An Introduction of the Generalized Wiener-Hopf Technique for Coupled Angular and Planar Regions

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***Abstract*— This paper presents the formulation of the electromagnetic problem constituted of coupled angular and planar regions by using the generalized Wiener-Hopf technique. The paper introduce also the technique to obtain a solution of the problem by reducing the factorization problem to Fredholm integral equation. The test case of a PEC planar waveguide filled by a dielectric medium that opens to a PEC angular region is presented.**

I. INTRODUCTION

While scattering problems in structures constituted of only angular regions or only layered regions can be approached by a number of consolidated techniques in spectral domain, the problem of scattering by coupled angular and planar regions requires novel tools.

According to our opinion, one of the most powerful technique capable of handling this problem is the Generalized Wiener-Hopf Technique (GWHT) which has been recently introduced to solve new unsolved problems [1-8].

The generalized Wiener-Hopf technique has been introduced to extend the classical Wiener-Hopf technique first to solve electromagnetic problems involving angular regions [9-11]. This technique has the same capacity of the popular Sommerfeld-Malyuzhinets (SM) method [12-14] to obtain solutions of wedge problems in closed form [15].

However there are many differences between the two methods. In particular, from a mathematical point of view, the GWHT yields to a factorization problem, instead SM requires the solution of systems of difference equations. It means that in case of absence of exact solution we need to resort to efficient approximate technique to factorize WH kernels or to solve system of SM difference equations.

Fortunately concerning with the GWHT an efficient approximate factorization technique (Fredholm factorization) has been proposed [16-18]. The Fredholm factorization consists of eliminating the minus functions in the Wiener-Hopf (WH) equations of the problem by using the well-known additive decomposition based on Cauchy integration formula. The elimination of the minus functions yields equations in plus unknowns that can be systematically rephrased in terms of Fredholm integral equations of second type. It is remarkable that no singular integral equations with complicated kernel (i.e. no special function) are involved so that efficient approximate solution method are easily applicable. The Fredholm factorization works with classical Wiener-Hopf equations as well as with Generalized Wiener equations. In particular by using the Fredholm factorization it has been possible to solve successful several problem of diffraction involving complex wedge geometries [10-11]. These results have given new vitality to the studies on the Wiener-Hopf formulation reinforcing in these authors the belief that this technique is a very powerful tool for solving fundamental electromagnetic problems [19].

The aim of this paper is to introduce the GWHT to solve the class of problem constituted of coupled angular and planar regions, see for example Fig.1.

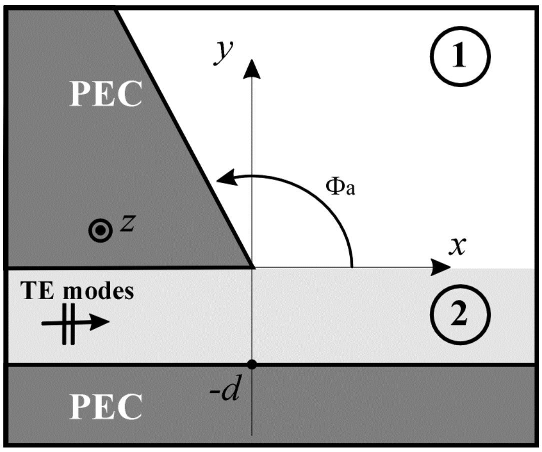

Fig. 1. PEC wedge over finete dielectric layer terminated by PEC.

II. THE GENERALIZED WIENER-HOPF EQUATIONS OF THE PROBLEM

Figure 1 shows a region where the planar layer 2 filled by dielectric with relative permittivity ε_r is coupled to the angular region 1 through the aperture (x>0,y=0), both terminated by PEC faces.

The region (-d<y<0, x<0) constitutes a planar waveguide. Without loss of generality we suppose that the source is constituted by the progressive first TE mode. The TE modes have propagation constant

$$\eta_i = \sqrt{\varepsilon_r k^2 - (\frac{i\pi}{d})^2} \qquad (1)$$

with $i \in \mathbb{N}_0$ and where k is the free space propagation constant.

The source defines the incident magnetic field and in particular with the first TE mode we have

$$H_x^i(x,y) = \cos\frac{\pi}{d} y\, e^{-j\eta_1 x} \quad (-\mathrm{d} < \mathrm{y} < 0, \mathrm{x} < 0) \qquad (2)$$

We note that the discontinuity at x=0, -d<y<0 excites all the TE modes that propagate along the negative x direction.

The formulation of the problem is defined in terms of Laplace transforms $V_+(\eta)$ and $I_+(\eta)$ respectively of the E_z and H_x field components:

$$V_+(\eta) = \int_0^\infty E_z(x,0) e^{j\eta x} dx$$
$$I_+(\eta) = \int_0^\infty H_x(x,0) e^{j\eta x} dx \qquad (3)$$

The Wiener Hopf equation of the layered region 2 is readily obtained by considering its spectral transmission line [18]:

$$Y_l(\eta) V_+(\eta) = -I_-^i(\eta) - I_-^s(\eta) - I_+(\eta) \qquad (4)$$

where $Y_l(\eta) = -j\cot(\xi_l(\eta)d) Y_{cl}(\eta)$, $\xi_l(\eta) = \sqrt{\varepsilon_r k^2 - \eta^2}$, $Y_{cl}(\eta) = \xi_l(\eta)/k\, Z_o$, Z_o is the impedance of the free space and

$$I_-^i(\eta) = \int_{-\infty}^0 H_x^i(x,0_-) e^{j\eta x} dx = -j\frac{1}{\eta - \eta_1} \qquad (5)$$

and the minus unknown $I_-^s(\eta)$ is the Laplace transform of the magnetic field $H_x^s(x,0_-)$ due to all the regressive TE modes (i=1,2,..).

The WH equation of the angular region 1 is given by [9]

$$Y_c(\eta) V_+(\eta) - I_+(\eta) = -I_{a+}(-m) \qquad (6)$$

where $Y_c(\eta) = \xi(\eta)/k\, Z_o$, $\xi(\eta) = \sqrt{k^2 - \eta^2}$, $m = -\eta\cos\Phi + \xi(\eta)\sin\Phi$ and

$$I_{a+}(-m) = \int_0^\infty H_\rho(\rho,\Phi) e^{-jm\rho} d\rho \qquad (7)$$

Eq. (6) is a Generalized WH Equation (GWHE) since the unknown $I_{a+}(-m)$ is a minus function in the m-plane and not in the η -plane.

To solve (4) and (6), we need to eliminate the minus unknowns taking into account their analytical properties.

In 1931 Wiener and Hopf proposed a very ingenious expedient based on the multiplicative factorization of a suitable kernel.

However it is very difficult to obtain effective kernel factorizations especially in presence of generalized WH equations.

Conversely the Fredholm factorization is always possible.

III. Fredholm Factorization of the Problem

The elimination in (4) of $I_-^s(\eta)$ is readily obtained taking into account that its plus part evaluated by the Cauchy plus decomposition (8) is vanishing [16-18]

$$\frac{1}{2\pi j}\int_{\gamma_1} \frac{I_-^s(\eta')}{\eta' - \eta} d\eta' = 0 \qquad (8)$$

By applying the plus Cauchy decomposition formula to (4), after some algebraic manipulations that have the aim of eliminating the singularity in the kernel $1/(\eta' - \eta)$ we get [1]:

$$I_+(\eta) = -\mathcal{Y}_\ell V_+(\eta) + j\frac{1}{\eta - \eta_1} \qquad (9)$$

where the not singular integral operator $\mathcal{Y}_\ell$ is expressed by

$$\mathcal{Y}_\ell[..] = Y_l(\eta)[..] + \frac{1}{2\pi j}\int_{-\infty}^{\infty} \frac{Y_l(\eta') - Y_l(\eta)}{\eta' - \eta}[..] d\eta' \qquad (10)$$

Similar considerations apply for the generalized equations (6). The elimination of the minus unknown $I_{a+}(-m)$ can be obtained by using the plus Cauchy decomposition in the m plane. Now the process of decomposition is more difficult because two complex planes, m and η, are involved. To simplify it a third complex plane α is defined [9]

$$\alpha(\eta) = -k\cos\left[\frac{\pi}{\Phi}\arccos(-\frac{\eta}{k})\right] \qquad (11)$$

In general, the decomposition process is much more complicated in presence of acute aperture angle Φ. For obtuse angle we get [11]

$$I_+(\eta) = (Y_c(\eta) + \mathscr{Y}_a)V_+(\eta) \tag{12}$$

where

$$\mathscr{Y}_a = \frac{1}{2\pi j}\int_{-\infty}^{\infty}\left(\frac{Y_c(\eta')}{\alpha(\eta')-\alpha(\eta)}\frac{d\alpha}{d\eta'} - \frac{Y_c(\eta)}{\eta'-\eta} + \right)[...]d\eta' \tag{13}$$

By eliminating $I_+(\eta)$ in (9) and (12) we obtain the Fredholm integral equation in $V_+(\eta)$.

Since the equation presents compact operator, the solution can be obtained by simple numerical discretization.

However we observe that the numerical solution provides only an analytical element of $V_+(\eta)$. It is possible to get $V_+(\eta)$ for arbitrary values of η resorting to the original GWHE (4) and (6). In particular starting from these equations it is possible to provide simple analytical continuation in w-plane ($\eta = -k\cos w$) and thus to extend the solution. Several numerical simulations for the canonical problem reported in Fig. 1, also in presence of acute aperture angle Φ, will be presented during the oral presentation.

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