Synchronous vibration parameters identification by tip timing measurements

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Abstract

The Blade Tip Timing (BTT) measurement system is a technique to measure vibration parameters of a rotating bladed disk. In particular for synchronous vibrations the BTT provides signals versus the rotation speed of the disk starting from the measurement of the time of arrival (TOA) of each blade under the tip timing probes. The signals must be post processed in order to obtain the interesting parameters of each blade vibration. The paper presents a method to extract the main parameters (amplitude and frequency) in resonance condition from the tip timing measurements. The proposed method is a revision of the already existing well known Two-Parameter Plot (2PP) method which requires a minimum of two probes. Improvements to the existing 2PP method are here suggested mainly in the part of engine order identification.

The proposed method is then applied to the BTT measured signals coming from a rotating bladed disk excited at different engine orders. At the same time on the disk the vibration of one blade was detected by strain gauges. The strain gauges were calibrated and they provide the reference values of the vibration parameters. The vibration parameters derived by the proposed method are in agreement with those obtained by the strain gages methodology.

Keywords: vibrations, bladed disk, spinning tests, tip timing

1. Introductions

Turbomachinery blades undergo various types of excitations during operation and, for this reason, their fatigue life is reduced and there is an increase in the risk of crack formation that can lead to costly damages. A good design of turbo machinery should avoid the most dangerous excitation frequencies during operation. Therefore the measurement of blade vibration amplitude and frequency during rotation is a challenging topic in the frame of the blade monitoring to avoid high cycle fatigue failure. The Blade Tip Timing (BTT) measurement technique is presently the most promising technique for blade monitoring. It is a non-intrusive measurement procedure. The technology was born in 1970 but it is still under improvement in our days. The great advantages with respect to the standard method like strain gages are that the technique is not intrusive (the blades are not in contact with the measurement probes) and it can monitor each blade giving the possibility to identify if one blade vibrates more than the others (for typical mistuning problems).

The BTT detects the vibration amplitude and frequency of the blades of a rotating disk by using a set of probes. Number, type of probes and locations depend on the application and on the data that are needed. The basic principle of this technique is that, in absence of vibrations, the blade passes in front of the probe after each rotation in a time that depends only on the rotational speed. On the contrary, in the case in which the blade is vibrating, the vibration changes the time of arrival (TOA) and the new value can be higher or lower than the theoretical one.

The BTT works in three steps:

- acquisition of the TOA for each blade by the probes,
- calculation of the blade displacement in the measurement point,
- analysis of the data and displaying of the results.

Many methods of data analysis for BTT are available and they are divided into indirect and direct methods.

- *Indirect methods*. In this case the data are collected through one or two probes and the measurements are done during an acceleration (or deceleration) of the disk in a range of speed when one or more resonances are excited. These methods give the maximum amplitude (in resonance) and the corresponding frequency. One indirect method that has become the standard method for single synchronous resonances is the one proposed by Zablotsky and Korostelev (1970) and it is called Single Parameter method [2,4]. Another indirect method is the so-called Two Parameter Plot method [1,2,3] that will be deeply analyzed in this paper.
- *Direct methods*. They generally use four or more probes to collect the data and they operate over a chosen number of revolutions at a constant rotational speed of the system. These methods give the amplitude not only at resonance, but at each rotational speed. They have the limitation that they need to already know the engine order (*EO*) corresponding to each resonance. The basic procedure is a least-squares sine fit of the data starting from a given equation of the blade vibration displacement at a given *EO*. Examples of direct methods are the Determinant method and the Autoregressive method [3,5,6].

This paper is focused on the Two-Parameter Plot (2PP) indirect method developed by Heath and Imregun [1,2]. The method is here reconsidered and some possible improvements are proposed. The 2PP method has the advantage to be very simple, requiring only two probes and it can be used for a first analysis before applying any other direct methods, providing the *EO* of each resonance. In particular in this paper a more straightforward procedure than that proposed in [1,2], is developed to identify the engine order (*EO*). Once the *EO* is known, the resonance frequency can easily be determined. The blade vibration amplitude can be estimated by the 2PP method using the two probes or more than two probes used in pair and averaging the results of the different pairs of probes. Alternatively, once the *EO* is known, data coming from more than two sensors can be post processed together in a fitting data procedure using one of the direct methods proposed in literature which assume that the *EO* is already known [3,5,6].

In this paper the proposed method was applied to the tip timing data coming from an experimental set up where simultaneous strain gage measurements were carried out on one of the blades. The vibration displacements obtained by the strain gauges after a calibration procedure are considered as reference [10]. The accuracy of the values of the resonance parameters (frequency and amplitude) obtained by the tip timing measurement post processed by the proposed method is shown through the comparison with the same parameters obtained by the strain gauges measurements.

2. The Tip timing basic principle

The Blade Tip Timing (BTT) is a not-intrusive measurement technique for the blade vibration monitoring of blades during rotation. Probes positioned on the casing detect the time of arrival (TOA) of each blade. In absence of vibration and in case of constant rotation speed, each blade passes in front of the probe always after the same time delay as the blue blade in the scheme of Fig. 1a). This means that in case of constant rotation speed the signals detected by the probe at the passing of each blade are equally spaced in time as shown in the upper scheme of Fig. 1 b). On the contrary, in the case in which the blade is vibrating the TOA depends on the blade vibration amplitude. The signals detected by the probe at the passing of each vibrating blade are at different distance in time as shown in the lower scheme of Fig. 1 b). The system computes the difference ΔT between the TOA detected by the probe at the passing of the vibrating blade and the TOA of the not vibrating blade given by a reference sensor, which points on a not vibrating part of the disk. The time difference ΔT multiplied by the rotor speed and by the distance of the measurement point from the rotation center, gives the blade displacement amplitude at a given rotational speed. The measurements are normally taken during an increasing ramp of rotational speed. For each value of rotational speed the blade displacement amplitude is detected. The plot displacement versus rotational speed has a shape similar to that represented in Fig. 1 c) when a natural frequency of the blade is excited in the selected range of speed. The form of the plot displacement versus rotational speed of Fig. 1 c) depends on the position of the probe relative to the measured vibrating blade. It can be demonstrated that the Peak-to-Peak distance in Fig. 1 c) is a good approximation of the value of the maximum vibration amplitude *A* [4] in resonance of the blade. This property is used in many methods, even in the 2PP method that is analyzed in this paper.

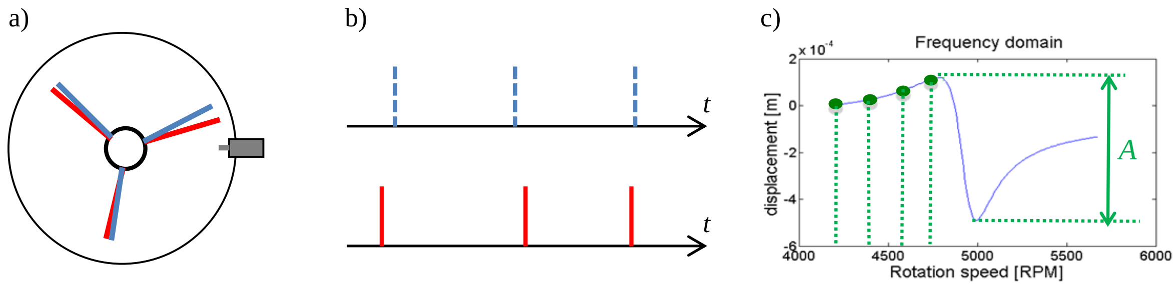

Fig. 1. Basic principles of tip timing measurement. a) Scheme of the blades passing in front of the probe. b) TOA detected by a probe for non-vibrating and vibrating blades. c) Peak-to-Peak plot.

3. The Two-Parameter Plot (2PP) method

The Two-Parameter Plot (2PP) is an indirect method for the Blade Tip Timing (BTT) data analysis developed by Heath and Imregun [1,2] to post process tip timing data for synchronous vibration. This method needs two probes on the same axial position on the bladed disk casing. The signals detected by the two probes have shape similar to that of Fig. 1c). The two probes *a* and *b* are located around the casing respectively at the position θ_a and θ_b, being $\Delta\theta$ the angle between them:

$$\Delta\theta = \theta_b - \theta_a \tag{1}$$

In a synchronous vibration the two blade tip displacements (x_a and x_b) detected by the two probes are assumed to be harmonic, and for a given engine order *EO*, they can be written as:

$$x_a = A(\omega)\, sin\big(EO\Omega t_a + \varphi(\omega)\big) + D$$
$$x_b = A(\omega)\, sin\big(EO\Omega t_b + \varphi(\omega)\big) + D \tag{2}$$

where Ω is the rotational speed, t_a and t_b are the time of arrivals (TOA) at which the blade passes in front of the two probes, $\varphi(\omega)$ is the phase of the response, $A(\omega)$ is the modulus of the response and D is the constant offset. The equations can be reformulated substituting Ωt with the angular position of the two probes θ_a and $\theta_b = \theta_a + \Delta\theta$:

$$x_a = A(\omega)\, sin\big(EO\theta_a + \varphi(\omega)\big) + D$$

$$x_b = A(\omega)\, sin(EO\theta_a + \varphi(\omega) + EO\Delta\theta) + D$$

By substituting to $A(\omega)$ and $\varphi(\omega)$ the typical frequency response and phase functions for a single degree of freedom at a given excitation force, the two theoretical plot of Fig. 2 a) are obtained. The plots have the typical shape of the tip timing signal shown in Fig. 1 c). Moreover by plotting the same two Eqn. (3) one against the other, the resulting plot is the ellipse of Fig. 2 b). It is then expected that by plotting the real measured data coming from the two probes, one against the other, a curve as much as similar to an ellipse is obtained. This is true if the hypothesis of one harmonic vibration is correct.

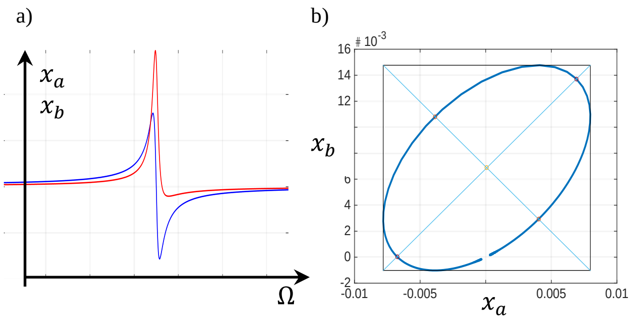

Fig. 2. a) Two theoretical sensor signals. b) Ellipse generated by plotting the two signals

The 2PP method is based on the analysis of this ellipse in order to obtain the vibration parameters. In the present paper some variations to the original 2PP method are proposed in particular in the part of the derivation of the engine order value (*EO*) of a synchronous vibration. The new approach, here proposed, is based, with the due modifications, on the theory of the ellipse treated as a Lissajous's figure [8] to determine the phase shift between two harmonic signals.

3.1. *Engine order determination*

As shown in Eqn. (3) the *EO* value is strictly related to the phase shift between the two signals detected by the two probes. The contribution to the phase difference given by the *EO* is expressed by the $EO\Delta\theta$ term. Once this term is obtained, the engine order can be easily computed, because the angle $\Delta\theta$ between the two probes is a constant.

In the 2PP method [2] $EO\Delta\theta$ has a specific name: *PSR* -
(3) Probe Spacing on the Resonance. It is obtained by using a third order polynomial formula in α, with α being the ratio between the minor and the major axes of the ellipse. From each ellipse two values of *PSR* (and than two values of *EO*) had to be determined. In order to choose the correct one between the two it was necessary to already know the expected engine order value.

The new version presented here avoids the polynomial procedure and allows obtaining directly the right value of *EO*. The procedure is based on the similarity of this case with the Lissajous's figure method. The Lissajous's figure is a parametric plot well known in literature [9] particularly used in laboratory practice to visualize two shifted harmonic signals coming from the acquisition of an oscilloscope. The two signals of the 2PP method (Eqn (3)) are described through two harmonics as well, even if there are two main differences between the 2PP signals and the Lissajous ones. In the 2PP method the variable that changes is the frequency, while in Lissajous's analysis the variable is the time. The other difference is the vibration amplitude that is constant in the Lissajous's figures, while in the case of the 2PP method it depends on the frequency.

Consider the equations of the two signals x_a and x_b detected by the two probes (Eqn. (3)) during the rotation: x_a is the signal detected by the first probe and x_b is the signal from the second probe. The first probe is the one with angular position θ_a lower than θ_b in the direction of the disk rotation.

The two signals have the same maximum amplitude $A(\omega)$ at each frequency, since they are detected by two different probes, but they come from the same blade.

The Eqn (3) of the two signals where D is set for simplicity equal to zero, become:

$$x_a = A(\omega)\, sin\big(c + \varphi(\omega)\big) \qquad (4)$$

$$x_b = A(\omega)\, sin(c + \varphi(\omega) + \delta)$$

where $c = EO\theta_a$ and $\delta = EO\Delta\theta$.

The plot of x_a versus x_b is the ellipse of Fig. 2 b).

Dividing both equation by $A(\omega)$ and computing the arcsine, the two resulting equations are:

$$c + \varphi(\omega) = \arcsin\big(x_a/A(\omega)\big) \qquad a)$$
$$c + \varphi(\omega) + \delta = \arcsin(x_b/A(\omega)) \qquad b)$$

Calculating b) – a):

$$\delta = arcsin\big(x_b/A(\omega)\big) - arcsin\big(x_a/A(\omega)\big) \qquad (5)$$

Suppose to consider the value of frequency at which the vibration x_a reaches the maximum value (*A1* in Fig. 3). The corresponding value of the vibration x_b is from Fig. 3 upper left LH that is the vertical distance tangent to the ellipse.

At that value of frequency Eqn. (5) results:

$$\delta = arcsin(LH/A1) - \pi/2 \qquad (6)$$

Assuming $\delta^* = \arcsin(LH/A1)$ with $-\pi/2 < \delta^* < \pi/2$ depending on *LH* value, the possible solution of Eqn. (6) could be:

$$\delta = \delta^* - \pi/2$$

$$\delta = 2\pi + \delta^* - \pi/2 = \delta^* + 3\pi/2 \quad (1) \qquad (7)$$

$$\delta = \pi - \delta^* - \pi/2 = \pi/2 - \delta^* \quad (2)$$

The first of the (7) is not acceptable since δ mast have a positive sign, from Eqn. (4) the signal x_b is detected after x_a. The possible solutions of δ could be the (1) and (2) of Eqn. (7).

In order to choose between the two possibilities the method adopted in the Lissajous's figure [9] based on the direction of the plot of the ellipse is here adopted. When the direction of the ellipse by increasing ω is clockwise,the correct phase shift is obtained employing the solution (1). When the direction of the ellipse is counter-clockwise the correct phase shift is given by the solution (2).

Fig.3 shows the different ellipses for the two possible equations (7) of δ, for different values of LH,

Once the value of δ is computed the value of *EO* can be easily determined considering the relationship $\delta = EO\Delta\theta$ where $\Delta\theta$ is known since it is the angular distance of the two probes.

The angular distance between the two probes $\Delta\theta$ is not arbitrary, in order to find the right value of the *EO* from the proposed method it should be verified that :

$$\Delta\theta * EO < 360° \tag{8}$$

The meaning of Eqn. (8) is that the probes distance must be chosen in order that they see the vibration of the blade inside one period of vibration. If only two probes are available the $\Delta\theta$ should be chosen so that for the maximum possible EO_{max} $\Delta\theta * EO_{max} < 360°$. If more than two probes are available in order to apply the method to the different probe pairs it is suggested that the location of the probes is chosen according to the following rule: the angle between the first and the last probe, multiplied by the maximum engine order must be lower than 360°. If the previous suggestion cannot be applied due to the probes dimensions, it is necessary that at least the angle between two consecutive probes, multiplied by the maximum engine order must be lower than 360°.

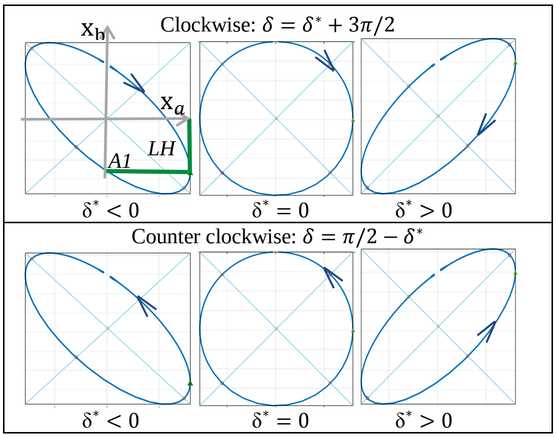

Fig. 3. Phase shift computation

3.2. *Maximum amplitude of vibration*

Once the *EO* is computed the maximum amplitude of vibration in resonance can be calculated with different methods. The basic one is the Single Parameter method [4] where the maximum vibration amplitude is estimated by the peak-to-peak value of the curves of Fig. 1. Heath suggests the following equation:

$$A_{max} = \frac{(max(x_a) - min(x_a)) + (max(x_b) - min(x_b))}{2} \tag{9}$$

where the amplitude is estimated as the average of the peak-to-peak values obtained by the two different probes.

Starting from the ellipse obtained by the two probe signals Dimitriadis et al. [3] proposed from the ellipse axes:

$$A_{max} = a + b \tag{10}$$

Where *a* and *b* are length of the two ellipse semi-major and semi-minor axis. Eqn. (10) is theoretically correct only when the (x_a , x_b) plot is a circle, i.e. when the minor axis and the major axis have the same value, in this case A_{max} is equal to the diameter of the circle. In the general case the ellipse obtained by the two probes is not a circle, but it is always an ellipse inscribed in a square that is its axis are always along the square bisectors [5]. Consider this particular property of the ellipse inscribed in a square a different equation is here proposed to compute the maximum vibration amplitude starting from the ellise semi axis length *a* and *b* :

$$A_{max} = \sqrt{2(a^2 + b^2)} \tag{11}$$

The derivation of Eqn. (11) is shown in appendix. Other methods can be found in literature to determine the vibration amplitude, once the *EO* is known [3], [4], [6] and [7], based on data fitting. These methods are based on Eqn. (2) that are used to fit the measured values of more than two probes for a given rotation speed Ω. The parameters *A*, φ and *D* can be obtained by the fitting procedure at each Ω value. The maximum value of the amplitude *A* corresponds to the resonance amplitude at the given *EO*.

3.3. *Natural frequency estimation*

Once the *EO* is determined the natural frequency (ω_n) can be deduced by the plot of the two probes "rotational speed vs. blade displacement" as shown in Fig. 4. A simple procedure is proposed here to determine the rotational speed Ω_n at which the resonance occurs. The value of the natural frequency is:

$$\omega_n = EO \cdot \Omega_n \tag{12}$$

Ω_n can be derived as a weighted average on the peak amplitude of the rotational speeds. In detail Ω_n can be estimated as the mean of the value from the first probe and the value Ω_2 from the second probe. Ω_1 and Ω_2 are calculated considering the peak amplitudes A and B, C and D of Fig. 4 as:

$$\begin{aligned}
\Omega_1 &= \Omega_A \left(\frac{A}{A+B}\right) + \Omega_B \left(\frac{B}{A+B}\right) \\
\Omega_2 &= \Omega_C \left(\frac{C}{C+D}\right) + \Omega_D \left(\frac{D}{C+D}\right) \\
\Omega_n &= (\Omega_1 + \Omega_2)/2
\end{aligned} \tag{13}$$

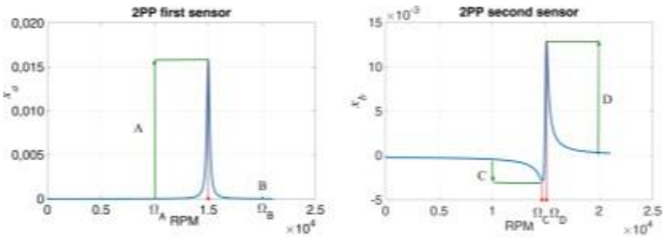

Fig. 4. Amplitude and frequency of the peaks in the probe signals.

4. Experimental measurements

The proposed method of extracting the vibration parameters has been applied to the tip timing data detected by five optical probes on a rotating bladed disk with 12 blades and with a diameter of 440 mm (Fig. 5 a). The disk is in aluminum with the simple geometry of a flat plate where each blade has the shape of a cantilever beam. The measured vibration is out of plane. The probes position during the measurement is shown in Fig. 6 a) and b). The details of the measurement technique are described in [10]. In order to verify the tip timing measurements one of the blades was instrumented by strain gauges (Fig. 5 b)), which were attached at the two sides of the blade root in order to measure the out-of-plane bending (1F) mode belonging to the first modal family. As explained in [10] the strain gauges were calibrated by means of calibrated masses positioned on the free end of the instrumented blade. A tuned FE model was used to derive from the strain gage measurement the corresponding displacement at the tip timing measurement point on the blade. After this calibration procedure the measurement derived from strain gauge was considered as the reference measurement to which compare the tip timing data.
For each blade and in particular for the blade instrumented with strain gage a typical plot detected by the tip timing probes is shown in Fig. 7. Each signal (each color) is the measurement of one probe on the same blade during a spinning test at increasing rotating speed. Each peak corresponds to a resonance that is to a given *EO*. The peaks are the typical peaks detected by the tip timing probe with a shape similar to that shown in Fig. 2a).

a) b)

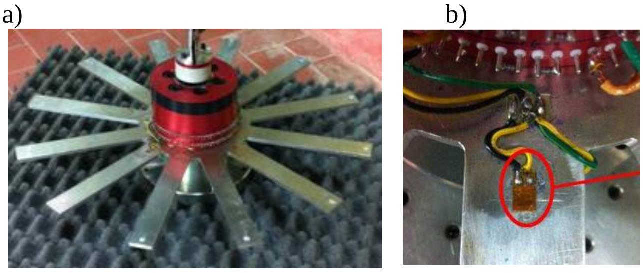

Fig. 5. a) Rotating disk. b) Blade with strain gage.

a) b)

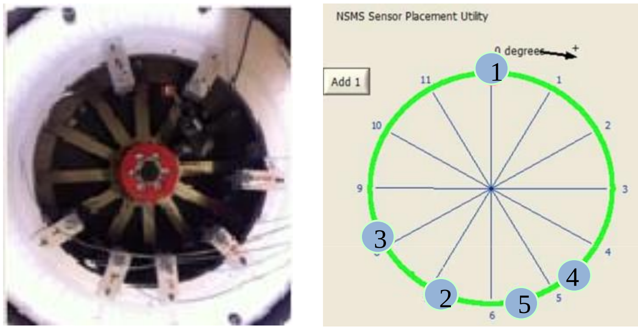

Fig. 6. a) View of the rotating rig. b) Probes position.

5. Application of the 2PP method to the tip timing signals

The proposed method is applied to identify at which *EO* number each peak corresponds. In order to apply the method the pairs of probes whose distance satisfy the rule expressed by the Eqn. (8) must be selected. The signal from the probes pair 4 – 5 ($\Delta\theta_{54}$= 29°) can be used to identify the EO from 1 to 11, the obtained results can be checked by the probes pairs 2 – 5 ($\Delta\theta_{25}$= 37°) for *EO* from 1 to 9, and by the probes 2 – 3 ($\Delta\theta_{32}$= 41°) for EO 1 to 8.

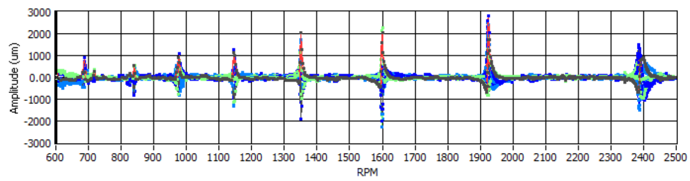

Fig. 7. Tip timing data signals, for one blade, all sensors

Fig. 8 a) and b) show an enlarged view of the signals coming from two sensors at one of the resonance of Fig. 7. It can be seen that the signals are disturbed by the noise and there is a distortion in the signal itself. There is a remaining wave after the main signal change. For these reason the resulting 2PP ellipse representing the plot of one signal against the other of Fig. 8 c) is not so regular as in the theoretical case. The determination of the length of *LH* could then be critical in this case of real experimental data. It can be observed that for rules of geometry of an ellipse inscribed in a square the distance LH between the tangent points and the ellipse centre (*LH*) must be the same for each of the four tangent points. It is than suggested to determine the four values of this distance, indicated as *LH*, *LH'*, *LH''*, *LH'''* in Fig. 8 c) and to perform the average value of the absolute values of these distances. A careful choice among these four values is important; for example in Fig. 8 c) it would be better to exclude in the mean the value of the distance *LH'''*, which is clearly affected by an error due to the noise in the signal. An improvement to calculate the LH distances could be the fitting of the ellipse of Fig. 8 and the calculation of the tangent segments, this should speed up and automize the procedure.

Concerning the sign of the final *LH* value according to the scheme of Fig. 3 it has to be chosen looking at the value in the right side of the square. In the example of Fig. 8 c) *LH* has a negative value, the rotation on the ellipse during plotting is clockwise and then the formula to determine δ according to the scheme of Fig. 3 will be $\delta = \delta^* + 3\pi/2$. For the determination of the value of $A1$ which is needed for the calculation of δ^* it is suggested to compute the mean of the sides of the square computed as tangent to the ellipse, which gives the value of $2A1$.

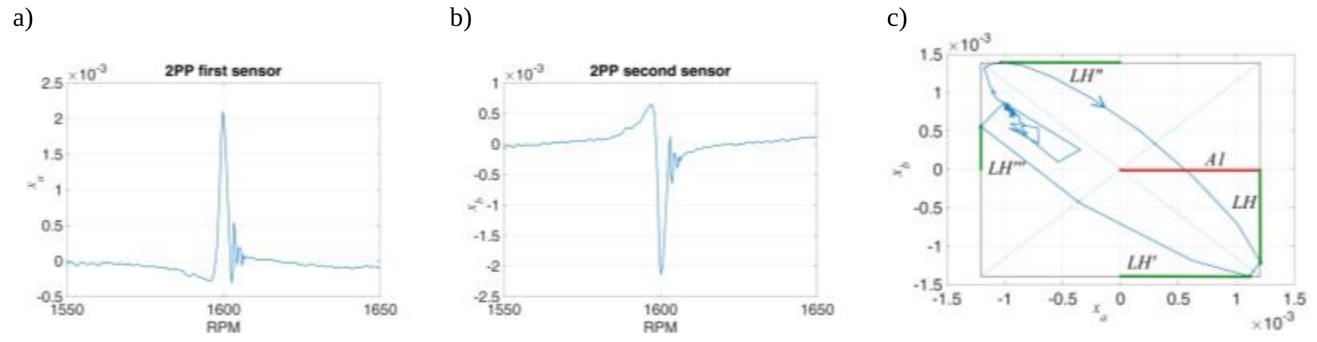

Fig. 8. Signals from tip timing sensors. a) Signal from first sensor. b) Signal from second sensor. c) Ellipse from the two signals.

The EO computation for the different peaks using the different pairs of probes is shown in Table 1. The first row of Table 1 shows the expected EO values. It can be seen that the proposed calculation gives a not integer value for the EO which rounded to the closest integer is the expected value

Table 1 Engine order computation.

Probes	*EO = 6*	*EO = 7*	*EO = 9*
2 – 5	5.7	6.7	8.9
4 – 5	5.8	6.8	9.1
2 – 3	6.1	7.1	

Once the *EO* is known the resonance frequency can be estimated following the procedure explained in section 3.3. In Table 2 the values of the mean frequency obtained by the different pairs of probes (2-5, 4-5, 2-3 for *EO* 6 and 7 and 2-5, 4-5 for *EO* 9) are listed and they are compared with those obtained by the strain gage on the blade.

Table 2 Natural frequency computation

	f_n *strain gages* [Hz]	f_n *2PP* [Hz]	$e(f_n)$ %
6	159.3	159.9	0.4
7	158.0	157.7	-0.2
9	148.5	146.9	-1.1

The last column of Table 2 is the difference between the frequencies values expressed in percentage of the frequency measured by the strain gauges. It can be seen that it is obtained a good estimation of the resonance frequency values with an error lower that 1.1%.

As shown in section 3.2 the vibration amplitude of the blades at resonance can be determined independently from the *EO* value. Using the 2PP method with the ellipse it can be computed following Eqn. (9) or Eqn. (11). Eqn. (9) requires only the estimation of the maximum and minimum value of displacement detected by the two probes that is the sides of the square in which the ellipse is inscribed, Eqn. (11) involves all the ellipse point since implies the calculation of the two ellipse semi-axis. The two Eqn. (9) and (11) lead to the same result in case of theoretical signals without any signal noise.
The results obtained by the experimental data are listed in the third and fifth columns of Table 3. The listed results are already a mean of the values obtained by the different pairs of probes.

It can be seen that the value obtained by Eqn. (9) are always slightly higher than those obtained by Eqn. (11). The comparison with the reference values obtained through the strain gauges is given in Table 3 by the error $e_{(11)}$ and $e_{(9)}$ expressed as percentage difference with respect to the strain gauge value. It seems from the observation of the $e_{(11)}$ and $e_{(8)}$ values that the amplitude values given by Eqn. (11) gives values more close to the reference. The case of *EO* 9 seems to be better estimated by Eqn. (9). This difference must be evaluated considering the repeatability error of the measurement. Repetitions of the measurement in the same condition proved that the values of amplitude estimated in different measurements gives values of amplitude ±3%. In the case *EO* 9 estimation with Eqn. (11) is inside the repeatability error and then acceptable

Table 3: Averaging of maximum vibration amplitude computation for Two-Parameter Plot method

EO	A_{max} **strain gages**	A_{max} **eqn.(11)**	$e_{(11)}$	A_{max} **eqn.(9)**	$e_{(9)}$
-	µm	µm	%	µm	%
6	2364	2485	5.1	2553	8.0
7	1662	1781	7.1	1822	9.6
9	960	935	-2.6	956	-0.4

6. Conclusions

The paper presents a method to extract the main parameters at the resonance of a synchronous vibration for a rotating bladed disk starting from tip timing data and using a minimum of two probes. The method presented here is based on the 2PP method [1][2] but some modifications to improve the original method are here suggested. The 2PP method consists in the derivation of the vibration parameters starting from the elliptical plot of the signals of the two probes one against the other. The main improvement proposed here is in the determination of the engine order (*EO*) associated to the different peaks of the tip timing signals. Alternatively to the original method which arrived to two values of *EO*, one to be discarded, it was shown that it is possible to determine directly the right value of *EO* by considering the rotation direction during the ellipse construction. The method was tested on a real measurement data set acquired by optical probes of a BTT system on a rotating bladed disk. The disk was instrumented with strain gages whose measurements were considered as reference. The comparison of the BTT measurements against the strain gages ones proved the goodness of the method in the estimation of the *EO* and consequently in the estimation of the resonance frequency. It was shown also an alternative procedure to the original one to calculate the vibration amplitude using the ellipse axis length instead of the length of the square inscribing the ellipse. This method gives slightly better results than the original one in presence of noise.

The procedure here presented is mainly suggested to determine the *EO* value and the resonance frequency, which is the basis for the application of any other methods. For the determination of the vibration amplitude when more than two probes are available one direct methods, which uses in a fitting procedure the signals coming from all the probes, can probably decrease the influence on the results of the signal noise.

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Appendix

In order to demonstrate Eqn. (11) consider the ellipse inscribed in a square of Fig. 9 where a is the semi major-axis, b is the semi minor-axis and μ is the semi diagonal of the square, the equation of the tangent line is

$$y = -x + \mu \tag{14}$$

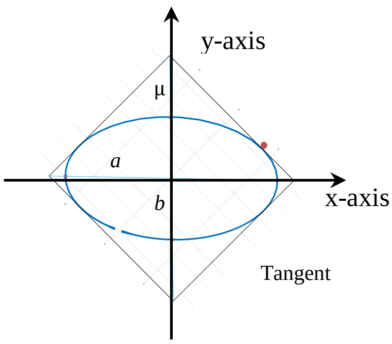

Fig. 9. Ellipse inscribed in a square

The equation of the ellipse centered in the origin of the reference system is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{15}$$

The tangent point between the two curves can be determined by the following system of equations that must satisfy the non negative constraints $x > 0$ and $y > 0$:

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\ y = -x + \mu \end{cases} \tag{16}$$

By substituting equation of the tangent line in equation of the ellipse:

$$x^2(a^2 + b^2) - 2\mu a^2 x \ + a^2(\mu^2 - b^2) = 0 \tag{17}$$

This equation is a polynomial of the second order in *x*. The two curves are tangent, so there must be only a single solution for polynomial (17) and therefore its determinant must be equal to zero.

$$\Delta = \ \mu^2 a^4 - a^2(a^2 + b^2)(\mu^2 - b^2) = 0 \tag{18}$$

this leads to:

$$a^2 + b^2 = \mu^2 \tag{19}$$

where μ is the semi-diagonal of the square and its relationship with the side of the square *A* is the following one:

$$\mu = A\frac{\sqrt{2}}{2} \tag{20}$$

Substituting (20) in (19)

$$A = \ \sqrt{2(a^2 + b^2)} \tag{21}$$

where *a* and *b* are respectively the semi minor-axis and the semi major-axis.