\mathbf{v}_\delta)$ , we can also compute the component-wise  $L^2(E)$ -projection of  $\mathbf{v}_\delta \in \mathbb{V}_{\delta,i}^E$  on the space of polynomials of order  $k$ :  $\Pi_k^0: \mathbb{V}_{\delta,i}^E \rightarrow [\mathbb{P}_k(E)]^2$ , defined by

$$(\Pi_k^0 \mathbf{v}_\delta, \mathbf{m})_E = (\mathbf{v}_\delta, \mathbf{m})_E \quad \forall \mathbf{m} \in [\mathbb{M}_k(E)]^2.$$

The right-hand side of this latter equation can be computed observing that, for any given  $\mathbf{m} \in \mathbb{M}_k(E)$ , we can decompose  $\mathbf{m} = \mathbf{m}^\nabla + \mathbf{m}^\perp$ , where  $\mathbf{m}^\nabla \in \nabla\mathbb{M}_{k+1}(E)$  and  $\mathbf{m}^\perp \in (\nabla\mathbb{M}_{k+1}(E))^\perp$ . Let  $\hat{m} \in \mathbb{M}_{k+1}(E)$  be such that  $\nabla\hat{m} = \mathbf{m}^\nabla$ ; then

$$\begin{aligned} (\mathbf{v}_\delta, \mathbf{m})_E &= (\mathbf{v}_\delta, \mathbf{m}^\nabla)_E + (\mathbf{v}_\delta, \mathbf{m}^\perp)_E = (\mathbf{v}_\delta, \nabla\hat{m})_E + (\mathbf{v}_\delta, \mathbf{m}^\perp)_E = \\ &= -(\text{div}(\mathbf{v}_\delta), \hat{m})_E + (\mathbf{v}_\delta \cdot \mathbf{n}_{\partial E}, \hat{m})_{\partial E} + (\mathbf{v}_\delta, \mathbf{m}^\perp)_E, \end{aligned}$$

and all terms on the last line are computable knowing  $\text{div}(\mathbf{v}_\delta)$  and the first and last set of degrees of freedom of  $\mathbf{v}_\delta$ .

Concerning the pressure variable, we define the set of DOFs on each mesh element  $E$  as the  $n_k$  moments with respect to the monomial basis  $\mathbb{M}_k(E)$ , and the global discrete space for the pressure is  $\mathbb{Q}_\delta = \{q_\delta : q_\delta|_{F_i} \in \mathbb{Q}_{\delta,i}, i = 1, \dots, N\}$ , with

$$\mathbb{Q}_{\delta,i} := \{q_\delta \in L^2(F_i) : q_\delta|_E \in \mathbb{P}_k(E) \quad \forall E \in \mathcal{T}_{\delta,i}\}.$$

Note that no requirements of continuity are made for this space.

Let us now call  $\mathcal{M}^\ell = \{\mu_{\delta,\xi}^\ell \mid \xi = 1, \dots, n_{\text{dof}}^{\Gamma_\ell}\}$  the set of the degrees of freedom introduced on all the (duplicated) mesh edges  $e \in \mathcal{E}_\delta$  lying on trace  $\Gamma_\ell$ ,  $\ell \in \mathcal{L}$ , and then let us set the space  $\mathbb{G}_\delta$  as:

$$\mathbb{G}_\delta = \bigcup_{\ell \in \mathcal{L}} \mathcal{M}^\ell.$$

### 3.3. Mixed VEM discrete formulation

Let us fix  $i \in \mathcal{I}$  and let us define

$$a_{\delta,i}: \mathbb{V}_{\delta,i} \times \mathbb{V}_{\delta,i} \rightarrow \mathbb{R}, \quad b_{\delta,i}: \mathbb{Q}_{\delta,i} \times \mathbb{V}_{\delta,i} \rightarrow \mathbb{R}$$

such that,  $\forall \mathbf{v}_\delta, \mathbf{w}_\delta \in \mathbb{V}_{\delta,i}$ ,  $q_\delta \in \mathbb{Q}_{\delta,i}$ :

$$a_{\delta,i}(\mathbf{w}_\delta, \mathbf{v}_\delta) := \sum_{E \in \mathcal{T}_{\delta,i}} (\nu \Pi_k^0 \mathbf{w}_\delta, \Pi_k^0 \mathbf{v}_\delta)_E + S^E((I - \Pi_k^0) \mathbf{w}_\delta, (I - \Pi_k^0) \mathbf{v}_\delta), \quad (12)$$

$$b_{\delta,i}(q_\delta, \mathbf{v}_\delta) := \sum_{E \in \mathcal{T}_{\delta,i}} (q_\delta, \beta \cdot \Pi_k^0 \mathbf{v}_\delta)_E, \quad (13)$$

where  $S^E: \mathbb{V}_{\delta,i} \times \mathbb{V}_{\delta,i} \rightarrow \mathbb{R}$  is a bilinear form such that  $\forall \mathbf{v}_\delta \in \mathbb{V}_{\delta,i}$ :

$$\exists \alpha_*, \alpha^* > 0: \alpha_* a_i^E(\mathbf{v}_\delta, \mathbf{v}_\delta) \leq S^E(\mathbf{v}_\delta, \mathbf{v}_\delta) \leq \alpha^* a_i^E(\mathbf{v}_\delta, \mathbf{v}_\delta) \quad \forall \mathbf{v}_\delta \in \ker \Pi_k^0,$$

in which  $a_i^E(\cdot, \cdot)$  is the restriction of the bilinear form  $a_i(\cdot, \cdot)$  to element  $E \subset \mathcal{T}_{\delta,i}$ . Then, setting  $\forall \mathbf{v}_\delta, \mathbf{w}_\delta \in \mathbb{V}_\delta$ , and  $q_\delta \in \mathbb{Q}_\delta$

$$a_\delta(\mathbf{w}_\delta, \mathbf{v}_\delta) := \sum_{i \in \mathcal{I}} a_{\delta,i}(\mathbf{w}_\delta|_{F_i}, \mathbf{v}_\delta|_{F_i}), \quad b_\delta(q_\delta, \mathbf{v}_\delta) := \sum_{i \in \mathcal{I}} b_{\delta,i}(q_\delta|_{F_i}, \mathbf{v}_\delta|_{F_i}),$$

we write the following discrete problem: *find*  $\mathbf{u}_{0\delta} \in \mathbb{V}_\delta$ ,  $h_\delta \in \mathbb{Q}_\delta$ ,  $\lambda_\delta \in \mathbb{G}_\delta$  *such that*:

$$\begin{cases} a_\delta(\mathbf{u}_{0\delta}, \mathbf{v}_\delta) - b_\delta(h_\delta, \mathbf{v}_\delta) - d(\mathbf{v}_\delta, h_\delta) + g(\lambda_\delta, \mathbf{v}_\delta) = \\ \quad = -(h_{\delta D}, \mathbf{v}_\delta \cdot \mathbf{n})_{\Gamma_D} - a_\delta(\mathbf{u}_{\delta N}, \mathbf{v}_\delta) & \forall \mathbf{v}_\delta \in \mathbb{V}_\delta, \\ d(\mathbf{u}_{0\delta}, q_\delta) + c(h_\delta, q_\delta) = (f, q_\delta) - d(\mathbf{u}_{\delta N}, q_\delta) & \forall q_\delta \in \mathbb{Q}_\delta, \\ g(\mu_\delta, \mathbf{u}_{\delta 0}) = -g(\mu_\delta, \mathbf{u}_{\delta N}) & \forall \mu_\delta \in \mathbb{G}_\delta. \end{cases} \quad (14)$$

### 3.4. Well-posedness of the discrete problem

In order to prove the well posedness of (14), let us observe that according to the definition of the space  $\mathbb{G}_\delta$  an inf-sup condition holds for the operator  $g: \mathbb{G}_\delta \times \mathbb{V}_\delta \mapsto \mathbb{R}$ . Indeed we have:

$$\forall \mu_\delta \in \mathbb{G}_\delta, \exists \mathbf{v}_\delta^* \in \mathbb{V}_\delta: \frac{|g(\mu_\delta, \mathbf{v}_\delta^*)|}{\|\mathbf{v}_\delta^*\|_{\mathbb{V}_\delta}} = 1 \quad (15)$$

choosing  $\mathbf{v}_\delta^*$  as the basis function of the VEM space corresponding to the degree of freedom  $\mu_\delta$ , and recalling that basis functions in  $\mathbb{V}_\delta$  have discontinuous normal components across the traces on each fracture.

Then we place ourselves in the space  $\mathcal{V}_\delta = \mathbb{V}_\delta \cap \ker(g)$ , having the same local approximation properties of  $\mathbb{V}_\delta$ , and we re-write (14) as: *find*  $\mathbf{u}_{0\delta} \in \mathcal{V}_\delta$ ,  $h_\delta \in \mathbb{Q}_\delta$  *such that*:

$$\begin{cases} a_\delta(\mathbf{u}_{0\delta}, \mathbf{v}_\delta) - b_\delta(h_\delta, \mathbf{v}_\delta) - d(\mathbf{v}_\delta, h_\delta) = \\ \quad = g(\mu_\delta, \mathbf{u}_{\delta N}) - (h_{\delta D}, \mathbf{v}_\delta \cdot \mathbf{n})_{\Gamma_D} - a_\delta(\mathbf{u}_{\delta N}, \mathbf{v}_\delta) & \forall \mathbf{v}_\delta \in \mathcal{V}_\delta, \\ d(\mathbf{u}_{0\delta}, q_\delta) + c(h_\delta, q_\delta) = (f, q_\delta) - d(\mathbf{u}_{\delta N}, q_\delta) & \forall q_\delta \in \mathbb{Q}_\delta, \end{cases}$$

whose well-posedness is proven in [32].

#### 4. Implementation

This section is aimed at proving practical details of the algorithm, from the discretization strategy to the assembly of matrices that have to be computed.

For every fracture  $F_i$ , with  $i \in \mathcal{I}$ , let us denote by  $n_{\text{dof},i}^v$  the number of DOFs for the velocity space and  $n_{\text{dof},i}^p$  the number of DOFs for the pressure space on fracture  $F_i$  and let  $n_{\text{dof},i} := n_{\text{dof},i}^v + n_{\text{dof},i}^p$ . It is also  $n_{\text{dof}}^v = \sum_{i \in \mathcal{I}} n_{\text{dof},i}^v$  and  $n_{\text{dof}}^p = \sum_{i \in \mathcal{I}} n_{\text{dof},i}^p$  defined as the total number of DOFs throughout the whole network for the velocity and pressure spaces, respectively. We collect the degrees of freedom for the velocity space in a vector  $\bar{\mathbf{u}} \in \mathbb{R}^{n_{\text{dof}}^v}$ , and the degrees of freedom for the pressure in a vector  $\bar{\mathbf{h}} \in \mathbb{R}^{n_{\text{dof}}^p}$ .

Let us introduce matrix  $A_i \in \mathbb{R}^{n_{\text{dof},i}^v \times n_{\text{dof},i}^v}$  defined as

$$(A_i)_{jk} = a_{\delta,i}(\boldsymbol{\varphi}_{i,k}, \boldsymbol{\varphi}_{i,j})$$

representing the discrete operator  $a_{\delta,i}$  on each fracture  $F_i$ ,  $i = 1, \dots, N$ , defined as in (12), and in which  $\boldsymbol{\varphi}_{i,j}$ ,  $j = 1, \dots, n_{\text{dof},i}^v$  are the VEM basis functions of the finite-dimensional discrete space  $\mathbb{V}_{\delta,i}$  for the velocity on  $F_i$ . The computation of the projection operator is as in (11), whereas the stabilization term  $S^E(\mathbf{v}_\delta, \mathbf{w}_\delta)$ , for  $\mathbf{v}_\delta, \mathbf{w}_\delta \in \mathbb{V}_{\delta,i}$  is defined as the euclidean scalar product of the degrees of freedom of  $\mathbf{v}_\delta, \mathbf{w}_\delta$  multiplied by a factor  $\nu^E := h_E^2 \nu(x_E, y_E)$ . Furthermore, being  $\chi_{i,k} \in \mathbb{Q}_{\delta,i}$ , for  $k = 1, \dots, n_{\text{dof},i}^p$ , the basis functions of the finite-dimensional space for the pressure variable on  $F_i$ , we denote by:

$$(B_i)_{jk} = b_{\delta,i}(\boldsymbol{\varphi}_{i,k}, \chi_{i,j}) \quad (C_i)_{jk} = c_i(\chi_{i,k}, \chi_{i,j}) \quad (D_i)_{jk} = d_i(\boldsymbol{\varphi}_{i,k}, \chi_{i,j})$$

the matrices associated with the bilinear forms  $b_{\delta,i}$  in (13),  $c_i$  in (5) and  $d_i$  in (6), respectively. It is then possible to write an independent saddle point problem on each fracture  $F_i$  of the network, expressing the advection diffusion problem on each fracture and corresponding to the first two equations in (14). The structure of the saddle point matrix  $K_i \in \mathbb{R}^{n_{\text{dof},i} \times n_{\text{dof},i}}$ ,  $i \in \mathcal{I}$  is:

$$K_i = \begin{bmatrix} A_i & -B_i - D_i \\ D_i^\top & C_i \end{bmatrix}. \quad (16)$$

We then construct column vectors  $\mathbf{f}_i \in \mathbb{R}^{n_{\text{dof},i}}$ ,  $i \in \mathcal{I}$ , as the vector of load values and boundary condition terms. Assembling matrices on the whole DFN we have:

$$K = \begin{bmatrix} K_1 & 0 & \cdots & 0 \\ 0 & K_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & K_N \end{bmatrix} \quad \text{and} \quad \mathbf{f} = \begin{bmatrix} \mathbf{f}_1 \\ \vdots \\ \vdots \\ \mathbf{f}_N \end{bmatrix}.$$

We remark that matrices  $K_i$  are singular for fractures  $F_i$ ,  $i \in \mathcal{I}$  with pure Neumann boundary conditions, the uniqueness of the solution deriving from the matching among all the fractures on the network, which is given by the last equation in (14), not yet enforced here.

To this end, let us recall that each trace  $\Gamma_\ell$ ,  $\ell \in \mathcal{L}$  is given by the union of segments  $\sigma_t^\ell$ ,  $t = 1, \dots, n_e^{\Gamma_\ell}$ . To each segment  $\sigma_t^\ell$  of  $\Gamma_\ell$  four mesh edges are associated, denoted by  $\{e_t^{\ell,i,+}, e_t^{\ell,i,-}, e_t^{\ell,j,+}, e_t^{\ell,j,-}\} := \mathcal{E}_{\delta,t}^\ell \subset \mathcal{E}_\delta$ , with  $i, j \in \mathcal{I}_\ell$  and the  $+$  or  $-$  sign is chosen depending if the mesh edge belongs to the element on the right or left side of  $\sigma_t^\ell$ , respectively, after fixing an orientation for  $\sigma_t^\ell$  on  $F_i$  and  $F_j$ , independently. On a trace  $\Gamma_\ell$ , for a given polynomial accuracy  $k$ , we associate  $k+1$  DOFs to each edge  $e_t^{\ell,i,\star} \in \mathcal{E}_{\delta,t}^\ell$ , with  $\star = +$  or  $-$  and we denote by  $u_{t,\xi}^{\ell,i,\star}$ ,  $\xi = 1, \dots, k+1$  each of these DOFs. It is possible to introduce a function  $\kappa$ , such that, for  $\ell \in \mathcal{L}$ ,  $i \in \mathcal{I}_\ell$ ,  $\star \in \{+, -\}$ ,  $t = 1, \dots, n_e^{\Gamma_\ell}$ , and  $\xi = 1, \dots, k+1$ , the number  $\kappa(\ell, t, \xi, i, \star) \in \{1, \dots, n_{\text{dof}}^v\}$  corresponds to the index of the degree of freedom  $u_{t,\xi}^{\ell,i,\star}$  in the global numbering. For each trace  $\Gamma_\ell$ , for each  $t = 1, \dots, n_e^{\Gamma_\ell}$  it is possible to write  $k+1$  conditions enforcing the balance of the normal component of the velocity at fracture intersections, by simply setting to zero the sum of the basis functions on the edges in  $\mathcal{E}_{\delta,t}^\ell$  as follows: for  $\xi = 1, \dots, k+1$ :

$$\Lambda_{\ell,t,\xi} \bar{\mathbf{u}} = 0,$$

being  $\Lambda_{\ell,t,\xi} \in \mathbb{R}^{n_{\text{dof}}^v}$  defined as:

$$\Lambda_{\ell,t,\xi} := \begin{pmatrix} \kappa_1 & & \kappa_2 & & \kappa_3 & & \kappa_4 & & & \\ 0 & \cdots & 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 & \cdots & 0 \end{pmatrix},$$

and, for  $i, j \in \mathcal{I}_\ell$ ,  $(\kappa_\varrho)_{\varrho=1,\dots,4}$  is the ordered set of the values:

$$\{\kappa(\ell, t, \xi, i, +), \kappa(\ell, t, \xi, i, -), \kappa(\ell, t, \xi, j, +), \kappa(\ell, t, \xi, j, -)\}.$$

Then it is possible to collect column-wise row vectors  $\Lambda_{\ell,t,\xi}$  for  $\ell \in \mathcal{L}$ ,  $t = 1, \dots, n_e^{\Gamma_\ell}$ , and  $\xi = 1, \dots, k+1$  as to form matrix  $\Lambda \in \mathbb{R}^{((k+1)\sum_{\ell \in \mathcal{L}} n_e^{\Gamma_\ell}) \times n_{\text{dof}}^v}$ , and then matrix  $\mathcal{L} = [\Lambda, \mathbf{0}_p]$ , with  $\mathbf{0}_p \in \mathbb{R}^{n_{\text{dof}}^p \times n_{\text{dof}}^p}$  being the null matrix. Finally, if we let  $\boldsymbol{\lambda}$  be the vector of values of the Lagrange multipliers of the system, and setting  $\mathbf{s} = [\bar{\mathbf{u}}^\top, \bar{\mathbf{h}}^\top]^\top$  we have that the final linear system corresponding to the discrete problem (14) is:

$$\begin{bmatrix} K & \mathcal{L}^\top \\ \mathcal{L} & O \end{bmatrix} \begin{bmatrix} \mathbf{s} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix}. \quad (17)$$

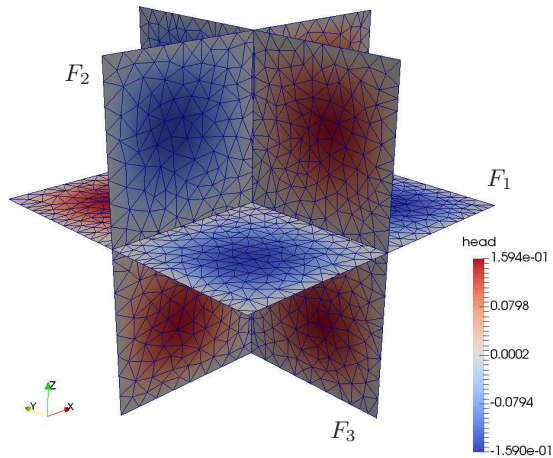


Figure 3: BP1: spatial distribution of fractures

This type of system is in the standard form that arises when applying domain decomposition methods. We refer the reader to [47, 48, 49, 50].

## 5. Numerical results

Let us now show some numerical tests performed on networks of increasing complexity. Approximation spaces with polynomial accuracy up to order  $k = 5$  are considered: the mixed discrete VEM space is used for the space of the velocity and the space of element-wise polynomials of the same order is used for the pressure space.

The first two considered problems take into account simple networks of up to three fractures, for which an analytic solution is known, and convergence curves are presented to compare the obtained convergence rates with the optimal ones reported in [32]. Subsequently some simulations on more complex DFNs are proposed, aimed at showing the applicability of the method, and also to provide some qualitative insight on the behaviour of the method.

### 5.1. Benchmark problems

As a first example, a 3 fracture, 3 trace network is analyzed, labeled BP1. The domain is the same as the one in [35], and is shown in Figure 3. Here an advection-diffusion-reaction problem is considered, with non constant coefficients and it is solved in mixed formulation.

The computational domain  $\Omega$  is given by the union of fractures  $F_1$ ,  $F_2$  and  $F_3$ , defined as follows:

$$\begin{aligned}
 F_1 &= \{(x, y, z) \in \mathbb{R}^3 : -1 \leq x \leq 1, -1 \leq y \leq 1, z = 0\}, \\
 F_2 &= \{(x, y, z) \in \mathbb{R}^3 : -1 \leq y \leq 1, -1 \leq z \leq 1, x = 0\}, \\
 F_3 &= \{(x, y, z) \in \mathbb{R}^3 : -1 \leq z \leq 0, -1 \leq x \leq 1, y = 0\},
 \end{aligned}$$

with three intersections:

$$\begin{aligned}\Gamma_1 &= \{(x, y, z) \in \mathbb{R}^3 : -1 \leq x \leq 1, y = 0, z = 0\}, \\ \Gamma_2 &= \{(x, y, z) \in \mathbb{R}^3 : -1 \leq y \leq 1, z = 0, x = 0\}, \\ \Gamma_3 &= \{(x, y, z) \in \mathbb{R}^3 : -1 \leq z \leq 0, x = 0, y = 0\}.\end{aligned}$$

Introducing a fracture-local reference system  $(\hat{x}, \hat{y})$  we define, on each fracture  $F_i$ ,  $i = 1, \dots, 3$ :

$$\begin{aligned}\mathbf{K}_i(\hat{x}, \hat{y}) &= \begin{pmatrix} 1 + \hat{y}^2 & -\hat{x}\hat{y}/2 \\ -\hat{x}\hat{y}/2 & 1 + \hat{x}^2 \end{pmatrix}, \\ \mathbf{b}_i(\hat{x}, \hat{y}) &= (\hat{x} - \hat{y}, \hat{y} - 1), \\ \gamma_i(\hat{x}, \hat{y}) &= \hat{x}^3 + \hat{y},\end{aligned}$$

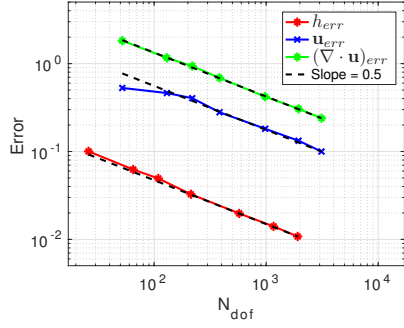
and the problem is formulated as in (1), being the load terms  $f_i$ ,  $i = 1, \dots, 3$  such that the exact solution on each fracture is:

$$\begin{aligned}h_1(x, y) &= |x|(1+x)(1-x)y(1+y)(1-y), \\ h_2(y, z) &= y(1+y)(1-y)|z|(1+z)(1-z), \\ h_3(z, x) &= z(1+z)(1-z)x(1+x)(1-x),\end{aligned}$$

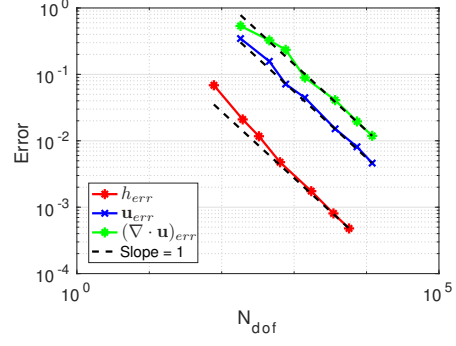
which is shown in Figure 3 interpolated on an example mesh, with coloring proportional to the hydraulic head values. The problem is solved on meshes with a number of degrees of freedom ranging from about  $1 \times 10^2$  to about  $6 \times 10^4$  and the following error norms are considered to evaluate accuracy of the numerical solution:

$$\begin{aligned}h_{\text{err}} &= \left( \sum_{E \in \mathcal{T}_\delta} \|h - h_\delta\|_E^2 \right)^{\frac{1}{2}}, \\ \mathbf{u}_{\text{err}} &= \left( \sum_{E \in \mathcal{T}_\delta} \|\mathbf{u} - \Pi_k^0 \mathbf{u}_\delta\|_E^2 \right)^{\frac{1}{2}}, \\ (\nabla \cdot \mathbf{u})_{\text{err}} &= \left( \sum_{E \in \mathcal{T}_\delta} \|\nabla \cdot (\mathbf{u} - \mathbf{u}_\delta)\|_E^2 \right)^{\frac{1}{2}}.\end{aligned}$$

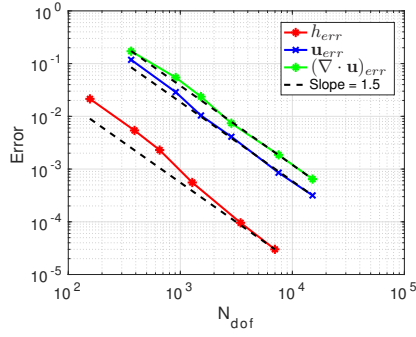
Figure 4 shows convergence rates of the above quantities against the number of DOFs when the computational mesh is refined. The various considered polynomial accuracy orders, ranging from  $k = 0$  to  $k = 5$  are reported in different plots. In all cases the expected convergence rates are obtained, with some instances of superconvergence for the pressure variable for coarser meshes. Figure 5 displays, instead, convergence curves of  $h_{\text{err}}$ ,  $\mathbf{u}_{\text{err}}$  and  $(\nabla \cdot \mathbf{u})_{\text{err}}$  against DOF numbers when the polynomial accuracy order is increased from  $k = 0$  to  $k = 5$ . Four different fixed geometrical meshes are reported. In all cases it is possible to observe exponential convergence rates.



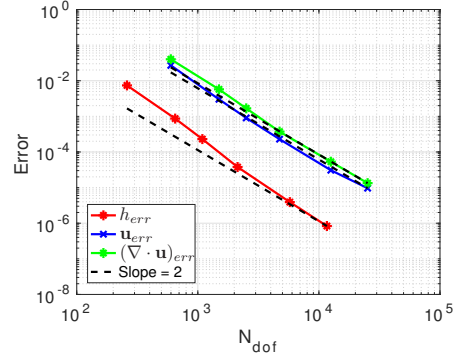
(a)  $k = 0$



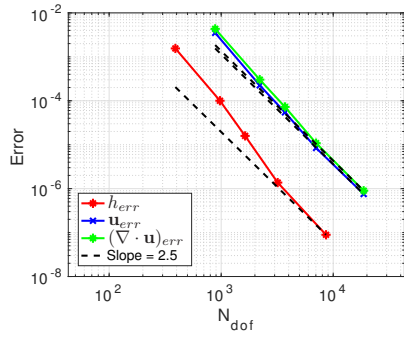
(b)  $k = 1$



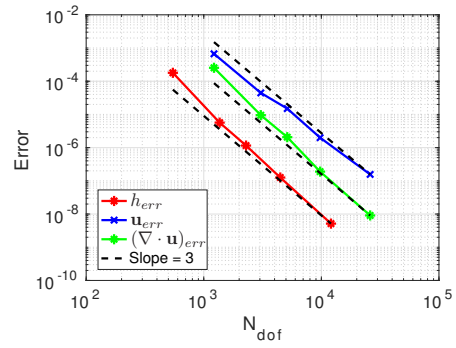
(c)  $k = 2$



(d)  $k = 3$

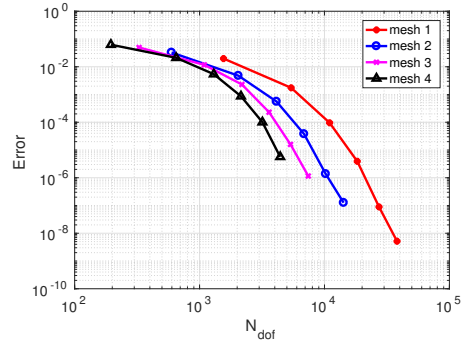


(e)  $k = 4$

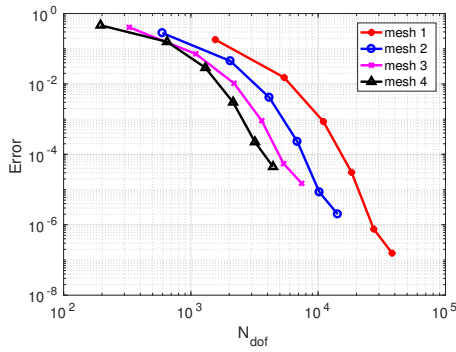


(f)  $k = 5$

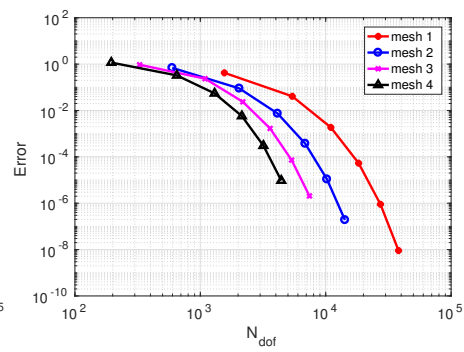
Figure 4: BP1: convergence curves of the computed error norms against the number of DOFs at refining the mesh



(a)  $h_{\text{err}}$



(b)  $\mathbf{u}_{\text{err}}$



(c)  $(\nabla \cdot \mathbf{u})_{\text{err}}$

Figure 5: BP1: convergence curves of the computed error norms against the number of DOFs at increasing values of the polynomial accuracy on four meshes

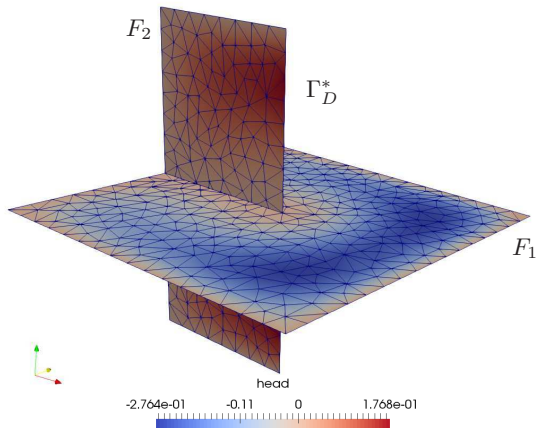


Figure 6: BP2: spatial distribution of fractures

Table 1: BP2: experimental convergence rates against DOF number

Order	$h_{\text{err}}$	$\mathbf{u}_{\text{err}}$	$(\nabla \cdot \mathbf{u})_{\text{err}}$
0	0.5210	0.5144	0.5015
1	1.0376	0.9989	0.6021
2	1.5171	1.1006	0.5135
3	1.5540	1.1569	0.5550
4	1.4054	1.1267	0.5329
5	1.4794	1.1592	0.4907

The second benchmark problem, labeled BP2, revisits a 2 fracture DFN, also studied in [35]. This problem is interesting as it allows us to test the method when applied to geometrical configurations with the presence of a trace ending in the interior of some fracture, which causes a loss of regularity in the solution, also away from the traces. The computational domain  $\Omega = F_1 \cup F_2$  is displayed in Figure 6, and the two fractures,  $F_1$  and  $F_2$  are defined as:

$$F_1 = \{(x, y, z) \in \mathbb{R}^3 : -1 \leq x \leq 1, -1 \leq y \leq 1, z = 0\},$$

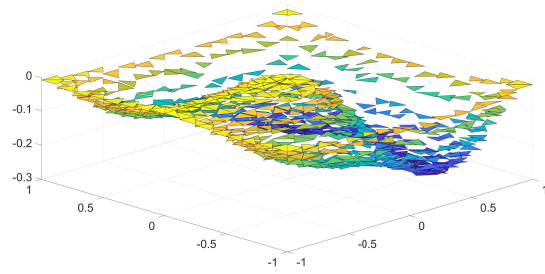
$$F_2 = \{(x, y, z) \in \mathbb{R}^3 : -1 \leq x \leq 0, -1 \leq z \leq 1, y = 0\},$$

intersecting in the trace  $\Gamma = \{(x, y, z) \in \mathbb{R}^3 : y = 0, z = 0 \text{ and } -1 \leq x \leq 0\}$ , whose tip is in the interior of  $F_1$ .

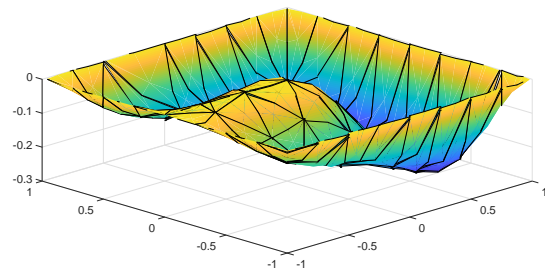
The problem is as follows:

$$\begin{cases} -\Delta h = f_1 & \text{in } F_1 \setminus \Gamma, \\ -\Delta h = f_2 & \text{in } F_2 \setminus \Gamma, \\ h = 0 & \text{on } \partial\Omega \setminus \Gamma_D^*, \\ h = (z^2 - z^4) \cos(\pi/4) & \text{on } \Gamma_D^*, \end{cases}$$

being  $\Gamma_D^*$  the portion of the boundary of  $\Omega$  with a non-null Dirichlet boundary

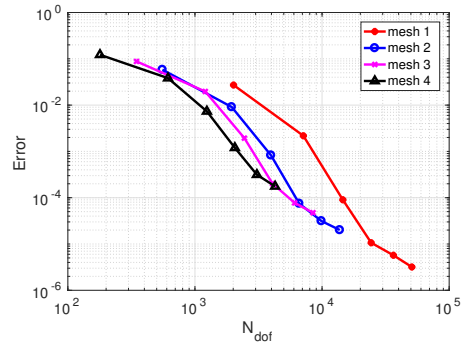


(a)  $k = 0$

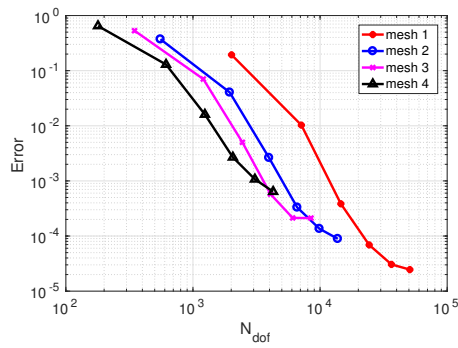


(b)  $k = 2$

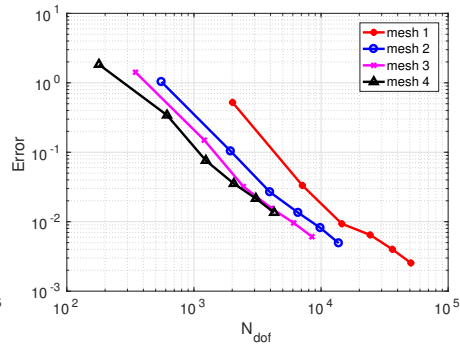
Figure 7: BP2: discrete solutions for fracture 1 and orders  $k = 0$  and  $k = 2$  on two different geometrical meshes with approximately the same number of DOFs



(a)  $h_{\text{err}}$



(b)  $\mathbf{u}_{\text{err}}$



(c)  $(\nabla \cdot \mathbf{u})_{\text{err}}$

Figure 8: BP2: convergence curves of the computed error norms against the number of DOFs at increasing values of the polynomial accuracy on four meshes

condition, as shown in Figure 6. The forcing terms  $f_i$ ,  $i = 1, 2$  are such that the exact solution for this problem is:

$$\begin{aligned} h_1(x, y, z) &= -\cos\left(\frac{1}{2}\arctan2(x, y)\right)(x^2 - 1)(y^2 - 1)(x^2 + y^2), \\ h_2(x, y, z) &= \cos\left(\frac{1}{2}\arctan2(z, x)\right)(z^2 - 1)(x^2 - 1)(z^2 + x^2), \end{aligned}$$

where  $\arctan2(x, y)$  is the inverse tangent function with 2 arguments, that returns the appropriate quadrant of the angle whose tangent is  $y/x$ . We remark that the regularity of the solution is such that  $h_1 \in H^2(F_1 \setminus \Gamma)$  but  $h_1 \notin H^3(F_1 \setminus \Gamma)$ . The exact solution is reported in Figure 6, where the coloring is proportional to the hydraulic head, and an example mesh is also shown. The computed numerical solution  $h_\delta$  is reported in Figure 7(a)-(b) for orders of polynomial accuracy  $k = 0$  and  $k = 2$ , respectively, on fracture  $F_1$ . A finer mesh is used for the lower order approximation in order to have a similar number of DOFs for the two reported solutions (3109 and 3071 DOFs respectively).

Convergence rates against DOFs for approximation spaces of polynomial accuracy of orders from 0 to 5 are shown in Table 1, whereas convergence curves for the error norms  $h_{\text{err}}$ ,  $\mathbf{u}_{\text{err}}$  and  $(\nabla \cdot \mathbf{u})_{\text{err}}$  against DOF numbers for increasing polynomial order  $k = 0, \dots, 5$  are displayed in Figure 8, for four different fixed polygonal meshes. As expected, given the low regularity of the solution, convergence rates for the error in the pressure variable does not improve increasing the accuracy of the approximation space from order 2 onward, whereas the convergence rates concerning the velocity field and its divergence do not improve from approximation orders  $k = 1$  and  $k = 0$  onward, respectively. It can be noticed that in practical DFN configurations, a high regularity of the solution should not be expected due to the presence of traces ending in the interior of the fractures.

## 5.2. DFN simulations

This subsection is devoted to the description of results obtained applying the method to the study of bigger networks. Throughout this subsection a maximum level of polynomial accuracy  $k = 2$  is used. At first, a small-sized 6 fracture and 6 trace DFN is considered, labeled DFN6. Despite its simplicity, it serves to give a clear description on how the method performs, and the insight it provides can be easily carried over to more complex problems. The geometrical domain  $\Omega$  for this experiment is shown in Figure 9, along with an example computed solution in which the grading colors are proportional to hydraulic head values. The Laplace problem in mixed form is solved in  $\Omega$  and, with reference to Figure 9, a unitary Neumann boundary condition is imposed on the edge marked with  $\Gamma_{N1}$  on  $F_1$  (also called source fracture), whereas a zero Dirichlet boundary condition is imposed on the edge marked by  $\Gamma_{D5}$  on  $F_5$  (the sink fracture). All other fracture edges are insulated. Three different meshes are considered, as detailed in Table 2, in which the mesh parameter  $\delta$  corresponds to the maximum element diameter of the original triangular mesh.

The results reported in Figures 10-12 are obtained on the finest mesh considered. Figure 10 displays the velocity field on fracture  $F_2$  for a solution with a polynomial accuracy  $k = 1$ ; the velocity fields obtained for values of  $k = 0$  and  $k = 2$  are practically indistinguishable. Fracture  $F_2$  has four traces  $\Gamma_\ell$ ,  $\ell = 1, \dots, 4$  given by the intersections of  $F_2$  with fractures  $F_1$ ,  $F_3$ ,  $F_4$  and  $F_6$ , respectively. As expected, flow enters  $F_2$  through trace  $\Gamma_1$  that is shared by  $F_2$  and the source fracture, and leaves the fracture through traces  $\Gamma_2$  and  $\Gamma_4$ . Trace  $\Gamma_3$  connects  $F_2$  to  $F_4$ , which is an insulated fracture: indeed it can be seen that there is both flux entering trace  $\Gamma_4$  and leaving it, with a net balance of zero flux (see the following Table 4). Figures 11(a)-(c) show the solution  $h_\delta$  on fracture  $F_2$  for the three different polynomial accuracy values. It can be noticed that pressure head values are quite similar among the three pictures and, even though there is no condition for pressure head continuity imposed by problem formulation (the functions in the pressure space are discontinuous across mesh elements), the solutions for orders 1 and 2 nevertheless exhibit a practically continuous behavior.

We introduce a quantity  $\Delta_h$  defined as:

$$\Delta_h^2 = \frac{1}{L^2} \sum_{\ell \in \mathcal{L}} \|h_{\text{av},i,\ell} - h_{\text{av},j,\ell}\|_0^2$$

with  $i, j \in \mathcal{I}_\ell$ , and  $L$  is the number of traces in the network. The quantity  $h_{\text{av},i,\ell}$  is the average of the discrete pressure on the two sides of trace  $\Gamma_\ell$  on fracture  $F_i$ , for  $i \in \mathcal{I}$ ,  $\ell \in \mathcal{L}_i$ , and thus  $\Delta_h$  is a measure of how discontinuous the pressure head is in the obtained solution across the traces, averaged by the number of traces. Since the discrete solution is discontinuous across each mesh edge, it is not possible to attain a zero value for  $\Delta_h$ , but it is expected that its value decreases when the approximation space is increased. The obtained values of  $\Delta_h$  are shown in Table 3 for the various meshes and the different orders of accuracy  $k$ . It can be seen that, as expected, the quantity  $\Delta_h$  decreases both when the mesh is refined and when the order of polynomial accuracy is increased, showing a desired converging behavior towards a solution that is continuous at the traces.

In Figure 12 the flux along trace  $\Gamma_4 = F_2 \cap F_6$  is presented for the finest mesh ( $\delta = 0.26$ ), with a comparison among the different accuracy orders. In all cases the approximations are similar and, as expected, second order elements are able to better reproduce details in the flux function, such as larger variations over small distances, while lower order elements are capable at capturing the mean value of the flux exchange. Practically the same net flux is obtained when integrating the complete profile in all cases.

Table 4 details the flux exchange in fractures and traces for the finest considered mesh and a polynomial accuracy  $k = 0$ : rows correspond to traces, whereas columns to fractures. The last row contains the sum of all the incoming and outgoing flux for each fracture, while the last column shows, for each trace, the balance in flux exchange between the two fractures that share the trace. An almost exact flux balance is obtained, both within fractures and in trace

Table 2: DFN6: number of DOFs for the considered mesh parameters  $\delta$  and polynomial accuracy levels  $k$

Order \ Mesh	$\delta = 1.09$	$\delta = 0.70$	$\delta = 0.26$
0	975	2039	8458
1	3190	6798	29304
2	6335	13597	59441

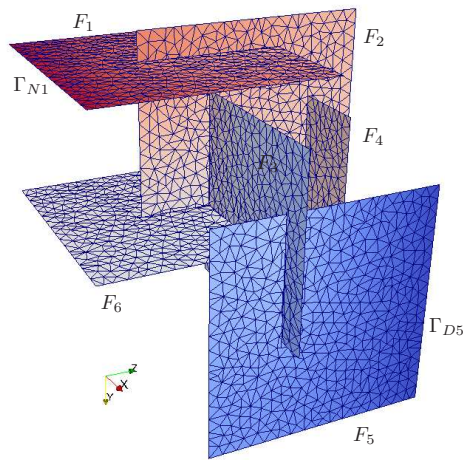


Figure 9: DFN6: spatial geometry with source( $F_1$ ) and sink ( $F_5$ ) fracture

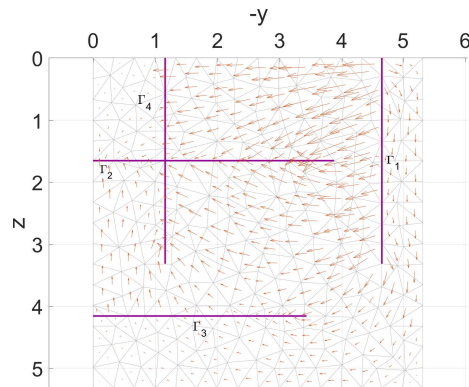


Figure 10: DFN6: velocity field on fracture  $F_2$ ,  $\delta = 0.26$ ,  $k = 1$

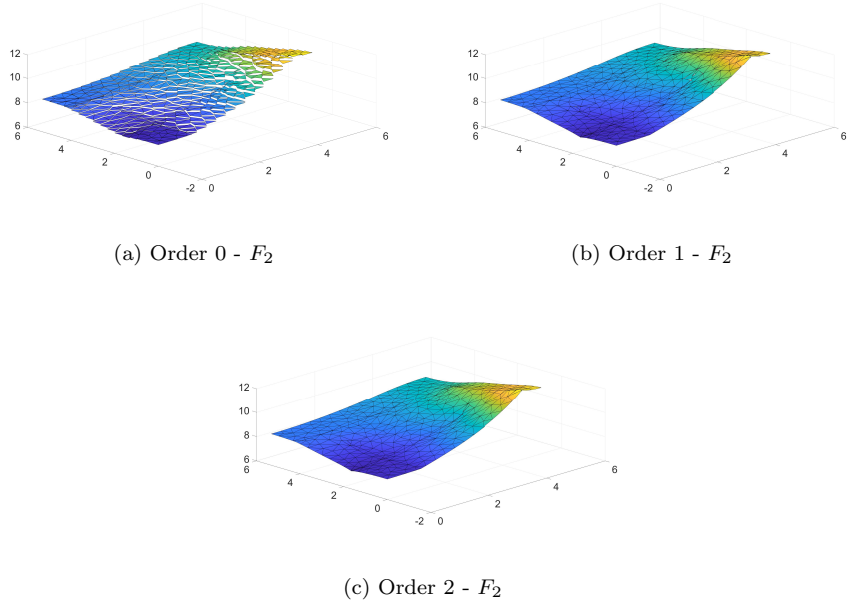


Figure 11: DFN6: solution  $h_\delta$  on fracture  $F_2$  for orders 0 to 2,  $\delta = 0.26$

Table 3: DFN6: values of  $\Delta_h$  for the various considered meshes and polynomial accuracy orders

Order	Mesh	$\delta = 1.09$	$\delta = 0.70$	$\delta = 0.26$
		0	$5.2 \times 10^{-1}$	$2.9 \times 10^{-1}$
1		$1.4 \times 10^{-1}$	$9.4 \times 10^{-2}$	$2.5 \times 10^{-2}$
2		$9.5 \times 10^{-2}$	$5.5 \times 10^{-2}$	$1.4 \times 10^{-2}$

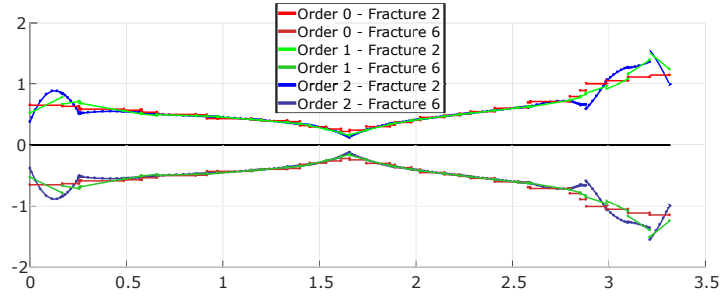


Figure 12: DFN6: normal velocity on the trace  $\Gamma_4$  for orders 0 to 2,  $\delta = 0.26$

Table 4: DFN6: flux data and flux mismatches across traces (last column) and flux balance on fractures (last row) for order  $k = 0$ ,  $\delta = 0.26$

	$F_1$	$F_2$	$F_3$	$F_4$	$F_5$	$F_6$	balance
$\Gamma_1$	-5.3174	5.3174	0	0	0	0	0
$\Gamma_2$	0	-3.4141	3.4141	0	0	0	0
$\Gamma_3$	0	6.5e-12	0	-6.5e-12	0	0	-6.9e-17
$\Gamma_4$	0	-1.9032	0	0	0	1.9032	-4.4e-16
$\Gamma_5$	0	0	-5.3174	0	5.3174	0	8.9e-16
$\Gamma_6$	0	0	1.9032	0	0	-1.9032	0
balance	-5.3174	4.7e-10	4.4e-16	-6.5e-12	5.3174	1.7e-11	4.4e-16

exchanges, for all orders of accuracy. The incoming flux from fracture  $F_1$  is perfectly balanced with the outgoing flux in  $F_5$ : fracture  $F_1$  acts as a source that provides a total flux of 5.3174 to the system from the Neumann edge (negative values represent flux leaving the fracture through the traces), which leaves the system at fracture  $F_5$  with a practical 0 unbalance reported in the bottom-right cell of the table. All other fractures show an almost null net flow, which agrees with the homogeneous Neumann boundary condition. As already mentioned, the flux exchange in trace  $T_3$  is almost zero since fracture  $F_4$  is neither a source nor a sink fracture and since it has only one intersection with another fracture ( $F_2$ ).

According to the previous analysis it can be concluded that the mixed formulation shows a very high accuracy when computing flux exchange between fractures. The flux balance is almost exact for any order of accuracy of the method. Low order elements also show a very good performance with much less computational demand.

The last proposed example considers a larger network consisting of 134 fractures and 604 traces that more closely resembles a real network. This problem is labeled DFN134 and its geometry is shown in Figure 13. In this network there is a wider range of fracture sizes and it has fractures intersecting at varying angles that originate several geometrical difficulties such as almost parallel and very close traces, small angles, small edges and badly shaped elements. Boundary conditions are imposed, with a homogeneous Dirichlet boundary condition on one edge of a sink fracture (labeled  $Si$  in the picture) and three prescribed incoming flux values of 100, 200 and 200 on one edge of three different source fractures ( $So$  in the picture). The Laplace problem is solved in mixed form with approximation spaces of orders 0, 1 and 2 on four different meshes, as detailed in Table 5.

An example solution on the whole network is shown in Figure 13, in which the coloring is proportional to  $h_\delta$  values. In Figure 14 results for  $h_\delta$  and orders  $k = 0, 1, 2$  for a selected fracture are presented on the mesh with  $\delta = 3.68$ , whereas the velocity field is shown in Figure 15 on the same mesh and a order  $k = 2$ . There are 18 traces present in the fracture generating a complex geometry that nevertheless is handled smoothly by the Virtual Element discretization. In Table 6 the quantity  $\Delta_h$  is reported for the various meshes and orders of

Table 5: DFN134: Number of DOFs for the considered meshes and polynomial accuracy values

Order	Mesh	$\delta = 11.34$	$\delta = 8.09$	$\delta = 5.95$	$\delta = 3.68$
	0		43630	53080	65149
1		125020	153764	190742	295594
2		234730	290151	361668	567001

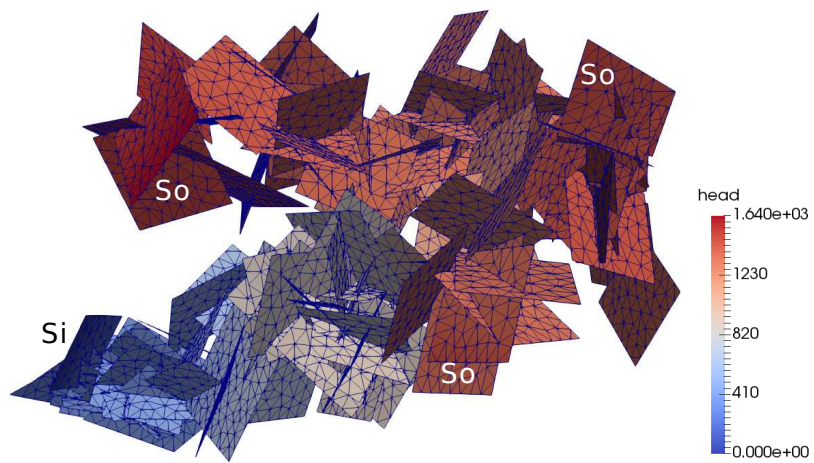
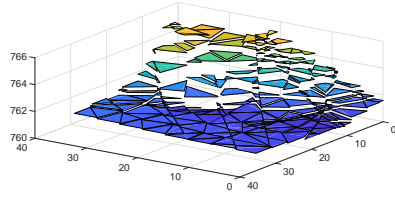
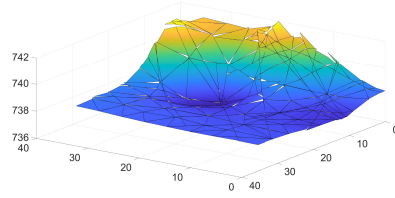


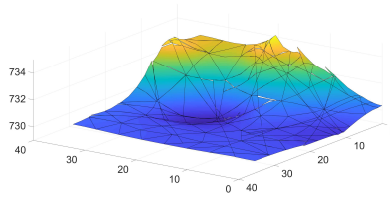
Figure 13: DFN134: spatial geometry



(a) Order 0 - Pressure head



(b) Order 1 - Pressure head



(c) Order 2 - Pressure head

Figure 14: DFN134: solution  $h_\delta$  for a selected fracture for different various of polynomial accuracy and  $\delta = 3.68$

Table 6: DFN134: values of  $\Delta_h$  for the various considered meshes and polynomial accuracy orders

Order	Mesh	$\delta = 11.34$	$\delta = 8.09$	$\delta = 5.95$	$\delta = 3.68$
		0	1.81	1.44	1.15
1		1.01	0.70	0.54	0.32
2		0.67	0.45	0.42	0.22

accuracy considered, showing again a clear convergent trend towards a solution continuous at the traces, when either the mesh is refined or the polynomial accuracy is increased. In Table 7 results for flux balance are presented, on the mesh with  $\delta = 3.68$  and for various polynomial accuracy levels. It can be seen that the right order of magnitude of the flux at the traces is seized already with the lowest order of accuracy. The flux balance results for all orders show again a perfect match between the total incoming flow through the source fractures (equal to a total flux of 400, given by the boundary conditions) and the flux entering the sink fracture through its traces, with a mismatch near to the machine-error order of magnitude.

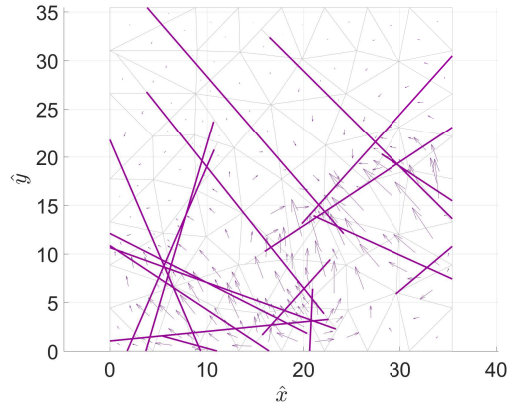


Figure 15: DFN134: velocity field for the 134 fracture network on a selected fracture with order 2,  $\delta = 3.68$

Table 7: DFN134: flux entering the sink fracture through its traces for various polynomial accuracy levels,  $\delta = 3.68$

Local trace	Order 0	Order 1	Order 2
$\Gamma_1$	7.7695	8.7141	7.9296
$\Gamma_2$	22.6830	22.7356	22.6577
$\Gamma_3$	44.1668	45.9451	41.8777
$\Gamma_4$	20.9764	25.6750	31.8235
$\Gamma_5$	4.2227	3.3396	3.0034
$\Gamma_6$	1.2128	1.1298	1.5661
$\Gamma_7$	141.0733	142.6016	142.8069
$\Gamma_8$	97.4382	90.4480	89.2446
$\Gamma_9$	43.0454	45.2085	44.6129
$\Gamma_{10}$	17.4119	14.2027	14.4776
Balance	400.0000	400.0000	400.0000

## 6. Conclusions

In the present work a numerical method for the resolution of the advection-diffusion-reaction problem in mixed form in network of fractures is proposed. The method exploits the flexibility of virtual elements in handling almost general polygonal meshes to easily generate a conforming mesh of the network. Suitable approximation spaces are introduced, and the well posedness of the resulting numerical scheme is shown.

Benchmark problems are proposed on simple networks with known analytic solutions, showing optimal convergence properties for the method for discrete spaces of increasing polynomial accuracy ranging from 0 to 5. Qualitative results on more complex networks show good approximation performances already for the lower polynomial accuracy levels. The direct computation of the velocity variable allows for a very accurate description of the flux, which is of paramount importance in view of the use of such velocity field as an input for simulations of dispersion phenomena.

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