Analysis of Coupled Angular Regions in Spectral Domain

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Abstract— **This paper analyses the problem of coupling multiple angular regions in spectral domain by using the generalized Wiener-Hopf technique. The paper introduces also the technique to obtain a solution of the problem by reducing the factorization problem to Fredholm integral equation. We present a test case constituted by two PEC wedges.**

I. Introduction

Recently the authors of this paper has extended the Generalized Wiener-Hopf Technique (GWHT) to solve electromagnetic problems involving coupled angular and planar regions [1-8].

According to our opinion, the GWHT is able to contemporary handle planar stratified structures together with wedge structures [9].

This procedure allows to handle the problem of multiple wedges where angular regions and layers are alternating (see Fig. 1 for example).

In order to illustrate the formulation, we first recall how in literature canonical problems constituted of only planar regions or only angular regions are addressed.

Concerning the planar regions, there are two very excellent books on this topics [10, 11]. However in these books the presence of planar discontinuities is not systematically considered. An unified theory, that starts from the fundamental half plane problem, is presented in [12], where the stratified regions in presence of planar discontinuity is studied. The theory is based on the Wiener-Hopf (WH) technique where the unknowns are Laplace/Fourier transforms of the field components.

Concerning the angular regions, several methods have been proposed to solve the wedge problem.

For instance the canonical PEC wedge problem has been studied in the natural domain by separation of variables [11]. However, the natural domain limits the modelization to simple problems, and to overcome these difficulties, spectral representations have been developed in the past.

An exhaustive analysis of literature shows that the most important representations of angular region problems in spectral domain are:

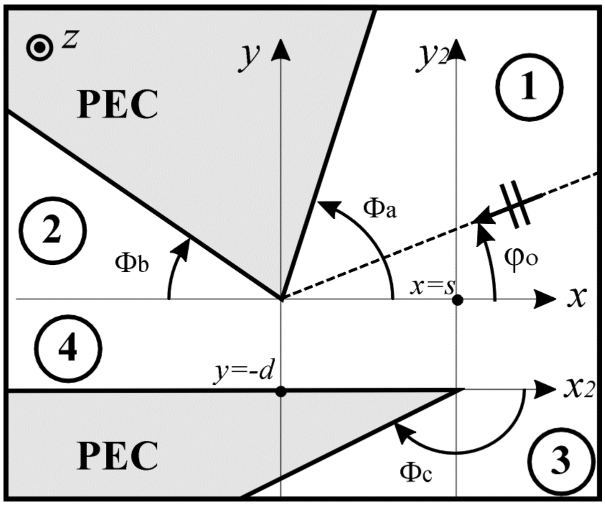

Fig. 1. Mutiple angular region problem: the two PEC wedges. Two cartesian reference systems are reported $(x,y)=(x_2+s,y_2-d)$ together with cylindrical coordinate system. Four regions are defined: region 1 $0<\varphi<\Phi a$ with center $(x,y)=(0,0)$, region 2 $\pi-\Phi b<\varphi<0$ with center $(x,y)=(0,0)$, region 3 $-\Phi c<\varphi<0$ with center $(x_2, y_2)=(0,0)$, region 4 $-d<y<0$.

1) The Sommerfeld Malyuzhinets functions
2) The Kontorovich Lebedev transform
3) The Laplace transform in radial direction.

Using Laplace transforms, the WH technique has been extended to angular regions too [13-16].

We note that for long time the WH technique was considered inapplicable to wedge problems of arbitrary aperture angle.

Since WH technique is now suitable to handle problem with rectangular and angular geometry, we state that GWHT is one of the best mathematical tool to handle coupled angular and planar regions problem since the Laplace transform of the field components can be defined in both kind of regions. Successful applications have already been reported in [1-9,16].

II. FORMULATION AND SOLUTION PROCEDURE

To briefly illustrate as the GWHT formulation works in the case of coupling multiple angular regions, let us consider the scattering problem constituted of two wedges in free space with E-polarized incident plane wave (1)

$$E_z^i = E_o e^{jk\rho\cos(\varphi-\varphi_o)} \tag{1}$$

where k is the free space propagation constant, see Fig. 1.

In this geometry we have three angular regions that couple with a planar layer. For each of these regions we can write a Generalized Wiener-Hopf Equation (GWHE) that involves plus and minus functions defined in suitable complex planes.

The complexity of the problem together with the arbitrariness of geometrical/material parameters do not permit a closed form factorization.

However, the solution of a GWHE problem can be obtained through Fredholm factorization [17-19]. It consists of the elimination of the minus functions in the WH equations through Cauchy decomposition formulas.

After several mathematical elaborations, see [4], the Fredholm factorization yields to the system of Fredholm integral equation of order three:

$$\mathbf{V}_+(\eta) + \int_{-\infty}^{+\infty} \mathbf{M}(\eta,\eta')\mathbf{V}_+(\eta')d\eta' = \mathbf{N}(\eta) \tag{2}$$

where the three unknowns $V_{1+}(\eta)$, $V_{2+}(\eta)$ and $V_{\pi+}(\eta)$ that define the vector

$$\mathbf{V}_+(\eta) = \left| V_{1+}(\eta) \quad V_{2+}(\eta) \quad V_{\pi+}(\eta) \right|^t \tag{3}$$

are the Laplace transforms of the electrical field $E_z(x,y)$ in the three apertures where the angular regions couple with the layer region:

$$\begin{aligned} V_{1+}(\eta) &= \int_0^{\infty} E_z(x,0) e^{j\eta x} dx \\ V_{2+}(\eta) &= \int_0^{\infty} E_z(x_2+s,-d) e^{j\eta x_2} dx_2 \\ V_{\pi+}(\eta) &= \int_{-\infty}^{0} E_z(x,0) e^{-j\eta x} dx \end{aligned} \tag{4}$$

The kernel $\mathbf{M}(\eta,\eta')$ is defined by:

$$\mathbf{M}(\eta,\eta') = \begin{vmatrix} Z^e(\eta) & Z_m^e(\eta)e^{j\eta s} & 0 \\ Z_m^e(\eta)e^{-j\eta s} & Z^e(\eta) & 0 \\ 0 & 0 & Z_3^e(\eta) \end{vmatrix} \bullet Y_e(\eta,\eta') \tag{5}$$

where

$$\begin{aligned} Z^e(\eta) &= \frac{k Z_o e^{-j\xi(\eta)d}}{2\xi(\eta)} \\ Z_2^e(\eta) &= \frac{k Z_o}{2\xi(\eta)} \\ Z_3^e(\eta) &= j\frac{k Z_o e^{-j\xi(\eta)d}\sin(\xi(\eta)d}{\xi(\eta)} \end{aligned} \tag{6}$$

$\xi(\eta) = \sqrt{k^2-\eta^2}$ is the free space spectral propagation constant and Z_o is the free space impedance.

$Y_e(\eta,\eta')$ is a known not singular matrix (not reported here for reason of space).

It is remarkable that the elements of this matrix are expressed by elementary functions. They depend on the aperture angles Φ_a, Φ_b and Φ_c as well as the geometrical parameters d and s, see Fig. 1.

Also the known source term $\mathbf{N}(\eta)$ in (2) is expressed by elementary functions. $\mathbf{N}(\eta)$ depends on the aperture angles Φ_a, Φ_b and Φ_c, the geometrical parameters d and s, and also on the direction φ_o of the incident plane wave. For reason of space the expression of $\mathbf{N}(\eta)$ is not reported here. A deep analysis on $\mathbf{M}(\eta,\eta')$ shows that this kernel is compact, so that the obtained integral equation is a Fredholm integral equation of second kind, thus its numerical solution can be efficiently obtained by using simple quadrature methods.

To improve the numerical convergence of the integral equations we can warp the integration path in order to avoid the presence of singularities near the integration path.

One of the best possible choices is to counterclockwise rotate the original integration path (the real axis η) of an angle θ. Usually the best choice in single wedge problem is to take θ=45°.

In the two wedge problem the effect of the parameters d and s enforce a new constraint in the selection of θ.

The numerical solution of the integral equations provide an analytical element of the spectrum $\mathbf{V}_+(\eta)$ that is in the proper sheet of the η plane.

To obtain the diffraction coefficients, in general we need to know $\mathbf{V}_+(\eta)$ in the improper sheet of the η plane. Analytical continuation of $\mathbf{V}_+(\eta)$ is possible by resorting to the original GWHE of the problem reformulated in the w-plane ($\eta = -k\cos w$).

The efficiency and the validity of the formulation of the two wedge problem has been ascertained in [5] for a particular selection of parameters ($s = 0, \Phi_b = 0$).

Solution in more general cases will be presented during the oral session at the symposium and proposed in a future paper.

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