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*Original*

Nonlocal diffusion in porous media: A spatial fractional approach / Sapora, ALBERTO GIUSEPPE; Cornetti, Pietro; Chiaia, Bernardino; Lenzi, E. K.; Evangelista, L. R.. - In: JOURNAL OF ENGINEERING MECHANICS. - ISSN 0733-9399. - 143, D4016007:5(2017), pp. 1-7. [10.1061/(ASCE)EM.1943-7889.0001105]

*Availability:*

This version is available at: 11583/2669737 since: 2020-04-26T11:57:38Z

*Publisher:*

American Society of Civil Engineers (ASCE)

*Published*

DOI:10.1061/(ASCE)EM.1943-7889.0001105

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# 1 Nonlocal diffusion in porous media: a spatial fractional 2 approach

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## 4 Abstract

One dimensional diffusion problems in bounded porous media characterized by the presence of nonlocal interactions are investigated by a fractional calculus approach. Darcy's constitutive equation is assumed of convolution integral type and a power law attenuation function is implemented. Analogies and differences of the flow rate-pressure law with respect to other nonlocal and fractal models are outlined. By means of the continuity relationship, the fractional diffusion equation is then derived. It involves spatial Riemann-Liouville derivatives with non-integer order comprised between 1 and 2. The solution is obtained numerically via fractional finite differences and results are presented both in the transient and in the steady-state regimes. Eventually, the physical meaning of fractional operators is discussed and potential applications of the analysis are suggested.

5 *Keywords:* Nonlocal Darcy's law, long-range interactions, fractional diffusion  
6 equation

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## 7 Introduction

8 Understanding transport problems in porous media emerges nowadays as a  
9 primary concern, since it can have a fundamental impact on many different re-  
10 search fields, starting from the optimization of oil extraction to modeling scaffold

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11 geometry in tissue engineering, up to pharmaceutical and food industries.

12       The classical diffusion equation is obtained by combining Darcy's constitutive  
13 law, stating the proportionality of the flux to the pressure gradient, with the equa-  
14 tion of continuity, describing the conservation of the mass. On the other hand,  
15 several discrepancies have been found between the related solution, which is gen-  
16 erally described by exponential type functions, and experimental data (Nielsen  
17 et al., 1962; Bell and Nur, 1978; Roeloffs, 1988). There are several reasons re-  
18 sponsible of these deviations: as a matter of fact, fluids can react chemically with  
19 the medium increasing or diminishing the pore size, or they can interact with the  
20 solid part by carrying some particles which may obstruct some channels (Caputo,  
21 2000). Eventually, the network of channels can result in a complicated intercon-  
22 nected hierarchical geometry.

23       In order to overcome these drawbacks, different approaches have been pro-  
24 posed since the middle of the last century, mainly based on a modification of  
25 constitutive Darcy's relationship. In the framework of soil-water flow theory, for  
26 instance, a diffusion coefficient dependent on the water content was firstly con-  
27 sidered, leading to a nonlinear partial differential equation known as Richards'  
28 equation (Van Dam and Feddes, 2000; Lewandowska and Auriault, 2002; Jacques  
29 et al., 2002). Boltzmann's transformation reduces the expression into an ordinary  
30 differential equation, allowing the possibility of getting analytical solutions. Nev-  
31 ertheless, in many cases significant deviations from the real behavior were still ob-  
32 served (Taylor et al., 1999; Kunt and Lavallee, 2001). Among different attempts  
33 to generalize Richard's equation, let us cite those based on adding the dependence  
34 of the diffusion coefficient on time (Guerrini and Swartzendruber, 1992) or space  
35 (Pachepsky and Timlin, 1998).

36 More recent models involve the replacement of the first order time derivative  
37 with a fractional one in the final differential equation, to take memory effects into  
38 account (Pachepsky et al., 2003; Logvinova and Nel, 2004). A slightly differ-  
39 ent approach was followed by Gerolymatou et al. (2006), reformulating Richards'  
40 equation as a time integral relationship. On the other hand, (Caputo, 2000) applied  
41 fractional derivatives (with two different orders, each comprised between 0 and 1)  
42 to both members of Darcy's law to consider the temporal variation of the perme-  
43 ability during the process (see also (Caputo and Plastino, 2004)). In the spirit  
44 of this approach, it was recently proved that assuming the physical properties of  
45 a porous solid varying with a power law is equivalent to consider a dependence  
46 of the flux on the time fractional derivative of the pressure with order comprised  
47 between 0 and 1 (Deseri and Zingales, 2015; Alaimo and Zingales, 2015).

48 Regarding engineering applications, an important work to be mentioned is that  
49 of (Monteiro et al., 2012) where a mathematical model of the flow in nanoporous  
50 rocks was proposed. It is based on the hypothesis that the permeability of the  
51 inclusions depends substantially on the pressure gradient. The model, applied to  
52 shale oil extraction, showed that the production rate of the oil deposits decays  
53 with time following a power law whose exponent lies between  $-1/2$  and  $-1$ , in  
54 agreement with experimental data.

55 On the contrary, the approach performed by Sen and Ramos (2012) is com-  
56 pletely different, since it assumes the flux to be proportional to the pore pressure  
57 by means of a spatial convolution integral. By considering the attenuation function  
58 of power-law type, nonlocal Darcy's law was rewritten by means of spatial frac-  
59 tional operators. The problem was limited to infinite domains and an interesting  
60 interpretation of the entire porous medium as a network of channels with short-,

61 medium-, and long-distance connections was furnished. However, the resulting  
62 fractional diffusion equation was not given.

63 In the present work, the model proposed in (Sen and Ramos, 2012) is revisited  
64 and generalized to investigate the diffusion process on finite porous domains. The  
65 approach represents somehow an extension of the well-established method pro-  
66 posed in the framework of nonlocal elasticity (Carpinteri et al., 2011, 2014) (see  
67 also (Di Paola and Zingales, 2008; Drapaca and Sivaloganathan, 2012)) and later  
68 implemented also to study wave propagation in nonlocal media (Atanackovic and  
69 Stankovic, 2009; Sapora et al., 2013; Challamel et al., 2013; Aksoy , 2016).

70 Before proceeding, it is worth observing that: i) the present analysis does not  
71 involve an explicit connection between the order of the fractional derivative and  
72 the fractal geometry (if any) of the medium where diffusion takes place; indeed,  
73 some potential connections are suggested on the basis of some recent advances  
74 (Carpinteri and Sapora, 2010; Balankin an Elizarraraz , 2012; Zingales, 2014); ii)  
75 fractional diffusion equations are strictly related to continuous random walk ap-  
76 proaches. They generalize the standard Brownian motion by taking waiting times  
77 (which accounts for non-Markovian effects) and anomalous long particle displace-  
78 ments (known as Levy flights, which consider non-Gaussian displacements) into  
79 account (Gorenflo et al., 2002; Zoia et al., 2007; Berkowitz et al., 2006); iii) non-  
80 local diffusion constitutes a broad class of problems of interest in mathematics  
81 suited to wide variety of applications, including biological contexts, image pro-  
82 cessing, particle systems, coagulation models, nonlocal anisotropic approaches  
83 for phase transition and mathematical finance using optimal control theory, among  
84 others (Andreu-Vaillo et al., 2010). In this context, fractional calculus has proved  
85 to be a synthetic and efficient tool to model both memory effects and nonlocal

86 interactions (Scalas et al., 2000; Metzler and Nonnenmacher, 2002; del-Castillo-  
 87 Negrete, 2006; Magin et al., 2008; Lenzi et al., 2008; Gorenflo and Mainardi,  
 88 2009; Evangelista et al., 2011; Tarasov and Trujillo, 2013; Atanackovic et al.,  
 89 2014; Zingales, 2014; Sobolev, 2014).

## 90 **Nonlocal Darcy's equation**

91 Let us consider a diffusion process in a one-dimensional porous bar of length  
 92  $l$ . Assume that the volumetric flow rate per unit area  $q$  [ $m/s$ ] in one point depends  
 93 on the gradient of the pore pressure  $p$  [ $N/m^2$ ] all over the domain by means of a  
 94 convolution integral:

$$q(x) = -\frac{k}{\mu} \int_0^l g(y-x) \nabla p(y) dy, \quad (1)$$

95 being  $k$  the permeability [ $m^2$ ],  $\mu$  the fluid viscosity [ $Ns/m^2$ ], and  $g$  an attenuation  
 96 function. It describes the relationship between non-adjacent points of the medium  
 97 and it must be a decaying function in space. Equation (1) represents the spatial  
 98 nonlocal form of Darcy's constitutive equation, and it was firstly proposed in (Sen  
 99 and Ramos, 2012). Indeed, similar expressions had been put forward even before  
 100 to investigate Eringen's nonlocal elasticity and nondiffusive transport in magnetically  
 101 confined plasma (see (Lazar et al., 2006; del-Castillo-Negrete, 2006) and  
 102 related references).

103 Different attenuation functions  $g$  can be inserted into Eq. (1), leading to dif-  
 104 ferent nonlocal models. The attention is focused here on: i) a cone function, as  
 105 an example of standard nonlocal models (of course other choices, such as bell-  
 106 shaped or Gaussian functions are possible); ii) a power law expression, leading to  
 107 a fractional approach.

108 Let us start by considering the following cone function  $g$ :

$$g(\xi) = \begin{cases} \frac{1}{l_{ch}} \left(1 - \frac{|\xi|}{l_{ch}}\right) & \text{for } |\xi| < l_{ch} \\ 0 & \text{for } |\xi| > l_{ch} \end{cases} \quad (2)$$

109 where  $l_{ch}$  is a parameter characteristic of the material and  $\xi = y - x$ . Of course, if  
 110  $l_{ch}$  tends to zero the attenuation function (2) tends to Dirac function  $\delta(x)$ : in this  
 111 case, the nonlocal constitutive law (1) tends to the local one,  $q = -k/\mu \nabla p(x)$ .  
 112 Furthermore, the particular form of (2) has been chosen according to the fact that,  
 113 if the gradient of the pressure is constant, no differences should be observed from  
 114 the local model. In other words, the following relationship has to be satisfied:

$$\int_{-\infty}^{+\infty} g(\xi) d\xi = 1. \quad (3)$$

115 The computation of the flow rate generating a constant pressure gradient  $\nabla \bar{p}$  is  
 116 just a matter of integration. By inserting Eq. (2) into Eq. (1), simple analytical  
 117 manipulations yield:

$$q(x) = \begin{cases} -\frac{k\nabla \bar{p}}{2\mu} \left[1 + 2\left(\frac{x}{l_{ch}}\right) - \left(\frac{x}{l_{ch}}\right)^2\right] & \text{for } 0 < x < l_{ch} \\ -\frac{k\nabla \bar{p}}{\mu} & \text{for } l_{ch} < x < l - l_{ch} \\ -\frac{k\nabla \bar{p}}{2\mu} \left[1 + 2\left(\frac{l-x}{l_{ch}}\right) - \left(\frac{l-x}{l_{ch}}\right)^2\right] & \text{for } l - l_{ch} < x < l \end{cases} \quad (4)$$

118 The dimensionless flow  $q^* = q\mu/(k\nabla \bar{p})$  versus the dimensionless space  $x^* = x/l$   
 119 is plotted in Fig.1 for different  $l_{ch}^* = l_{ch}/l$  values: the flow decreases (in modulus)  
 120 at the edges, whereas it matches the local solution ( $q^* = -1$ ) on the the central  
 121 core of the bar. The size of the domain affected by nonlocality depends clearly on  
 122 the value of  $l_{ch}^*$ .

123 Let us assume now the following power-law expression for function  $g$  (Tarasov  
124 and Zaslavky, 2006; Atanackovic and Stankovic, 2009; Carpinteri et al., 2011):

$$g(\xi) = \frac{1}{2\Gamma(2-\alpha)|\xi|^{\alpha-1}}, \quad (5)$$

125 with  $1 < \alpha < 2$ . In this case  $\alpha$  is the material parameter governing the transition  
126 from a smooth behavior (lower  $\alpha$ ) to a sharp one (higher  $\alpha$ ), and it accounts for  
127 non-Gaussian displacements of the particles inside the media. Relation (1) takes  
128 thus the following form:

$$q(x) = -\frac{k_\alpha}{\mu} I_{0,l}^{2-\alpha}(\nabla p). \quad (6)$$

129 The operator  $I_{0,l}^\beta$  represents the fractional Riesz integral ( $\beta > 0$ , Samko et al.  
130 (1993))

$$I_{0,l}^\beta f(x) = \frac{1}{2} \left[ I_{0+}^\beta f(x) + I_{l-}^\beta f(x) \right] = \frac{1}{2\Gamma(\beta)} \int_0^l \frac{f(y)}{|x-y|^{1-\beta}} dy, \quad (7)$$

131 where  $I_{0+}^\beta$  and  $I_{l-}^\beta$  are the left and right Riemann-Liouville fractional integrals,  
132 respectively:

$$I_{0+}^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^x \frac{f(y)}{(x-y)^{1-\beta}} dy, \quad (8)$$

$$I_{l-}^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_x^l \frac{f(y)}{(y-x)^{1-\beta}} dy. \quad (9)$$

134 According to choice (5), the fractional permeability  $k_\alpha$  possesses anomalous  
135 physical dimensions  $[m]^\alpha$ . An intriguing possibility, as will be outlined later,  
136 would be that of linking them with the fractal features of the medium where dif-  
137 fusion takes place (Chang and Yortos, 1990; Carpinteri and Mainardi, 1997; Yao  
138 et al., 2012; Carpinteri et al., 2009). Hereinafter, the following condition will

139 be supposed to hold, for the sake of completeness:  $k_\alpha = k$  for  $\alpha = 2$ . Thus, in  
140 this case, Eq. (6) reverts consistently to local Darcy's relationship  $q = -k/\mu \nabla p$ .  
141 Eventually, observe that also in the present fractional approach we can normalize  
142 the coordinate  $r$  with respect to an intrinsic length  $l_{ch}$ , as done in Eq. (2). On the  
143 other hand, handling two parameters ( $\alpha$  and  $l_{ch}$ ) governing the transition from a  
144 nonlocal behavior to a local behavior would represent a not trivial task, at least at  
145 this preliminary stage. For a first attempt in this framework, the reader can refer  
146 to (Sumelka and Blaszczyk, 2014).

147 As done before, let us now consider the flow  $q$  associated to a constant pressure  
148 gradient  $\nabla \bar{p}$ :

$$q(x) = -\frac{k_\alpha l^{2-\alpha}}{2\mu\Gamma(3-\alpha)} \nabla \bar{p} \left[ \left(\frac{x}{l}\right)^{2-\alpha} - \left(1 - \frac{x}{l}\right)^{2-\alpha} \right] \quad (10)$$

149 By denoting  $q^* = q\mu/(k_\alpha l^{2-\alpha} \nabla \bar{p})$ , results are presented in Fig.2 for different  
150  $\alpha$ -values. Once again, observe that the flow decreases in correspondence to the  
151 bar extremes. Moreover, when  $\alpha \rightarrow 2$  (as when  $l_{ch} \rightarrow 0$ ) the classical local solution  
152 ( $q^* = -1$ ) is recovered. Nevertheless, there are some differences with respect to  
153 the previous case (Fig.1): whereas the nonlocal model based on a cone attenuation  
154 function always provides the local solution at a certain distance from the borders,  
155 according to the fractional approach all the structure is affected by non-locality.  
156 This is imputable to the long tails of the power law expression (5). Furthermore,  
157 it is evident from (10) that the flux increases along the bar length as  $l^{2-\alpha}$ .

158 By means of dimensional analysis, it is possible to prove that the fractional  
159 permeability  $k_\alpha$  decreases as  $l^{1-\alpha}$  ( $1 < \alpha < 2$ ) instead of as  $l^{-1}$ , the latter condi-  
160 tion holding both for local or other nonlocal models: this means that the pressure  
161 increases less than linearly with the bar length, as occurs in the classical case.  
162 The interested reader is referred to (Carpinteri and Sapora, 2010), where diffusion

163 problems in fractal media (and more specifically, in a Cantor bar) were investi-  
 164 gated, proving that the field variable scales as  $l^\beta$ ,  $\beta = \alpha - 1$  being the non-integer  
 165 dimension of the fractal set inside the bar where the gradient concentrates. The  
 166 fractional nonlocal model and fractal model, although different, are thus char-  
 167 acterized by the same scaling properties. Further studies are in progress. For  
 168 recent advances on the relations between fractal geometry and fractional calcu-  
 169 lus in transport problems, see also Balankin and Elizarraraz (2012); Alaimo and  
 170 Zingales (2015).

### 171 **Fractional diffusion equation**

172 Let us now introduce the time variable  $t$  and consider the continuity equation:

$$\frac{\partial p(x,t)}{\partial t} = -\frac{1}{c\phi} \frac{\partial q(x,t)}{\partial x}, \quad (11)$$

173 where  $c$  is the compressibility [ $m^2/N$ ] and  $\phi$  is the porosity.

174 By substituting Eq.(6) into (11), we get:

$$\frac{\partial p(x,t)}{\partial t} = \frac{d_\alpha}{2} \{D_{0+}^\alpha [p(x,t) - p(0,t)] + D_{l-}^\alpha [p(x,t) - p(l,t)]\}, \quad (12)$$

175 where  $d_\alpha = k_\alpha/\mu c\phi$  is the fractional diffusivity coefficient [ $m^\alpha/s$ ], and  $D_{a+}^\beta$  and  
 176  $D_{b-}^\beta$  are the left and right Riemann-Liouville fractional derivatives with respect to  
 177 the spatial variable  $x$ . They write:

$$D_{0+}^\beta f(x) = \sum_{k=0}^{n-1} \frac{f^k(0)}{\Gamma(1+k-\beta)} (x)^{k-\beta} + \frac{1}{\Gamma(n-\beta)} \int_0^x \frac{f^n(y)}{(x-y)^{\beta-n+1}} dy, \quad (13)$$

178

$$D_{l-}^\beta f(x) = \sum_{k=0}^{n-1} \frac{(-1)^k f^k(l)}{\Gamma(1+k-\beta)} (l-x)^{k-\beta} + \frac{(-1)^n}{\Gamma(n-\beta)} \int_x^l \frac{f^n(y)}{(y-x)^{\beta-n+1}} dy, \quad (14)$$

179  $n$  being the smallest integer larger than  $\beta$ , i.e.  $n = 2$  in the present case.

180 Equation (12) represents a fractional differential equation (Podlubny, 1999)  
 181 in space. Note that, whereas the left fractional derivative coincides always with  
 182 its integer order counterpart when the order of derivation is an integer number,  
 183 the right fractional derivative coincides with the corresponding integer derivative  
 184 only when the order of derivation is even; otherwise, it is equal to its opposite.  
 185 Thus, the term in the curly brackets (which coincides with the Riesz fractional  
 186 derivative up to a multiplicative factor, (Samko et al., 1993)) is equal to  $2\partial^2 p/\partial x^2$   
 187 when  $\alpha = 2$  ( Eq.(12) reverting to the classical diffusion equation), and vanishes  
 188 when  $\alpha = 1$  (thus leading to a trivial condition providing a constant pressure field  
 189 in time throughout the body).

190 Suitable initial and boundary conditions must be assigned to Eq. (12). By  
 191 analogy of what presented in Carpinteri et al. (2014), they write:

$$p(x, t = 0) = p_0(x), \quad (15)$$

$$192 \quad p(x = 0, t) = p_0(t), \quad \text{or} \quad q(x = 0, t) = \frac{k\alpha}{\mu} D_{l-}^{\alpha-1} [p(x) - p(l)]_{x=0} = q_0(t), \quad (16)$$

$$193 \quad p(x = l, t) = p_l(t), \quad \text{or} \quad q(x = l, t) = -\frac{k\alpha}{\mu} D_{0+}^{\alpha-1} [p(x) - p(0)]_{x=l} = -q_l(t), \quad (17)$$

194 In other words, the boundary conditions on the flow rate (16) and (17) are ex-  
 195 pressed by Caputo's right fractional derivative (with order  $\alpha - 1 \in (0, 1]$ ) eval-  
 196 uated in the left extreme and by Caputo's left fractional derivative (with order  
 197  $\alpha - 1 \in (0, 1]$ ) evaluated in the right extreme. Of course, they are integral-type  
 198 boundary conditions.

199 **Numerical solution and discussion of results**

200 If the diffusion problem described by (12) was set on an infinite medium,  
 201 analytical solutions could be achieved by Laplace-Fourier transforms (Gorenflo  
 202 and Mainardi, 2009; Atanackovic and Stankovic, 2009). On the other hand, if  
 203 the analysis refers to finite domains as in the present case, numerical schemes  
 204 have to be implemented. Different expressions were proposed to approximate  
 205 fractional operators with order comprised between 1 and 2 (Lynch et al., 2003;  
 206 Meerschaert et al., 2006; Ortigueira, 2008; Yang et al., 2010): the so-called L2  
 207 algorithm by Oldham and Spanier (1974) is here adopted. Let us introduce a  
 208 partition of the interval  $[0, l]$  on the  $x$  axis made of  $n$  ( $n \in \mathbb{N}$ ) intervals of length  
 209  $\Delta x = l/n$ . The generic point of the partition has the abscissa  $x_i$ , with  $i = 1, \dots, n+1$   
 210 and  $x_1 = 0$ ,  $x_{n+1} = l$ ; that is,  $x_i = (i-1)\Delta x$ . Hence, for the inner points of the  
 211 domain ( $i = 2, \dots, n$ ), the discrete form of Eq.(12) reads ( $1 \leq \alpha < 2$ ):

$$\begin{aligned} \frac{p_{i,j+1} - p_{i,j}}{\Delta t} \approx \frac{d_\alpha}{2} \frac{(\Delta x)^{-\alpha}}{\Gamma(3-\alpha)} \times \\ \left\{ \frac{2-\alpha}{(i-1)^{\alpha-1}} (p_{2,j} - p_{1,j}) + \sum_{k=0}^{i-2} (p_{i-k+1,j} - 2p_{i-k,j} + p_{i-k-1,j}) [(k+1)^{2-\alpha} - k^{2-\alpha}] + \right. \\ \left. - \frac{2-\alpha}{(n-i+1)^{\alpha-1}} (p_{n+1,j} - p_{n,j}) + \sum_{k=0}^{n-i} (p_{i+k+1,j} - 2p_{i+k,j} + p_{i+k-1,j}) [(k+1)^{2-\alpha} - k^{2-\alpha}] \right\}, \end{aligned} \quad (18)$$

212 where  $p_{i,j} = p(x_i, t_j)$  and  $t_j = j\Delta t$ ,  $\Delta t$  representing the discrete time step.

213 Let us now introduce dimensionless time  $t^* = td_\alpha/l^\alpha$ . Suppose the following  
 214 initial shape for the pressure field:  $p(x^*, t=0) = p_0 \exp(-(x^* - 0.5)^2/0.01)$ , being  
 215  $p_0$  a reference pressure, and homogeneous boundary conditions. The space-time  
 216 dimensionless solution  $p^* = p/p_0$  related to Eq. (18) is reported in Fig. 3 for  
 217  $\alpha = 1.25$  and  $2.00$ .

218 As can be seen, nonlocal interactions affect the solution, influencing both the  
 219 shape and the global diffusion velocity (the end of the transient regime results de-  
 220 layed). The situation is described at a fixed time  $t^* = 0.031$  in Fig. 4 for different  
 221 fractional orders  $\alpha$ .

222 Eventually, we consider the case of a constant pressure difference between the  
 223 bar extremes,  $p(x = l, t) > p(x = 0, t)$  in the steady-state regime. The contribution  
 224 of the left-hand term in Eq. (18) (i.e. the time derivative) vanishes. The solution,  
 225 in terms of dimensionless pressure gradient  $\nabla p^* = \nabla p \times l / (p_l - p_0)$  is plotted in  
 226 Fig. 5 for different fractional orders  $\alpha$ . In the classical local case ( $\alpha = 2$ ), the  
 227 pressure gradient is obviously constant throughout the body. On the other hand,  
 228 for fractional orders  $\alpha$ , the pressure gradient concentrates at the extremes of the  
 229 domain due to a lower presence of nonlocal interactions, i.e. to boundary effects.  
 230 Lower values with respect to the local solution are attained on the central bar, this  
 231 effect being more pronounced for decreasing fractional orders  $\alpha$ .

### 232 *Physical meaning of fractional operators*

233 For sufficiently regular functions, Riemann-Liouville fractional derivatives co-  
 234 incide with those defined by Marchaud (Samko et al., 1993), where the derivatives  
 235 are replaced by the corresponding incremental ratios. In the steady-state regime  
 236 and in absence of external forces, Eq. (12) can be thus put in the following form:

$$\frac{k_\alpha (\alpha - 1)}{2 \Gamma(2 - \alpha)} \left[ \frac{p(x) - p(0)}{(x)^\alpha} + \frac{p(x) - p(l)}{(l - x)^\alpha} + \alpha \int_0^l \frac{p(x) - p(y)}{|x - y|^{1 + \alpha}} dy \right] = 0. \quad (19)$$

237 For the inner points of the domain ( $i = 2, \dots, n$ ), the discrete form of Eq. (19)  
 238 reads:

$$k_{i,1}^{ys}(p_i - p_1) + k_{i,n+1}^{ys}(p_i - p_{n+1}) + \sum_{j=1, j \neq i}^{n+1} k_{i,j}^{yv}(p_i - p_j) = 0, \quad (20)$$

239 It is evident how the nonlocal fractional model is equivalent to a discrete model  
 240 where two channels appear: the former connecting the inner material pores with  
 241 the bar edges, ruling the volume-surface long-range interactions, with permeabil-  
 242 ity  $k^{vs}$ ; the latter connecting the inner material pores with each other, describing  
 243 the nonlocal interactions between non-adjacent volumes, with permeability  $k^{vv}$ .  
 244 Provided that the indexes are never equal one to the other, the following expres-  
 245 sions for the permeabilities hold ( $i = 1, \dots, n+1$ ):

$$k_{i,1}^{vs} = k_{1,i}^{vs} = \frac{k_\alpha}{2} \frac{\alpha - 1}{\Gamma(2 - \alpha)} \frac{A\Delta x}{(x_i - x_1)^\alpha}, \quad (21)$$

$$k_{i,n+1}^{vs} = k_{n+1,i}^{vs} = \frac{k_\alpha}{2} \frac{\alpha - 1}{\Gamma(2 - \alpha)} \frac{A\Delta x}{(x_{n+1} - x_i)^\alpha}, \quad (22)$$

$$k_{i,j}^{vv} = k_{j,i}^{vv} = \frac{k_\alpha}{2} \frac{\alpha(\alpha - 1)}{\Gamma(2 - \alpha)} \frac{A(\Delta x)^2}{|x_i - x_j|^{1+\alpha}}, \quad (23)$$

248 being  $A$  the area cross-section. Furthermore, by looking at the boundary con-  
 249 ditions (16)-(17), it is possible to state that a fourth set of elements has to be  
 250 introduced: it is composed by a single channel connecting the bar extremes, with  
 251 permeability

$$k_{1,n+1}^{ss} = k_{n+1,1}^{ss} = \frac{k_\alpha}{2} \frac{A}{\Gamma(2 - \alpha) (x_{n+1} - x_1)^{\alpha-1}}. \quad (24)$$

252 The superscript  $ss$  for the permeability (24) is used since the element connecting  
 253 the bar edges can be seen as modeling the interactions between material pores  
 254 lying on the surface, which, in the simple one-dimensional model under exami-  
 255 nation, reduce to the two points  $x=0, l$ . Note that the presence of such a channel  
 256 was implicitly embedded in the constitutive equation (6). However, since it pro-  
 257 vides a constant flow contribution throughout the domain, its presence was lost by  
 258 derivation when passing from Eq. (6) to Eq. (12).

259 To summarize, the constitutive fractional relationship (6) is equivalent to a  
260 discrete pore-channel model with three sets of nonlocal elements. Note that their  
261 permeabilities (21)-(24) all decay with the distance, although their decaying ve-  
262 locity is different.

## 263 **Conclusions**

264 A Darcy's law of convolution integral type, describing the dependence of the  
265 flow rate in one point on the gradient of pressure of all nonadjacent points, was  
266 assumed. By choosing a power law expression for the attenuation function, mod-  
267 eling the decreasing flow rate along with the distance, the fractional diffusion  
268 equation for porous materials was derived. Fractional operators were limited to  
269 the space variable. The problem was investigated on finite domains, through frac-  
270 tional finite differences, both in the transient and steady-state regimes. The influ-  
271 ence of the fractional order  $1 < \alpha \leq 2$  on results was discussed, and a physically-  
272 sound interpretation of fractional operators was derived in terms of volume and  
273 surface channels with different permeability.

274 The results presented here may be useful to investigate pressure response of  
275 a well reservoir which in general is not homogenous (Chang and Yortos, 1990;  
276 Acuna et al., 1995; Yao et al., 2012; Camacho-Velázquez et al., 2008; Yang et  
277 al., 2014) and, consequently, is not well described in terms of the usual diffusion  
278 equation (Razminia et al., 2015a,b).

279 Eventually, the following step to extend the present investigation and to ana-  
280 lyze the delays of the fluid pressure at the boundary on the flow of fluid through  
281 the medium, seems that of further modifying nonlocal Darcy's law (1) by adding  
282 a fractional time derivative to the right member. The study can have a great im-

283 portance in the framework of oil fields, where patterns of mineralization and per-  
284 meability changes have to be modeled (Caputo, 2000).

285 The fractional model proposed herein is intrinsically multiscale. As it is well  
286 known, the structure of shales deposit has been reported to be multiscale, i.e.,  
287 ranging from the nanoscale up to the global scale of a deposit. We argue that  
288 the fractional mathematical modeling of the flow of fluids and gas in nanoporous  
289 geomaterials can create a new branch of subterranean fluid mechanics. Our future  
290 steps will include comparison with real field data.

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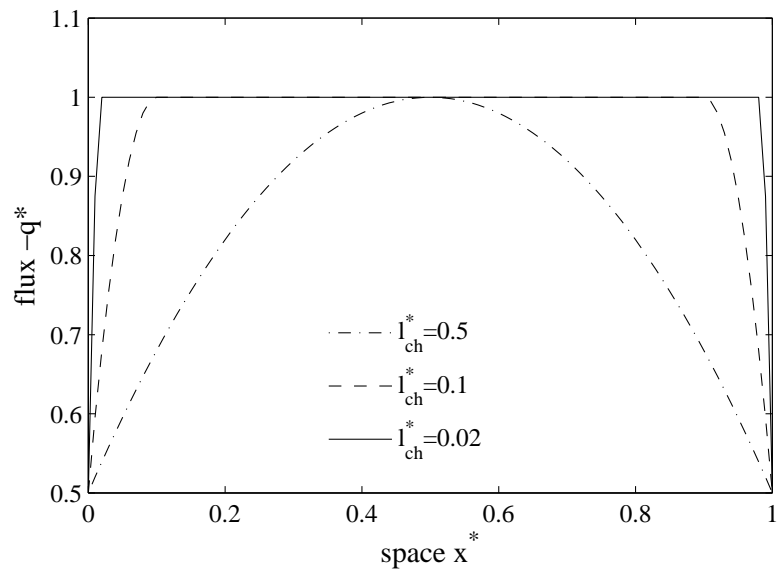


Figure 1: Nonlocal diffusion according to a cone attenuation function: dimensionless flow field providing a constant pressure gradient, for different  $l_{ch}^*$  values.

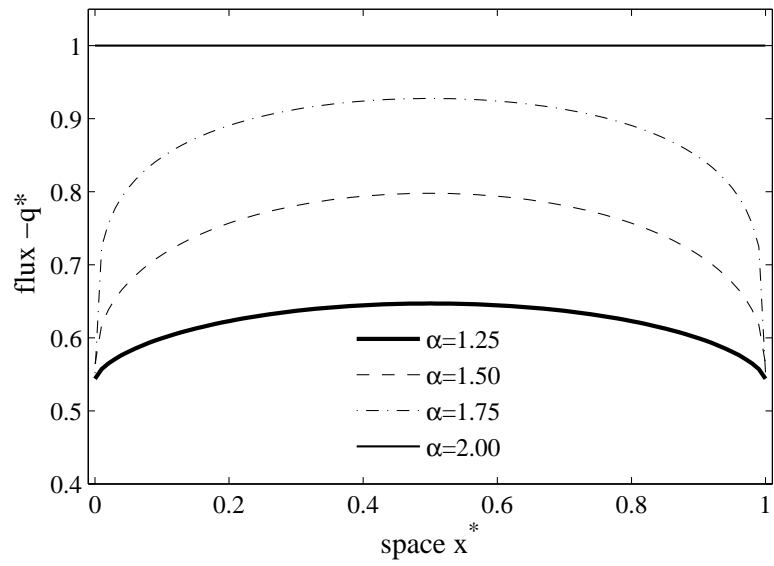


Figure 2: Nonlocal diffusion according to a power-law attenuation function (fractional model): dimensionless flow field providing a constant pressure gradient for different  $\alpha$  values.

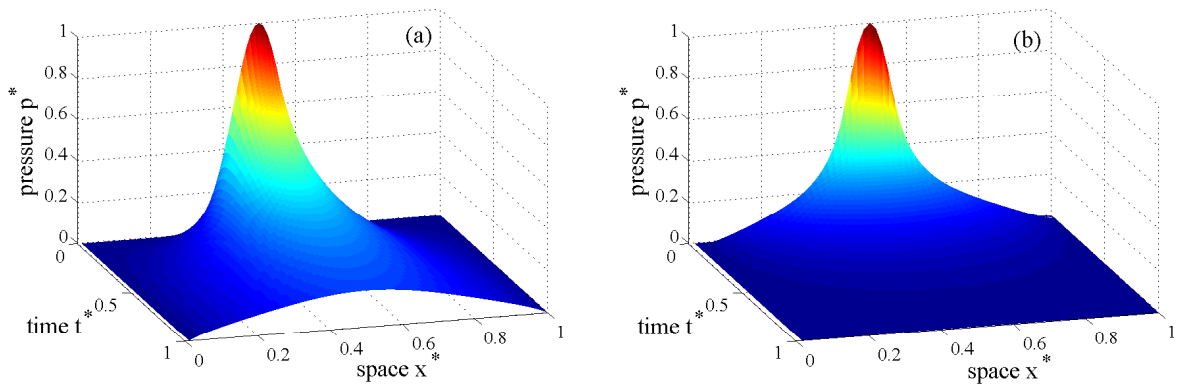


Figure 3: Dimensionless pressure field related to: (a) a nonlocal model ( $\alpha = 1.25$ ); (b) a local one ( $\alpha = 2.00$ ).

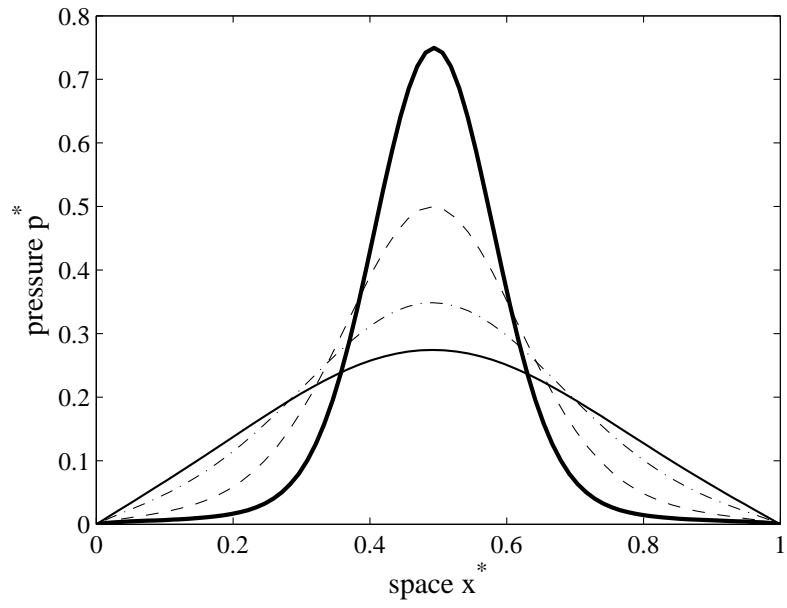


Figure 4: Dimensionless pressure field at  $t^* = 0.031$  for different  $\alpha$ -orders.

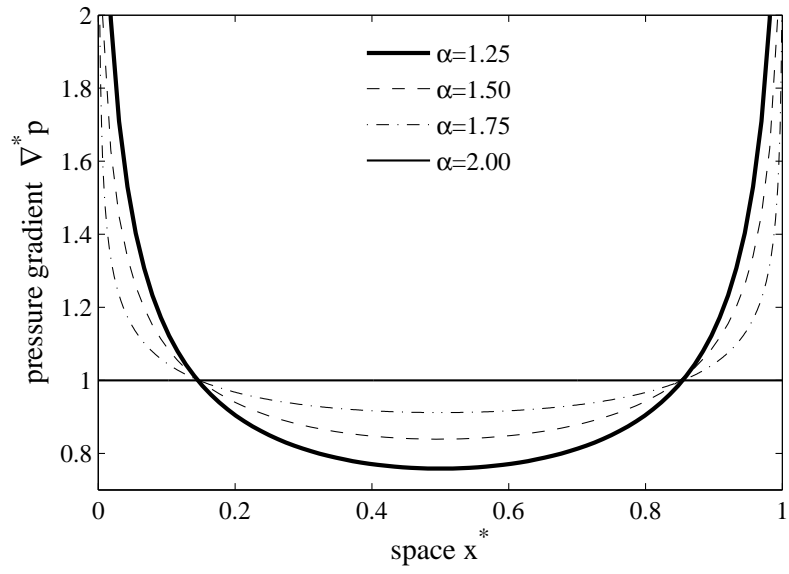


Figure 5: Dimensionless pressure gradient in the steady-state regime for different fractional orders  $\alpha$ .