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S-N curves in the Very-High-Cycle Fatigue regime: statistical modeling based on the hydrogen embrittlement consideration

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Abstract:

The continuous increment of durability and reliability requirements for many machinery components is significantly enhancing the research activity in the Very-High-Cycle Fatigue (VHCF) characterization of metallic materials, in particular of high-strength steels for critical structural applications.

According to the $\sqrt{area}$ model, the VHCF strength of high-strength steels can be estimated from the projected area of the 'Optically Dark Area' (ODA), which plays a key role in the VHCF response of high-strength steels: more than 95% of the total VHCF life is consumed in the ODA formation, with crack growing even though the Stress Intensity Factor (SIF) is below the threshold for crack growth. Following the hydrogen embrittlement theory proposed by Murakami, hydrogen is supposed to assist crack growth within the ODA.

The present paper proposes a general SIF formulation for the analytical model of the hydrogen assisted crack growth within the ODA. Starting from the general SIF formulation, a general expression for the material fatigue limit is obtained in the paper. The statistical method for the estimation of the parameters involved in the proposed model is finally illustrated in the paper and numerically applied to an experimental dataset.

Keywords: VHCF, Hydrogen assistance, Random fatigue limit, P-S-N curves

Acronyms and nomenclature

cdf: cumulative distribution function

HV: Vickers Hardness

LEV: Largest Extreme Value

ODA: Optically Dark Area

pdf: probability density function

rv: random variable

SIF: Stress Intensity Factor

$\sqrt{a_d}$, $\sqrt{a_{d,final}}$, $\sqrt{a_{d,final,max}}$, $\sqrt{a_{d,0}}$, $\sqrt{a_{ODA}}$: characteristic defect sizes, deterministic values

$\sqrt{A_{d,0}}$: initial defect size, rv

$\sqrt{a_{finite,\overline{ODA}}}$, $\sqrt{a_c}$, $\sqrt{a_{\infty,\overline{ODA}}}$: threshold defect sizes

C_{th}: rv function of K_{th}

c_H, c_{s_l}, $c_{s_l,\alpha_H\geq 0}$, $c_{s_l,\alpha_H<0}$, c_{th}, c_Y, m_Y, n_Y, α_H, α_{th}: constant coefficients

$E[\cdot]$: expectation of a random variable

$f_{\sqrt{A_{d,0}}}$: pdf of $\sqrt{A_{d,0}}$

$F_{\sqrt{A_{d,0}}}$, F_{S_l}, $F_{S_l|\sqrt{a_{d,0}}}$, F_{X_l}, $F_{X_l|\sqrt{a_{d,0}}}$, F_Y, $F_{Y|\sqrt{a_{d,0}}}$, $F_{Y_f|\sqrt{a_{d,0}}}$: cdfs

k_d, k_H, k_{th}, k_T: characteristic SIFs, deterministic values

K_{th}: threshold SIF, rv

L: Likelihood function

$P[\cdot]$: probability of an event

s: applied stress amplitude, deterministic value

s_l: fatigue limit, deterministic value

S_l: fatigue limit, rv

$S_l|\sqrt{a_{d,0}}$: conditional fatigue limit, rv

$s_{finite,\overline{ODA}}$, $s_{\infty,\overline{ODA}}$: threshold stresses

$\boldsymbol{T}$: design matrix

$x = \log_{10}[s]$: logarithm of the applied stress amplitude, deterministic value

$x_l = \log_{10}[s_l]$: logarithm of the fatigue limit, deterministic value

$X_l = \log_{10}[S_l]$: logarithm of the fatigue limit, rv

$X_l|\sqrt{a_{d,0}} = \log_{10}[S_l|\sqrt{a_{d,0}}]$: conditional logarithm of the fatigue limit, rv

y: fatigue life (logarithm of the number of cycles to failure), deterministic value

Y: fatigue life, rv

$Y|\sqrt{a_{d,0}}$: conditional fatigue life, rv

$Y_f|\sqrt{a_{d,0}}$: conditional finite fatigue life, rv

Z: standardized Normal rv

$\varphi_{LEV}(\cdot)$: pdf of a standardized Largest Extreme Value distribution

Φ_{Gauss}: cdf of a standardized Normal distribution

$\Phi_{Gauss}^{-1}(\cdot)$: inverse cdf of a standardized Normal distribution

$\mu_{\sqrt{A}}$, $\mu_{lC_{th}}$, $\mu_{lK_{th}}$, μ_{X_l}, μ_Y, $\sigma_{\sqrt{A}}$, $\sigma_{lC_{th}}$, $\sigma_{lK_{th}}$, σ_{X_l}, σ_Y: parameters of the statistical distributions

$s_{l,\alpha}$, $s_{l,\sqrt{a_{d,0}},\alpha}$, $y_{\sqrt{a_{d,0}},\alpha}$: α-th quantiles

$\tilde{\cdot}$: parameter estimate

$\cdot|\cdot$: conditional event

1. Introduction

The continuous enhancement of durability and reliability targets of many machinery components is one of the major reasons for the increasing interest in the Very-High-Cycle Fatigue (VHCF) characterization of metallic materials. Among the metallic materials, high-strength steels are widely used for critical structural applications and, due to this, a particular attention is reserved to them in the VHCF literature (e.g., Refs.[1-4]).

According to the literature[5], the VHCF response of high-strength steels is strongly affected by the presence of internal defects and, in particular, by their size. In this respect, the $\sqrt{area}$ model proposed by Murakami[5] is generally adopted to evaluate the effect of the defect size on the VHCF of high-strength steels.

According to the $\sqrt{area}$ model, the VHCF strength of high-strength steels can be estimated from the projected area of the 'Optically Dark Area' (ODA)[1,6]. The ODA is a restricted region, dark at the optical microscope, which can be seen in the vicinity of the internal defect originating the VHCF failure. In order to explain the ODA formation, a number of different theories and corresponding ODA denominations ('Fine Granular Area' (FGA)[7] or 'Granular Bright Facet' (GBF)[8]) have been proposed in the literature (see Ref.[9] for a review and the recent theory proposed by Hong et al.[10]). Despite of the different theories concerning the ODA formation, it is generally acknowledged in the literature[3,4,11,12] that crack grows from the initial defect to the boundary of the ODA even if the initial defect size is not large enough to yield a Stress Intensity Factor (SIF) value larger than the threshold SIF. In particular, according to the hydrogen embrittlement theory, originally proposed by Murakami et al.[13] and then confirmed by different researchers[14-17], hydrogen assists crack growth within the ODA[1,3,4], thus inducing crack propagation even below the threshold SIF.

Hydrogen assisted crack growth has been differently modeled in the literature (see, e.g., the recent review volumes[18,19] and the references therein). From a macromechanical point of view, the hydrogen assistance has been taken into account by modifying the Paris' law or the threshold SIFs estimated without the hydrogen assistance (e.g., Refs.[20-29]). In some cases (e.g., Refs.[21,25,26,27]), the acceleration induced by hydrogen in crack growth is modeled by vertically translating the Paris' law or, equivalently, by amplifying the SIF through a correction factor. In some other cases (e.g., Refs.[23,24,28]), hydrogen assistance is modeled by subtracting a constant factor to the threshold SIFs. Randomness, which is intrinsically present when dealing with fatigue phenomena, is generally not taken into account by the macromechanical models proposed in the literature.

Recently, Liu et al.[17] proposed to model the hydrogen assistance within the ODA by adding a SIF due to the hydrogen concentration at the crack tip to the SIF associated to the internal defect size. The model has been then used to estimate the fatigue limit of the material[30] and to predict the S-N curves in the VHCF region[31]. The additional SIF proposed by Liu et al.[17] involves a number of unknown parameters that can be hardly estimated, thus preventing from the ease of use of the model.

The present paper proposes to generalize and simplify the model discussed in[17]. The general expression for the additional SIF includes the model in[17], as well as the typical correction factors proposed in the hydrogen embrittlement literature (multiplicative correction of the SIF in Paris' law[21,25-27] or subtractive correction of the threshold SIFs[23,24,28]). As a further generalization, the intrinsic randomness associated with VHCF phenomena is also introduced and statistically treated.

2. Methods

Starting from the hydrogen assistance as a theory for the ODA formation, a general formulation for the SIF contributions is proposed in Section 2.1. A general expression for the fatigue limit is then determined in Section 2.2. The procedure for the estimation of the fatigue limit statistical distribution is discussed in Section 2.3. Section 2.4 defines a procedure for the estimation of the Probabilistic-S-N (P-S-N) curves. Finally, the method for the estimation of the parameters involved in the P-S-N curves is presented in Section 2.5.

2.1. Hydrogen assisted crack growth: a general SIF formulation

In order to define a mechanistic model for the crack growth in the VHCF region from internal defects, the following assumptions are introduced:

1. According to the $\sqrt{area}$ model[5], the threshold SIF, referred to as k_{th}, can be generally expressed as follows:
   $$k_{th} = 10^{-3} c_{th} (HV + 120) \sqrt{a_d}^{\alpha_{th}}, \tag{1}$$
   where a_d (in $\mu\mathrm{m}^2$) denotes the projected area of the defect, HV (in $\mathrm{kg_f/mm^2}$) is the Vickers hardness of the material in the vicinity of the defect and c_{th} and α_{th} are two material coefficients, being $c_{th} > 0$ (according to Ref.[5], $c_{th} = 3.3/2$; according to Ref.[1], $c_{th} = 2.77/2$; according to Ref.[32], $c_{th} = 1.87$; according to Ref.[17], $c_{th} = 1.8$) and $0 < \alpha_{th} < 1/3$ (according to Refs.[1,5,17,32], $\alpha_{th} = 1/3$; according to Refs.[4,8,12,33], $\alpha_{th} = 0$).
2. According to the $\sqrt{area}$ model[5], the SIF associated to the defect size, referred to as k_d, is given by the following expression:
   $$k_d = 10^{-3} 0.5 s \sqrt{\pi} \sqrt{a_d}^{1/2}, \tag{2}$$
   where s (in MPa) denotes the local stress amplitude in the vicinity of the crack tip.
3. The hydrogen assistance is modeled through a SIF, referred to as k_H, which has the easiest formulation able to fulfill:
   i. The principle of dimensional homogeneity, which states that k_H must be proportional to the stress and to the square-root of the defect size:
   $$k_H = c_H s \sqrt{a_d}^{1/2}. \tag{3}$$
   ii. The initial conditions, which state that, when crack starts growing, k_H must be given by:
   $$k_H = 10^{-3} c_H s \sqrt{a_{d,0}}^{1/2}, \tag{4}$$
   where $\sqrt{a_{d,0}}$ (in $\mu\mathrm{m}$) denotes the initial defect size.
   iii. The defect size dependence, which states that k_H may vary with the defect size.

   According to the three conditions i)-iii), the most general expression for k_H is finally given by:
   $$k_H = 10^{-3} c_H s \sqrt{a_{d,0}}^{1/2-\alpha_H} \sqrt{a_d}^{\alpha_H}, \tag{5}$$
   where α_H is a constant coefficient.

   It is worth noting that Eq. (5) reduces to Eq. (4) when $\sqrt{a_d} = \sqrt{a_{d,0}}$ and to Eq. (3) if $\alpha_H = 1/2$. Furthermore, Eq. (5) can model any dependence of k_H with respect to the defect size: if $\alpha_H = 0$, k_H does not depend on $\sqrt{a_d}$ (i.e., $k_H = 10^{-3} c_H s \sqrt{a_{d,0}}^{1/2}$); if $\alpha_H > 0$, k_H increases with respect to $\sqrt{a_d}$; if $\alpha_H < 0$, k_H decreases with respect to $\sqrt{a_d}$.
4. The SIF associated to the defect size and the SIF associated to the hydrogen assistance contribute to the total SIF at the crack tip, referred to as k_T, through the following additive model[17]:
   $$k_T = k_d + k_H. \tag{6}$$
5. Crack growth occurs only if $k_T > k_{th}$; vice-versa, crack growth is stopped if $k_T \leq k_{th}$.
6. Hydrogen assistance occurs only if $k_d \leq k_{th}$:
   $$k_T = \begin{cases} k_d + k_H, & if\ k_d \leq k_{th} \\ k_d, & if\ k_d > k_{th} \end{cases}. \tag{7}$$

If $k_d \leq k_{th}$ and according to Eqs. (5)-(7), k_T can be expressed as follows:

$$k_T = 10^{-3} 0.5 s \sqrt{\pi} \sqrt{a_d}^{1/2} + 10^{-3} c_H s \sqrt{a_{d,0}}^{1/2-\alpha_H} \sqrt{a_d}^{\alpha_H}. \tag{8}$$

In case $\alpha_H = 1/2$, Eq. (8) yields $k_T = \left(1 + \frac{c_H}{0.5\sqrt{\pi}}\right) k_d$, which shows that the hydrogen assistance is modeled as an amplifying multiplicative factor of k_d (SIF without the hydrogen assistance). In this respect, the case $\alpha_H = 1/2$ can be considered equivalent to the multiplicative models proposed in Refs.[21,25-27] for modeling the hydrogen assisted crack growth above the threshold SIF. It is worth noting that, according to Refs.[12,32], the Paris' law can be properly adopted to model crack growth even below the threshold SIF. Therefore, the multiplicative models proposed in Refs.[21,25-27] can be reasonably applied to the ODA region, where the SIF values are below the threshold SIF.

In case $\alpha_H = 0$, Eq. (8) yields $k_T = k_d + constant$, which shows that the hydrogen assistance is modeled as an additional constant value to k_d (SIF without the hydrogen assistance). Since the critical condition for crack growth is $k_T = k_{th}$, to add a constant value to k_d is equivalent to subtracting a constant value to k_{th} (i.e., the equation $k_T = k_d + constant \; = k_{th}$ can be rearranged as $k_d = k_{th} - constant < k_{th}$). In this respect, the case $\alpha_H = 0$ can be considered equivalent to the subtractive model proposed in Refs.[23,24,28] for taking into account the hydrogen assistance.

In case $\alpha_H = -5/2$, Eq. (8) yields $k_T = 10^{-3} 0.5 s\sqrt{\pi}\sqrt{a_d}^{1/2} + 10^{-3} c_H s \sqrt{a_{d,0}}^{3} / \sqrt{a_d}^{5/2}$, which is equivalent to the model proposed in Ref.[17] for taking into account the hydrogen assistance.

2.2. Hydrogen assisted crack growth: a general fatigue limit expression

The total SIF in Eq (8) depends on the values assumed by the exponent α_H. Let us first consider $\alpha_H \geq 0$. Three possible cases may occur when $\alpha_H \geq 0$ (i.e., k_T increases with the defect size):

a. The initial defect size is such that $k_d(\sqrt{a_{d,0}}) > k_{th}(\sqrt{a_{d,0}})$. Failure occurs and fatigue life is finite.
   In this case, crack starts growing according to the Paris' law without the hydrogen assistance from the initial defect size up to failure.
   No ODA appears on the fracture surface.
b. The initial defect size is such that $k_d(\sqrt{a_{d,0}}) \leq k_{th}(\sqrt{a_{d,0}})$ and $k_T(\sqrt{a_{d,0}}) > k_{th}(\sqrt{a_{d,0}})$. Failure occurs and fatigue life is finite.
   In this case, crack starts growing with the hydrogen assistance up to the threshold value for a crack growth without the hydrogen assistance. When the threshold value, referred to as $\sqrt{a_{ODA}}$, is reached, then crack grows up to failure according to the Paris' law and without the hydrogen assistance.
   An ODA with size equal to the threshold value appears on the fracture surface.
   The k_H is larger than zero just before the threshold value is reached (crack growth must proceed with the hydrogen assistance) and it zeroes when the threshold is reached (crack growth can proceed without the hydrogen assistance). As a consequence, k_T is larger than k_{th} just before the threshold value is reached and $k_d(\sqrt{a_{ODA}}) = k_{th}(\sqrt{a_{ODA}})$ when the threshold is reached.
c. The initial defect size is such that $k_T(\sqrt{a_{d,0}}) \leq k_{th}(\sqrt{a_{d,0}})$. No failure occurs and fatigue life is infinite.
   In this case, crack does not grow at all since the hydrogen assistance is not sufficient to induce the crack growth.
   The ODA does not form and the final defect size is equal to the initial defect size.

For a given initial defect size $\sqrt{a_{d,0}}$, the three possible cases a)-c) are separated by two different stress thresholds.

Case a) and case b) are separated by a first stress threshold, referred to as $s_{finite,\overline{ODA}}$, that represents the stress above which the ODA does not start forming in failed specimens. By taking into account Eqs. (1) and (2) and the condition $k_d\left(\sqrt{a_{d,0}}\right) = k_{th}\left(\sqrt{a_{d,0}}\right)$, it can be shown that $s_{finite,\overline{ODA}}$ takes the following form:

$$s_{finite,\overline{ODA}} = \frac{\frac{c_{th}}{0.5\sqrt{\pi}}(HV+120)}{\sqrt{a_{d,0}}^{1/2-\alpha_{th}}}. \tag{9}$$

Case b) and case c) are separated by a second stress threshold, referred to as s_l, that distinguishes between finite and infinite fatigue life (i.e., the material fatigue limit). By taking into account Eqs. (1) and (8) and the condition $k_T\left(\sqrt{a_{d,0}}\right) = k_{th}\left(\sqrt{a_{d,0}}\right)$, it can be shown that s_l takes the following form:

$$s_l = \frac{\frac{1}{0.5\sqrt{\pi}+c_H}c_{th}(HV+120)}{\sqrt{a_{d,0}}^{1/2-\alpha_{th}}} = \frac{c_{s_l,\alpha_H\geq 0}c_{th}(HV+120)}{\sqrt{a_{d,0}}^{1/2-\alpha_{th}}}, \tag{10}$$

where $c_{s_l,\alpha_H\geq 0}$ is a constant coefficient, which depends on the value assumed by c_H.

Let us now consider $\alpha_H < 0$ (i.e., k_T decreases with the defect size).

Four distinct cases may occur when $\alpha_H < 0$:

a. The initial defect size is such that $k_d\left(\sqrt{a_{d,0}}\right) > k_{th}\left(\sqrt{a_{d,0}}\right)$. Failure occurs and fatigue life is finite.
The same considerations of case a), for $\alpha_H \geq 0$, apply.
b. The initial defect size is such that $k_d\left(\sqrt{a_{d,0}}\right) \leq k_{th}\left(\sqrt{a_{d,0}}\right)$ and $k_T\left(\sqrt{a_{d,0}}\right) > k_{th}\left(\sqrt{a_{d,0}}\right)$. Failure occurs and fatigue life is finite.
The same considerations of case b), for $\alpha_H \geq 0$, apply.
c. The initial defect size is such that $k_d\left(\sqrt{a_{d,0}}\right) \leq k_{th}\left(\sqrt{a_{d,0}}\right)$ and $k_T\left(\sqrt{a_{d,0}}\right) > k_{th}\left(\sqrt{a_{d,0}}\right)$. No failure occurs and fatigue life is infinite.
In this case, crack starts growing with the hydrogen assistance but it finally arrests.
The ODA does not reach $\sqrt{a_{ODA}}$.
At the end of the ODA formation, the defect size reaches a value, referred to as $\sqrt{a_{d,final}}$, such that $k_T\left(\sqrt{a_{d,final}}\right) = k_{th}\left(\sqrt{a_{d,final}}\right)$.
d. The initial defect size is such that $k_T\left(\sqrt{a_{d,0}}\right) \leq k_{th}\left(\sqrt{a_{d,0}}\right)$. No failure occurs and fatigue life is infinite.
The same considerations of case c), for $\alpha_H \geq 0$, apply.

For a given initial defect size $\sqrt{a_{d,0}}$, the four possible cases a)-d) are separated by three different stress thresholds.

Case a) and case b) are separated by the $s_{finite,\overline{ODA}}$ given in Eq. (9).

Case c) and case d) are separated by a stress threshold, referred to as $s_{\infty,\overline{ODA}}$, that represents, for an infinite fatigue life, the stress below which the ODA does not start forming. With few passages, it can be shown that $s_{\infty,\overline{ODA}}$ takes the following form:

$$s_{\infty,\overline{ODA}} = \frac{\frac{1}{0.5\sqrt{\pi}+c_H}c_{th}(HV+120)}{\sqrt{a_{d,0}}^{1/2-\alpha_{th}}}, \tag{11}$$

which is equivalent to the expression of s_l for $\alpha_H \geq 0$ (Eq. (10)).

Finally, case b) and case c) are separated by the material fatigue limit s_l. In order to find an expression for s_l, the function $\Delta k = k_T - k_{th}$ must be taken into account and analyzed. In particular, the two following conditions must be fulfilled at the transition from infinite to finite fatigue life:

$$\begin{cases} \Delta k\left(\sqrt{a_d}\right) = 0 \\ \frac{\partial \Delta k\left(\sqrt{a_d}\right)}{\partial \sqrt{a_d}} = 0 \end{cases}. \tag{12}$$

Eq. (12) states that, in the $\left(\Delta k - \sqrt{a_d}\right)$ plane (Fig. 1), the transition from infinite to finite fatigue life occurs when, for a given stress amplitude s, the Δk curve is tangent to the $\sqrt{a_d}$ axis. In particular, the transition from infinite to finite fatigue life occurs when the initial defect size $\sqrt{a_{d,0}}$ equals the critical value $\sqrt{a_c}$ (dashed grey line in Fig. 1). If the $\sqrt{a_{d,0}}$ value is larger than $\sqrt{a_c}$, failure occurs and the Δk curve does not intersect the $\sqrt{a_d}$ axis (dotted black line in Fig. 1); while, if the $\sqrt{a_{d,0}}$ value is smaller than $\sqrt{a_c}$, no failure occurs and the Δk curve intersects the $\sqrt{a_d}$ axis at $\sqrt{a_{d,final}}$ (continuous black line in Fig. 1). The largest $\sqrt{a_{d,final}}$, referred to as $\sqrt{a_{d,final,max}}$, occurs at the transition from infinite to finite fatigue life. In case of finite fatigue life, the hydrogen assistance is present until k_d reaches k_{th} and the defect size attains the threshold defect size $\sqrt{a_{ODA}}$, according to assumption 6).

Figure 1

It can be shown (see Appendix A) that, by solving the system in Eq. (12), the value of the fatigue limit takes the following form:

$$s_l = \frac{c_{s_l,\alpha_H<0}\, c_{th}(HV+120)}{\sqrt{a_{d,0}}^{\,1/2-\alpha_{th}}}, \tag{13}$$

where $c_{s_l,\alpha_H<0}$ is a constant coefficient, which depends on the values assumed by c_H, α_{th} and α_H. It is worth noting that Eq. (13) differs from the fatigue limit expression proposed by Liu et al.[30] even if $\alpha_H = -5/2$ and $\alpha_{th} = 1/3$, as in Refs.[17,30] (see Appendix B).

Eq. (13) show that s_l is inversely proportional to $\sqrt{a_{d,0}}^{\,1/2-\alpha_{th}}$ as in Eq. (10), for the $\alpha_H \geq 0$ case. Therefore, it can be concluded that, according to the assumptions 1)-6), the fatigue limit takes the following general expression:

$$s_l = \frac{c_{s_l} c_{th}(HV+120)}{\sqrt{a_{d,0}}^{\,1/2-\alpha_{th}}}, \tag{14}$$

where c_{s_l} is a constant coefficient that must be properly estimated from the experimental dataset. It is worth noting that Eq. (14) reduces to the well-known formulation proposed by Murakami[5], if $\alpha_{th} = 1/3$ and can be thus considered as a generalization of the Murakami's model[5].

Fig. 2 shows, for the $\alpha_H < 0$ case, the three threshold stresses, $s_{finite,\overline{ODA}}$ (Eq. (9)), $s_{\infty,\overline{ODA}}$ (Eq. (11)) and s_l (Eq. (14)), as well as a schematic of the four regions a)-d) in a $\left(\log_{10}[s] - \log_{10}\left[\sqrt{a_{d,0}}\right]\right)$ plot. In the $\alpha_H \geq 0$ case, the s_l curve overlaps the $s_{\infty,\overline{ODA}}$ curve and case c) overlaps case d).

Figure 2

Fig. 2 also shows the maximum attainable limit for the fatigue strength under VHCF, which is equal to $1.6HV$, according to Murakami[5].

2.3. Statistical distribution of the fatigue limit

If, according to the literature (e.g., Refs.[34,35]), $\log_{10}[K_{th}]$ is assumed to be Normal with mean $\mu_{lK_{th}}$ and standard deviation $\sigma_{lK_{th}}$, $\log_{10}[K_{th}]$ can be expressed in terms of the distribution parameters as follows:

$$\log_{10}[K_{th}] = \mu_{lK_{th}} + \sigma_{lK_{th}} Z, \tag{15}$$

where Z is a standardized Normal random variable (rv). The mean value of $\log_{10}[K_{th}]$ must be consistent with the deterministic model proposed in Eq. (1). Therefore, according to Eq. (1), $\mu_{lK_{th}}$ is given by:

$$\mu_{lK_{th}} = \log_{10}\left[10^{-3} c_{th} (HV + 120) \sqrt{a_d}^{\alpha_{th}}\right]. \tag{16}$$

If, in Eq. (1), the deterministic k_{th} is substituted with the corresponding rv K_{th}, the deterministic c_{th} becomes a rv, too. By inverting Eq. (1) and by substituting the deterministic values with the corresponding rvs, it is possible to obtain an expression for the rv C_{th}:

$$C_{th} = K_{th}\left(10^{-3}(HV + 120)\sqrt{a_d}^{\alpha_{th}}\right)^{-1}. \tag{17}$$

By taking the logarithm of Eq. (17) and by taking into account Eq. (15), the rv $\log_{10}[C_{th}]$ becomes:

$$\log_{10}[C_{th}] = \left(\mu_{lK_{th}} + \sigma_{lK_{th}} Z\right) - \left(\log_{10}\left[10^{-3}(HV + 120)\sqrt{a_d}^{\alpha_{th}}\right]\right). \tag{18}$$

If Eq. (16) is substituted in Eq. (18), $\log_{10}[C_{th}]$ finally becomes:

$$\log_{10}[C_{th}] = \log_{10}[c_{th}] + \sigma_{lK_{th}} Z = \mu_{lC_{th}} + \sigma_{lC_{th}} Z, \tag{19}$$

which shows that $\log_{10}[C_{th}]$ is a Normal rv with mean $\mu_{lC_{th}} = \log_{10}[c_{th}]$ and standard deviation $\sigma_{lC_{th}} = \sigma_{lK_{th}}$.

If, in Eq. (14), the deterministic c_{th} is substituted with the rv C_{th}, the deterministic s_l becomes the conditional rv $S_l|\sqrt{a_{d,0}}$ (i.e., fatigue limit given the initial defect size):

$$S_l|\sqrt{a_{d,0}} = C_{th} \frac{c_{s_l}(HV+120)}{\sqrt{a_{d,0}}^{1/2-\alpha_{th}}}. \tag{20}$$

By taking the logarithm of Eq. (20) and by taking into account Eq. (19), the rv $\log_{10}\left[S_l|\sqrt{a_{d,0}}\right]$, referred to as $X_l|\sqrt{a_{d,0}}$, finally becomes:

$$X_l|\sqrt{a_{d,0}} = \log_{10}\left[\frac{c_{th} c_{s_l}(HV+120)}{\sqrt{a_{d,0}}^{1/2-\alpha_{th}}}\right] + \sigma_{lK_{th}} Z = \mu_{X_l}\left(\sqrt{a_{d,0}}\right) + \sigma_{X_l} Z. \tag{21}$$

Eq. (21) shows that $X_l|\sqrt{a_{d,0}}$ is a Normal rv with mean $\mu_{X_l}\left(\sqrt{a_{d,0}}\right) = \log_{10}\left[\frac{c_{th} c_{s_l}(HV+120)}{\sqrt{a_{d,0}}^{1/2-\alpha_{th}}}\right]$ and standard deviation $\sigma_{X_l} = \sigma_{lK_{th}}$. According to Eq. (21), the cumulative distribution function (cdf) of $X_l|\sqrt{a_{d,0}}$ is therefore given by:

$$F_{X_l|\sqrt{a_{d,0}}}\left(x_l; \sqrt{a_{d,0}}\right) = \Phi_{Gauss}\left(\frac{x_l - \mu_{X_l}\left(\sqrt{a_{d,0}}\right)}{\sigma_{X_l}}\right), \tag{22}$$

where $F_{X_l|\sqrt{a_{d,0}}}$ denotes the cdf of $X_l|\sqrt{a_{d,0}}$ and $\Phi_{Gauss}(\cdot)$ is a standardized Normal cdf.

According to Eq. (22), the α-th quantile of the conditional rv $S_l|\sqrt{a_{d,0}}$, referred to as $s_{l,\sqrt{a_{d,0}},\alpha}$, takes the following form:

$$s_{l,\sqrt{a_{d,0}},\alpha} = 10^{\mu_{X_l}\left(\sqrt{a_{d,0}}\right) + \sigma_{X_l} \Phi_{Gauss}^{-1}(\alpha)} = \frac{c_{th} c_{s_l}(HV+120)}{\sqrt{a_{d,0}}^{1/2-\alpha_{th}}} 10^{\sigma_{lK_{th}} \Phi_{Gauss}^{-1}(\alpha)}, \tag{23}$$

where $\Phi_{Gauss}^{-1}(\cdot)$ denotes the inverse of a standardized Normal cdf.

In order to define the statistical distribution of X_l (no more conditioned to the value assumed by $\sqrt{a_{d,0}}$), the randomness of the initial defect size must be taken into account. Since, following the literature[5], the initial defect size can be considered as the size of the largest defect present in the specimen, it can be assumed that the initial defect size rv, referred to as $\sqrt{A_{d,0}}$, follows a Type I Largest Extreme Value (LEV) distribution[5] with probability density function (pdf):

$$f_{\sqrt{A_{d,0}}}\left(\sqrt{a_{d,0}}\right) = \frac{\varphi_{LEV}\left(\frac{\sqrt{a_{d,0}}-\mu_{\sqrt{A}}}{\sigma_{\sqrt{A}}}\right)}{\sigma_{\sqrt{A}}} = \frac{e^{-\frac{\sqrt{a_{d,0}}-\mu_{\sqrt{A}}}{\sigma_{\sqrt{A}}}-e^{-\frac{\sqrt{a_{d,0}}-\mu_{\sqrt{A}}}{\sigma_{\sqrt{A}}}}}}{\sigma_{\sqrt{A}}}, \quad (24)$$

where $f_{\sqrt{A_{d,0}}}$ denotes the pdf of $\sqrt{A_{d,0}}$, $\varphi_{LEV}(\cdot)$ is a standardized LEV pdf and $\mu_{\sqrt{A}}$ and $\sigma_{\sqrt{A}}$ are the two constant parameters of the distribution.

The cdf of X_l can be obtained from the definition of marginal cdf:

$$F_{X_l}(x_l) = \int F_{X_l|\sqrt{a_{d,0}}}\left(x_l; \sqrt{a_{d,0}}\right) f_{\sqrt{A_{d,0}}}\left(\sqrt{a_{d,0}}\right) d\sqrt{a_{d,0}}, \quad (25)$$

where F_{X_l} denotes the cdf of X_l.

Therefore, by taking into account the assumed distributions (i.e., Eqs. (22) and (24)), Eq. (25) finally becomes:

$$F_{X_l}(x_l) = \int_0^\infty \Phi_{Gauss}\left(\frac{x_l - \mu_{X_l}\left(\sqrt{a_{d,0}}\right)}{\sigma_{X_l}}\right) \frac{\varphi_{LEV}\left(\frac{\sqrt{a_{d,0}}-\mu_{\sqrt{A}}}{\sigma_{\sqrt{A}}}\right)}{\sigma_{\sqrt{A}}} d\sqrt{a_{d,0}}. \quad (26)$$

The α-th quantile of the logarithm of the fatigue limit can be obtained by substituting $F_{X_l}(x_l)$ with α and by solving Eq. (26) with respect to x_l.

An approximate cdf of S_l, which would avoid the numerical computation of the integral in Eq. (26), can be obtained by assuming the rv $\log_{10}\left[\sqrt{A_{d,0}}\right]$ as approximately Normal with mean $\mu_{\sqrt{A}}$ and standard deviation $\sigma_{\sqrt{A}}$:

$$\log_{10}\left[\sqrt{A_{d,0}}\right] \cong \mu_{\sqrt{A}} + \sigma_{\sqrt{A}} Z. \quad (27)$$

By substituting Eq. (27) in Eq. (21), the rv X_l becomes:

$$X_l \cong \mu_{X_l}\left(10^{\mu_{\sqrt{A}}+Z\sigma_{\sqrt{A}}}\right) + \sigma_{X_l} Z = \log_{10}\left[\frac{c_{th} c_{s_l}(HV+120)}{10^{(1/2-\alpha_{th})\left(\mu_{\sqrt{A}}+Z\sigma_{\sqrt{A}}\right)}}\right] + \sigma_{lK_{th}} Z, \quad (28)$$

where the deterministic $\sqrt{a_{d,0}}$ has been substituted with the rv $\sqrt{A_{d,0}}$.

Eq. (28) can be further rearranged as follows:

$$X_l \cong \left(\log_{10}\left[c_{th} c_{s_l}(HV+120)\right] - (1/2-\alpha_{th})\mu_{\sqrt{A}}\right) + \sqrt{(1/2-\alpha_{th})^2 \sigma_{\sqrt{A}}^2 + \sigma_{lK_{th}}^2}\, Z, \quad (29)$$

by exploiting the well-known properties of a sum of independent Normal rvs.

According to Eq. (29), an approximate α-th quantile of the rv S_l, referred to as $s_{l,\alpha}$, is finally given by:

$$s_{l,\alpha} \cong \frac{c_{th} c_{s_l}(HV+120)}{10^{(1/2-\alpha_{th})\mu_{\sqrt{A}}}} 10^{\sqrt{(1/2-\alpha_{th})^2 \sigma_{\sqrt{A}}^2 + \sigma_{lK_{th}}^2}\,\Phi_{Gauss}^{-1}(\alpha)}. \quad (30)$$

2.4. P-S-N curves

The Probabilistic-S-N (P-S-N) curves statistically model the VHCF material response in the fatigue limit region and in the finite fatigue life region.

The cdfs in Eqs. (22) and (26) model the randomness in the fatigue limit region. As for the finite fatigue life region, different types of continuous distribution have been proposed in the literature (see, e.g., Ref.[36] and the references therein) for the number of cycles to failure, referred to as N_f. Without loss of generality, the finite fatigue life rv, referred to as $Y_f = \log_{10}[N_f]$, is supposed to be Normal distributed. Therefore, let us suppose that the conditional finite fatigue life, referred to as $Y_f|\sqrt{a_{d,0}}$ (i.e., finite fatigue life given the initial defect size), has mean μ_Y and standard deviation σ_Y, then:

$$F_{Y_f|\sqrt{a_{d,0}}} = \Phi_{Gauss}\left(\frac{y-\mu_Y}{\sigma_Y}\right), \tag{31}$$

where $F_{Y_f|\sqrt{a_{d,0}}}$ denotes the cdf of $Y_f|\sqrt{a_{d,0}}$.

It is well-known that the parameters of $F_{Y_f|\sqrt{a_{d,0}}}$ depend on the applied stress amplitude and on the initial defect size[37-39]. In particular, any monotonic decreasing function of s and $\sqrt{a_{d,0}}$ can be adopted for the mean and any positive function can be used for the standard deviation. In the most simple case, the standard deviation is constant and the mean is a linear function of both the logarithm of the applied stress amplitude (Basquin's model), referred to as $x = \log_{10}[s]$, and the logarithm of the initial defect size, $\log_{10}[\sqrt{a_{d,0}}]$:

$$\mu_Y(x, \sqrt{a_{d,0}}) = c_Y + m_Y x + n_Y \log_{10}[\sqrt{a_{d,0}}], \tag{32}$$

where c_Y, m_Y and n_Y are three constant parameters.

According to the probabilistic model "One failure mode due to one cause with fatigue limit" described in Ref.[36], when the logarithm of the applied stress is equal to x and the initial defect size is equal to $\sqrt{a_{d,0}}$, the cdf of the conditional fatigue life $Y|\sqrt{a_{d,0}}$ is given by:

$$F_{Y|\sqrt{a_{d,0}}}(y; x, \sqrt{a_{d,0}}) = F_{X_l|\sqrt{a_{d,0}}}(x, \sqrt{a_{d,0}}) F_{Y_f|\sqrt{a_{d,0}}}(y; \sqrt{a_{d,0}}), \tag{33}$$

where $F_{Y|\sqrt{a_{d,0}}}$ denotes the cdf of $Y|\sqrt{a_{d,0}}$.

By taking into account the assumed distributions (Eqs. (22) and (31)), Eq. (33) becomes:

$$F_{Y|\sqrt{a_{d,0}}}(y; x, \sqrt{a_{d,0}}) = \Phi_{Gauss}\left(\frac{x-\mu_{X_l}(\sqrt{a_{d,0}})}{\sigma_{X_l}}\right) \Phi_{Gauss}\left(\frac{y-\mu_Y(x,\sqrt{a_{d,0}})}{\sigma_Y}\right). \tag{34}$$

The α-th quantile of the conditional fatigue life, referred to as $y_{\sqrt{a_{d,0}},\alpha}$, can be obtained by substituting $F_{Y|\sqrt{a_{d,0}}}(y; x, \sqrt{a_{d,0}})$ with α and by solving the equation with respect to y for different values of x:

$$y_{\sqrt{a_{d,0}},\alpha} = \mu_Y(x, \sqrt{a_{d,0}}) + \sigma_Y \Phi_{Gauss}^{-1}\left(\alpha / \Phi_{Gauss}\left(\frac{x-\mu_{X_l}(\sqrt{a_{d,0}})}{\sigma_{X_l}}\right)\right), \tag{35}$$

where x must be larger than $\mu_{X_l}(\sqrt{a_{d,0}}) + \sigma_{X_l}\Phi_{Gauss}^{-1}(\alpha)$ in order to have finite values of $y_{\sqrt{a_{d,0}},\alpha}$. Eq. (35) thus provides the P-S-N curves given the initial defect size.

In order to define the statistical distribution of Y (no more conditioned to the value assumed by $\sqrt{a_{d,0}}$), the pdf of the initial defect size in Eq. (24) must be taken into account.

The cdf of Y can be obtained from the definition of marginal cdf:

$$F_Y(y; x) = \int F_{Y|\sqrt{a_{d,0}}}(y; x, \sqrt{a_{d,0}}) f_{\sqrt{A_{d,0}}}(\sqrt{a_{d,0}}) d\sqrt{a_{d,0}}. \tag{36}$$

Therefore, by taking into account the assumed distributions (Eqs. (24) and Eq. (34)), Eq. (36) finally becomes:

$$F_Y(y;x) = \int_0^\infty \Phi_{Gauss}\left(\frac{x-\mu_{X_l}(\sqrt{a_{d,0}})}{\sigma_{X_l}}\right)\Phi_{Gauss}\left(\frac{y-\mu_Y(x,\sqrt{a_{d,0}})}{\sigma_Y}\right)\frac{\varphi_{LEV}\left(\frac{\sqrt{a_{d,0}}-\mu_{\sqrt{A}}}{\sigma_{\sqrt{A}}}\right)}{\sigma_{\sqrt{A}}}\, d\sqrt{a_{d,0}}. \tag{37}$$

The α-th quantile of the fatigue life can be obtained by substituting $F_Y(y;x)$ with α and by solving the equation with respect to x for different values of y. Eq. (37) thus provides the P-S-N curves of the material.

2.5. Parameter estimation

The cdf of the conditional rv $X_l|\sqrt{a_{d,0}}$ in Eq. (22) depends on the coefficients c_{th}, α_{th}, $\sigma_{lK_{th}}$ and c_{s_l}. The coefficients c_{th}, α_{th} and $\sigma_{lK_{th}}$ can be estimated by considering that, for each failed specimen exhibiting an ODA with projected area a_{ODA}, $k_d(\sqrt{a_{ODA}}) = k_{th}(\sqrt{a_{ODA}})$ when failure occurs. In particular, let $a_{ODA,i}$ denote the projected ODA area of the i-th specimen tested at s_i then, according to Eq. (1) and Eq. (2):

$$0.5 s_i \sqrt{\pi}\sqrt{a_{ODA,i}}^{1/2} = c_{th}(HV+120)\sqrt{a_{ODA,i}}^{\alpha_{th}}. \tag{38}$$

The coefficients c_{th}, α_{th} and $\sigma_{lK_{th}}$ can be estimated from the model in Eq. (38) through application of the Least Squares method. By taking the logarithm of Eq. (38) and by considering $\log_{10}[K_{th}]$ as the experimental response and $\log_{10}[\sqrt{a_{ODA}}]$ as the explanatory variable, the linear model for the expectation of $\log_{10}[K_{th}]$ is given by:

$$E[\log_{10}[K_{th}]] = c_T + \alpha_{th}\log_{10}[\sqrt{a_{ODA}}], \tag{39}$$

where $E[\cdot]$ denotes the expectation of the experimental response and $c_T = \log_{10}[c_{th}(HV+120)]$. As an easy application of the Least Squares method, c_T, α_{th} and $\sigma_{lK_{th}}$ can be estimated from the n_f failures and, finally, the estimates of α_{th}, c_{th} and $\sigma_{lK_{th}}$ can be obtained through the following expressions:

$$\begin{cases} \widetilde{\alpha_{th}} = \dfrac{\sum_{i=1}^{n_f}\left(\log_{10}[k_{th,i}]-\overline{\log_{10}[k_{th}]}\right)\left(\log_{10}[\sqrt{a_{ODA,i}}]-\overline{\log_{10}[\sqrt{a_{ODA}}]}\right)}{\sum_{i=1}^{n_f}\left(\log_{10}[\sqrt{a_{ODA,i}}]-\overline{\log_{10}[\sqrt{a_{ODA}}]}\right)^2} \\ \widetilde{c_{th}} = \dfrac{10^{\widetilde{c_T}}}{HV+120} = \dfrac{10^{\overline{\log_{10}[k_{th}]}-\widetilde{\alpha_{th}}\,\overline{\log_{10}[\sqrt{a_{ODA}}]}}}{HV+120} \\ \widetilde{\sigma_{lK_{th}}} = \sqrt{\dfrac{\sum_{i=1}^{n_f}\left(\log_{10}[k_{th,i}]-\log_{10}\left[\widetilde{c_{th}}(HV+120)\sqrt{a_{ODA,i}}^{\widetilde{\alpha_{th}}}\right]\right)^2}{n_f-2}} \end{cases}, \tag{40}$$

where $\tilde{\cdot}$ denotes the parameter estimate, $k_{th,i} = 0.5 s_i\sqrt{\pi}\sqrt{a_{ODA,i}}^{1/2}$, $\overline{\log_{10}[k_{th}]} = \sum_{i=1}^{n_f}\log_{10}[k_{th,i}]/n_f$ and $\overline{\log_{10}[\sqrt{a_{ODA}}]} = \sum_{i=1}^{n_f}\log_{10}[\sqrt{a_{ODA,i}}]/n_f$.

As for the parameter c_{s_l}, the Maximum Likelihood Principle can be adopted as estimation method. In order to define the Likelihood function, the probability associated to the experimental dataset must be defined. Let us first consider the probability of having a failure. If a failure occurs at s when the initial defect size is $\sqrt{a_{d,0}}$, then the applied stress s is necessarily larger than the conditional fatigue limit $S_l|\sqrt{a_{d,0}}$. Therefore, by taking the logarithm, when a failure occurs the event $X_l|\sqrt{a_{d,0}} < x$ occurs and the probability of having a failure at s when the initial defect size is $\sqrt{a_{d,0}}$ is finally given by:

$$P\left[\text{failure at } s, \text{given } \sqrt{a_{d,0}}\right] = P\left[X_l|\sqrt{a_{d,0}} < x\right] = F_{X_l|\sqrt{a_{d,0}}}(x). \tag{41}$$

Otherwise, two possible causes may originate a runout at y_r when the initial defect size is $\sqrt{a_{d,0}}$ and the specimen is tested at s: either the conditional fatigue life is infinite or the conditional fatigue life is finite but the test is stopped at y_r before failure occurs. From a probabilistic point of view, a runout at y_r when the initial defect size is $\sqrt{a_{d,0}}$ and the specimen is tested at s can be described as follows:

$$P\left[\text{runout at } (y_r; s), \text{given } \sqrt{a_{d,0}}\right] = P\left[\left(X_l|\sqrt{a_{d,0}} < x\right) \cap \left(Y_f\middle|\sqrt{a_{d,0}} > y_r\right)\right] + P\left[X_l|\sqrt{a_{d,0}} \geq x\right]. \quad (42)$$

By taking into account the definition of cdf of $X_l|\sqrt{a_{d,0}}$ and $Y_f\left|\sqrt{a_{d,0}}\right.$, Eq. (42) becomes:

$$P\left[\text{runout at } (y_r; s), \text{given } \sqrt{a_{d,0}}\right] = F_{X_l|\sqrt{a_{d,0}}}(x)\left(1 - F_{Y_f|\sqrt{a_{d,0}}}(y_r)\right) + \left(1 - F_{X_l|\sqrt{a_{d,0}}}(x)\right). \quad (43)$$

With few passages, Eq. (43) can be finally rearranged as follows:

$$P\left[\text{runout at } (y_r; s), \text{given } \sqrt{a_{d,0}}\right] = 1 - F_{X_l|\sqrt{a_{d,0}}}(x) F_{Y_f|\sqrt{a_{d,0}}}(y_r). \quad (44)$$

The Likelihood function L associated to the experimental dataset can be finally computed from the failure (Eq. (41)) and runout (Eq. (44)) probabilities of each specimen as follows:

$$L = \prod_{i=1}^{n_f} P\left[\text{failure at } s_i, \text{given } \sqrt{a_{d,0,i}}\right] \cdot \prod_{j=1}^{n_r} P\left[\text{runout at } \left(y_r; s_j\right), \text{given } \sqrt{a_{d,0,j}}\right], \quad (45)$$

where n_r is the number of runout specimens, $\left(s_i, \sqrt{a_{d,0,i}}\right)$ is the couple associated to the i-th failed specimen and $\left(y_r; s_j, \sqrt{a_{d,0,j}}\right)$ is the triplet associated to the j-the runout specimen. It is worth noting that, for the n_r runout specimens, the fracture surfaces cannot be analyzed and, consequently, the initial defect size is not known at the end of test. However, $\sqrt{a_{d,0,j}}$ must be measured for the computation of the Likelihood function. Therefore, in order to have a measure of the initial defect size in runout specimens, the specimens must be subsequently tested up to failure at a larger stress amplitude.

By taking into account the assumed distributions (Eq. (22) and Eq. (31)), the Likelihood function in Eq. (45) becomes:

$$L\left(c_{s_l}\right) = \prod_{i=1}^{n_f} \Phi_{Gauss}\left(\frac{x_i - \mu_{X_l}\left(c_{s_l}; \sqrt{a_{d,0,i}}\right)}{\widetilde{\sigma_{X_l}}}\right) \prod_{j=1}^{n_r}\left(1 - \Phi_{Gauss}\left(\frac{x_j - \mu_{X_l}\left(c_{s_l}; \sqrt{a_{d,0,j}}\right)}{\widetilde{\sigma_{X_l}}}\right) \Phi_{Gauss}\left(\frac{y_r - \widetilde{\mu_Y}\left(x_j, \sqrt{a_{d,0,j}}\right)}{\widetilde{\sigma_Y}}\right)\right), \quad (46)$$

where, according to the plug-in principle (see, e.g., Ref.[40] and the references therein), $\widetilde{\mu_Y}\left(x_j, \sqrt{a_{d,0,j}}\right)$, $\widetilde{\sigma_Y}$ and $\widetilde{\sigma_{X_l}}$ are the Least Squares estimates of $\mu_Y\left(x_j, \sqrt{a_{d,0,j}}\right)$, σ_Y and σ_{X_l} and $\mu_{X_l}\left(c_{s_l}; \sqrt{a_{d,0,\cdot}}\right) = \log_{10}\left[c_{s_l} \frac{\widetilde{c_{th}}(HV+120)}{\sqrt{a_{d,0,\cdot}}^{1/2-\widetilde{\alpha_{th}}}}\right]$, being $\widetilde{c_{th}}$ and $\widetilde{\alpha_{th}}$ the Least Squares estimates of c_{th} and α_{th}.

The estimates of $\mu_Y\left(x_j, \sqrt{a_{d,0,j}}\right)$ and σ_Y can be obtained by applying the Least Squares estimation method to the failure data. In particular, let $Y_f\left|\sqrt{a_{d,0}}\right.$ be considered as the response variable and x and $\log_{10}\left[\sqrt{a_{d,0}}\right]$ the two explanatory variables of the model (Eq. (32)), then:

$$E\left[Y_f\middle|\sqrt{a_{d,0}}\right] = c_Y + m_Y x + n_Y \log_{10}\left[\sqrt{a_{d,0}}\right]. \quad (47)$$

The parameters c_Y, m_Y and n_Y of the multiple linear regression in Eq. (47) can be estimated from the n_f failures, through the Least Squares Method:

$$\begin{Bmatrix} \widetilde{c_Y} \\ \widetilde{m_Y} \\ \widetilde{n_Y} \end{Bmatrix} = (\boldsymbol{T}'\boldsymbol{T})^{-1}\boldsymbol{T}'\boldsymbol{Y}, \quad (48)$$

where $\boldsymbol{T}=\begin{bmatrix}1 & x_1 & \log_{10}\left[\sqrt{a_{d,0,1}}\right]\\ \cdots & \cdots & \cdots\\ 1 & x_{n_f} & \log_{10}\left[\sqrt{a_{d,0,n_f}}\right]\end{bmatrix}$ is the design matrix of the multiple linear regression, $\boldsymbol{T}'$ is the transpose of $\boldsymbol{T}$ and $\boldsymbol{Y}=\begin{Bmatrix}y_1\\ \cdots\\ y_{n_f}\end{Bmatrix}$ is the vector of the experimental responses, being y_i $(i=1,\dots,n_f)$ the fatigue life of the i-th failed specimen. According to the plug-in principle, the parameter estimate of $\mu_Y\left(x_j,\sqrt{a_{d,0,j}}\right)$ is therefore given by:

$$\widetilde{\mu_Y}\left(x_j,\sqrt{a_{d,0,j}}\right)=\widetilde{c_Y}+\widetilde{m_Y}x_j+\widetilde{n_Y}\log_{10}\left[\sqrt{a_{d,0,j}}\right]. \tag{49}$$

The Least Squares estimate of σ_Y is the Root Mean Square Error associated to the regression model:

$$\widetilde{\sigma_Y}=\sqrt{\frac{\sum_{i=1}^{n_f}\left(y_i-\widetilde{\mu_Y}\left(x_i,\sqrt{a_{d,0,i}}\right)\right)^2}{n_f-3}}. \tag{50}$$

The value of c_{s_l} that maximizes the Likelihood function $L\left(c_{s_l}\right)$ in Eq. (46) provides the Maximum Likelihood estimate $\widetilde{c_{s_l}}$ and also permits the estimation of μ_{X_l}:

$$\widetilde{\mu_{X_l}}\left(\sqrt{a_{d,0}}\right)=\log_{10}\left[\frac{\widetilde{c_{s_l}}\,\widetilde{c_{th}}(HV+120)}{\sqrt{a_{d,0}}^{1/2-\widetilde{\alpha_{th}}}}\right]. \tag{51}$$

According to the plug-in principle, the α-th quantiles in Eqs. (23) and (35) can be estimated by substituting the parameters with the corresponding estimates.

The approximate α-th quantile in Eq. (30) requires the estimation of the parameters $\mu_{\sqrt{A}}$ and $\sigma_{\sqrt{A}}$. Since the rv $\log_{10}\left[\sqrt{A_{d,0}}\right]$ is approximately assumed to be Normal, the estimates $\widetilde{\mu_{\sqrt{A}}}$ and $\widetilde{\sigma_{\sqrt{A}}}$ are respectively given by the sample mean and sample standard deviation of the logarithm of the initial defect sizes. By substituting in Eq. (30) the parameters $\mu_{\sqrt{A}}$ and $\sigma_{\sqrt{A}}$ with the corresponding estimates, the approximate α-th quantile of S_l can be finally estimated.

In order to estimate the cdfs in Eqs. (26) and (37), the parameters $\mu_{\sqrt{A}}$ and $\sigma_{\sqrt{A}}$ must be estimated through a Gumbel plot of the initial defect sizes[5]. By substituting in Eqs. (26) and (37) the parameters with the corresponding estimates, the α-th quantiles of X_l and Y can be finally estimated.

3. Numerical application to an experimental dataset

In order to show the steps that must be followed to estimate the model parameters and the α-th quantiles, the procedure explained in Section 2.5 will be applied to an experimental dataset.

VHCF tests have been carried out on Gaussian specimens[41] made of an AISI H13 steel with Vickers hardness $560\ \mathrm{kg_f/mm^2}$. Details on the testing setup and on the tested material are reported in Refs.[42,43] and will not be recalled here for the sake of brevity. Specimens were loaded at a constant stress amplitude until failure or up to 10^{10} cycles (runout specimens). Eighteen out of twenty Gaussian specimens failed during the VHCF tests at a number of cycles to failure ranging from $4.2\cdot 10^7$ to $9.6\cdot 10^9$ cycles. After the VHCF test, the two runout specimens were subsequently tested up to failure at larger stress amplitudes in order to reveal the possible presence of defects. Fracture surfaces were seen through a Scanning-Electron-Microscope (SEM) in order to measure the initial defect size in each specimen and through an optical microscope in order to measure the ODA size in failed specimens. From the SEM analysis, all the fatigue fractures nucleated from non-metallic inclusions (oxide-type inclusions).

The location of the initial defects in the specimens was taken into consideration, in order to assess the local stress amplitude in the vicinity of the initial defect. Fig. 3 shows the distribution of the initial defects with respect to the longitudinal axis of the specimen. From the initial defect locations, it is possible to estimate the normalized local stress amplitudes s_{local}/s_{max}, being s_{max} the nominal maximum stress amplitude applied during the test. As shown in Fig. 3, all the failures occurred for stress amplitudes larger than the 96% of the nominal maximum stress amplitude applied during the test. The maximum volume of material in which failures occurred was estimated to be equal to $2013\ \mathrm{mm}^3$, more than two times larger than the maximum risk-volume investigated in the VHCF literature[44].

Figure 3

In the following, the local stress amplitude in the vicinity of the initial defect will be considered as the stress amplitude applied during the test. As shown in the S-N plot of the experimental dataset (Fig. 4), the local stress amplitudes are in the range $473 - 630\ \mathrm{MPa}$.

Figure 4

As a first step, the parameters α_{th}, c_{th} and σ_{X_l} are estimated from the ODA sizes measured in failed specimens. As shown in Fig. 5, the linear model of Eq. (39) is in good agreement with the experimental data. In particular, according to Eq. (39), the Least Squares estimates of α_{th}, c_{th} and σ_{X_l} are given by:

$$\begin{cases} \widetilde{\alpha_{th}} = 0.2965 \\ \widetilde{c_{th}} = 1.9054 \\ \widetilde{\sigma_{X_l}} = \widetilde{\sigma_{lK_{th}}} = 0.0214 \end{cases}, \tag{52}$$

where the $\widetilde{\alpha_{th}}$ and $\widetilde{c_{th}}$ values are close to the values proposed in the literature for α_{th} (i.e., $1/3$ according to Refs.[1,5,17,32]) and for c_{th} (i.e., 1.87 according to Ref.[32], and 1.8 according to Ref.[17]).

Figure 5

As a second step, the parameters c_Y, m_Y, n_Y and σ_Y are estimated from the triplet $\left(y; s, \sqrt{a_{d,0}}\right)$ associated to every failed specimen. Fig. 6 shows the number of cycles as a function of the variable $s \cdot \sqrt{a_{d,0}}^{\,n_Y/m_Y}$ in a double-logarithmic plot. As shown in Fig. 6, the assumed linear model of Eq. (47) is in good agreement with the experimental data. In particular, according to Eqs. (48) and (50), the Least Squares estimates of c_Y, m_Y, n_Y and σ_Y are given by:

$$\begin{cases} \widetilde{c_Y} = 56.9259 \\ \widetilde{m_Y} = -16.4492 \\ \widetilde{n_Y} = -1.7990 \\ \widetilde{\sigma_Y} = 0.3559 \end{cases}, \tag{53}$$

where the ratio $\widetilde{n_Y}/\widetilde{m_Y} \cong 1/9$ is close to the $1/6$ value proposed by Murakami[5].

Figure 6

As a third step, the parameter c_{s_l}, is estimated through the Maximum Likelihood Principle, by maximizing the Likelihood function in Eq. (46). The numerical maximization is carried out in Matlab® and provides for c_{s_l} the following estimate:

$$\widetilde{c_{s_l}} = 0.7278, \tag{54}$$

which finally permits the estimation of $\mu_{X_l}\left(\sqrt{a_{d,0}}\right)$, according to Eq. (51):

$$\widetilde{\mu_{X_l}}\left(\sqrt{a_{d,0}}\right) = \log_{10}\left[\frac{1.3868\cdot(HV+120)}{\sqrt{a_{d,0}}^{0.2035}}\right]. \tag{55}$$

Eq. (55) has a structure similar to the expression of fatigue limit proposed by Murakami[5]. Moreover, the exponent $0.2035 \cong 1/5$ is close to the $1/6$ proposed by Murakami[5].

The estimates $\widetilde{\sigma_{X_l}}$ and $\widetilde{\mu_{X_l}}\left(\sqrt{a_{d,0}}\right)$ can be used for estimating the α-th quantile of the conditional rv $S_l|\sqrt{a_{d,0}}$ from Eq. (23-25), according to the plug-in principle. Fig. 7 shows a plot of the estimated 0.1-th, 0.5-th and 0.9-th quantiles of the conditional fatigue limit, as a function of the initial defect size $\sqrt{a_{d,0}}$. As shown in Fig. 7, all failure data are above the 0.9 quantile of the conditional fatigue limit, in agreement with the definition of fatigue limit.

Figure 7

The median S-N curve for a given initial defect size can be estimated from Eq. (35), by substituting the parameter with the corresponding estimates and α with 0.5. Fig. 8 shows a plot of the estimated median S-N curves, for values of the initial defect size ranging from $18.6\ \mu\mathrm{m}$ to $56.3\ \mu\mathrm{m}$. As shown in Fig. 8, for each initial defect size a specific finite life and a specific fatigue limit can be estimated. In particular, the larger the initial defect size the smaller the median fatigue strength.

Figure 8

For a given initial defect size, the S-N curves can be also estimated at failure probabilities different from the median from Eq. (35). As an example, Fig. 9 shows a plot of the estimated 0.1-th, 0.5-th and 0.9-th S-N curves for an initial defect size equal to $32.4\ \mu\mathrm{m}$.

Figure 9

In order to estimate the marginal fatigue limit distribution and the marginal P-S-N curves, the statistical distribution of the initial defect size must be estimated through a Gumbel plot. Fig. 10 shows the Gumbel plot of the initial defect sizes. According to the procedure suggested in Ref.[5], the two parameters of the LEV distribution can be easily estimated as follows:

$$\begin{cases} \widetilde{\mu_{\sqrt{A}}} = 32.1697 \\ \widetilde{\sigma_{\sqrt{A}}} = 9.7799 \end{cases}, \tag{56}$$

Figure 10

Given the statistical distribution of the initial defect size, it is finally possible to estimate the P-S-N curves for different values of the failure probability. Fig. 11 shows the estimated 0.1-th, 0.5-th and 0.9-th S-N curves for the tested specimens. As shown in Fig. 11 the region between the 0.1-th and the 0.9-th S–N curve includes about the 88% (which is close to the expected 80%) of the failure data; while the 0.5-th S–N curve is almost median between failure data at each stress amplitude.

Figure 11

By taking into account Eqs. (26) and (30), it is also possible to estimate the fatigue limit for the tested specimens. As shown in Fig. 12, the two expressions provide very similar results in terms of interval plot: according to Eq. (26), the 80% interval plot is $[410; 505]$ MPa with median 455 MPa; according to Eq. (30), the 80% interval plot is $[412; 503]$ MPa with median 455 MPa. It is worth noting that the approximate expression of Eq. (30) is computed without any numerical integration: the parameters of the approximate distribution of the initial defect size are estimated by computing the sample mean and standard deviation of the logarithm of the initial defect sizes:

$$\begin{cases} \widetilde{\mu_{\sqrt{A}}} = 1.5544 \\ \widetilde{\sigma_{\sqrt{A}}} = 0.1282' \end{cases} \quad (57)$$

Figure 12

For the sake of comparison, Fig. 12 also shows the interval plots computed for two different fatigue limit models proposed in the VHCF literature. Differently from the models proposed in the present paper and from the Murakami's model[1], which are in good agreement with the experimental data, the model proposed by Liu et al.[30] provides risky non-conservative results. In particular, all failed specimens are above the upper confidence limit computed from Eqs. (26) and (30), 15 out of 18 failed specimens are above the upper confidence limit computed according from the Murakami's model[1], and no failed specimen is above the upper confidence limit computed from the model proposed by Liu et al.[30].

4. Conclusions

A general SIF formulation for the analytical model of the hydrogen assisted crack growth was defined in the paper. The proposed formulation is based on general basic assumptions related to the hydrogen assistance in ODA formation and includes different models proposed in the Very-High-Cycle Fatigue literature and in the hydrogen embrittlement literature.

Starting from the general SIF formulation, a general expression for the material fatigue limit was obtained in the paper.

The procedures for modeling the randomness of the fatigue limit and the Probabilistic-S-N (P-S-N) curves were also provided. The statistical method for the estimation of the parameters involved in the proposed model was finally illustrated in the paper and numerically applied to an experimental dataset.

The numerical application showed the potentialities of the proposed approach in terms of estimated statistical results (P-S-N curves and fatigue limit distribution). It also highlighted the ease of application of the method: it was shown that model parameters can be easily estimated step by step through straightforward applications of the Least Squares Method and of the Maximum Likelihood Principle to experimental data.

Appendix A: Solution of the system in Equation (12)

By substituting Eqs. (1), (2) and (5) in Eq. (12), the system of equations becomes:

$$\begin{cases} \left(0.5\sqrt{\pi}\sqrt{a_d}^{1/2} + c_H\sqrt{a_{d,0}}^{1/2-\alpha_H}\sqrt{a_d}^{\alpha_H}\right)s - c_{th}(HV+120)\sqrt{a_d}^{\alpha_{th}} = 0 \\ \left(1/2\,0.5\sqrt{\pi}\sqrt{a_d}^{-1/2} + \alpha_H c_H\sqrt{a_{d,0}}^{1/2-\alpha_H}\sqrt{a_d}^{\alpha_H-1}\right)s - c_{th}(HV+120)\alpha_{th}\sqrt{a_d}^{\alpha_{th}-1} = 0 \end{cases}. \tag{A.1}$$

The stress amplitude can be expressed as a function of the defect size from the first of the two equations in Eq. (A.1):

$$s = \frac{c_{th}(HV+120)}{0.5\sqrt{\pi}\sqrt{a_d}^{1/2-\alpha_{th}} + c_H\sqrt{a_{d,0}}^{1/2-\alpha_H}\sqrt{a_d}^{\alpha_H-\alpha_{th}}}. \tag{A.2}$$

If Eq. (A.2) is substituted in the second of the two equations in Eq. (A.1), it yields:

$$\frac{1/2 0.5\sqrt{\pi}\sqrt{a_d}^{-1/2} + \alpha_H c_H\sqrt{a_{d,0}}^{1/2-\alpha_H}\sqrt{a_d}^{\alpha_H-1}}{0.5\sqrt{\pi}\sqrt{a_d}^{1/2-\alpha_{th}} + c_H\sqrt{a_{d,0}}^{1/2-\alpha_H}\sqrt{a_d}^{\alpha_H-\alpha_{th}}} = \alpha_{th}\sqrt{a_d}^{\alpha_{th}-1}. \tag{A.3}$$

With few passages, the largest final defect size can be obtained from Eq. (A.3):

$$\sqrt{a_{d,final,max}} = \left(\frac{(\alpha_{th}-\alpha_H)c_H}{(1/2-\alpha_{th})0.5\sqrt{\pi}}\right)^{\frac{1}{1/2-\alpha_H}}\sqrt{a_{d,0}}, \tag{A.4}$$

which shows that the largest final defect size is proportional to the initial defect size.

Finally, by substituting Eq. (A.4) in Eq. (A.2) it is possible to show that the fatigue limit takes the following form:

$$s_l = \frac{\left(\frac{(1/2-\alpha_{th})0.5\sqrt{\pi}}{(\alpha_{th}-\alpha_H)c_H}\right)^{\frac{1/2-\alpha_{th}}{1/2-\alpha_H}}\frac{\alpha_{th}-\alpha_H}{0.5\sqrt{\pi}(1/2-\alpha_H)}c_{th}(HV+120)}{\sqrt{a_{d,0}}^{1/2-\alpha_{th}}} = \frac{c_{s_l,\alpha_H<0}c_{th}(HV+120)}{\sqrt{a_{d,0}}^{1/2-\alpha_{th}}}, \tag{A.5}$$

where $c_{s_l,\alpha_H<0}$ is a constant coefficient, which depends on the values assumed by c_{th}, c_H, α_{th} and α_H.

Appendix B: Numerical difference between Equation (13) and the fatigue limit expression in Liu[30]

According to Liu's model[17,30], $\alpha_H = -5/2$. By substituting $\alpha_H = -5/2$ in Eq. (13), the fatigue limit expression becomes:

$$s_l = \frac{\left(\frac{(1/2-\alpha_{th})0.5\sqrt{\pi}}{(\alpha_{th}+5/2)c_H}\right)^{\frac{1/2-\alpha_{th}}{3}} \frac{\alpha_{th}+5/2}{1.5\sqrt{\pi}} c_{th}(HV+120)}{\sqrt{a_{d,0}}^{1/2-\alpha_{th}}}. \tag{B.1}$$

It is worth noting that the fatigue limit expression in Eq. (B.1) differs from the expression proposed by Liu et al.[30]. The difference is due to a different fatigue limit condition considered in Ref.[30]: in Ref.[30] the critical transition between finite and infinite fatigue life occurs when the condition $k_d(\sqrt{a_{ODA}}) = k_{th}(\sqrt{a_{ODA}})$ is fulfilled with a k_H negligible and equal to a small quantity. Liu et al.[30] considered as negligible a value of k_H equal to $0.01k_d$. According to Liu et al.[30], the fatigue limit must thus fulfill the following system of equations:

$$\begin{cases} k_d(\sqrt{a_{ODA}}) = k_{th}(\sqrt{a_{ODA}}) \\ k_H(\sqrt{a_{ODA}}) = 0.01k_d(\sqrt{a_{ODA}}) \end{cases}. \tag{B.2}$$

By substituting Eqs. (1), (2) and (5) in Eq. (B.2), the system of equations becomes after few passages:

$$\begin{cases} 0.5s\sqrt{\pi}\sqrt{a_{ODA}}^{1/2-\alpha_{th}} = c_{th}(HV+120) \\ c_H\sqrt{a_{d,0}}^{3} = 5\cdot 10^{-3}\sqrt{\pi}\sqrt{a_{ODA}}^{3} \end{cases}, \tag{B.2}$$

where α_H has been substituted with $-5/2$.

The $\sqrt{a_{ODA}}$ can be expressed as a function of $\sqrt{a_{d,0}}$ from the second of the two equations in Eq. (B.2):

$$\sqrt{a_{ODA}} = \left(\frac{c_H}{5\cdot 10^{-3}\sqrt{\pi}}\right)^{\frac{1}{3}}\sqrt{a_{d,0}}. \tag{B.3}$$

If Eq. (B.3) is substituted in the first of the two equations in Eq. (B.2), it yields:

$$0.5s\sqrt{\pi}\left(\frac{c_H}{5\cdot 10^{-3}\sqrt{\pi}}\right)^{\frac{1/2-\alpha_{th}}{3}}\sqrt{a_{d,0}}^{1/2-\alpha_{th}} = c_{th}(HV+120), \tag{B.4}$$

With few passages, the fatigue limit expression according to Liu et al.[30] can be obtained from Eq. (B.4):

$$s_{l,Liu} = \frac{\frac{\left(\frac{5\cdot 10^{-3}\sqrt{\pi}}{c_H}\right)^{\frac{1/2-\alpha_{th}}{3}}}{0.5\sqrt{\pi}} c_{th}(HV+120)}{\sqrt{a_{d,0}}^{1/2-\alpha_{th}}}. \tag{B.5}$$

By taking the ratio of Eqs. (B.1) and (B.5), it can be shown that:

$$\frac{s_l}{s_{l,Liu}} = \left(0.01^{-1}\frac{1/2-\alpha_{th}}{5/2+\alpha_{th}}\right)^{\frac{1/2-\alpha_{th}}{3}}\frac{5/2+\alpha_{th}}{3}, \tag{B.6}$$

which is larger than 1 for any positive value of α_{th} below $1/3$. In particular, the maximum value (equal to 1.37) of the ratio in Eq. (B.6) is reached when $\alpha_{th} = 0$ and the minimum value (equal to 1.04) is reached when $\alpha_{th} = 1/3$. In any case, $s_{l,Liu}$ is below s_l. In this respect, $s_{l,Liu}$ is a conservative estimate of the fatigue limit that can be directly computed from the model proposed in Ref.[17]. The conservativeness of the estimation proposed in Ref.[30] probably descends from the arbitrary negligible value of k_H assumed in Ref.[30]. If, rather than equal to $0.01k_d$, k_H is more generally assumed equal to δk_d, Eq. (B.6) becomes:

$$\frac{s_l}{s_{l,Liu}} = \left(\delta^{-1}\frac{1/2-\alpha_{th}}{5/2+\alpha_{th}}\right)^{\frac{1/2-\alpha_{th}}{3}}\frac{5/2+\alpha_{th}}{3}, \tag{B.7}$$

which can be exploited to compute the expression of δ that verifies $s_{l,Liu} = s_l$. In particular, it can be shown form Eq. (B.7) that the expression of δ that verifies $s_{l,Liu} = s_l$ takes the following form:

$$\delta = \frac{1/2-\alpha_{th}}{5/2+\alpha_{th}}\left(\frac{5/2+\alpha_{th}}{3}\right)^{\frac{3}{1/2-\alpha_{th}}}, \tag{B.8}$$

which is in the range $[0.021; 0.067]$ (larger than 0.01) for any positive value of α_{th} below $1/3$.

References

1 Matsunaga, H., Sun, C., Hong Y. and Murakami, Y. (2015) Dominant factors for very-high-cycle fatigue of high-strength steels and a new design method for components. *Fatigue Fract. Engng. Mater. Struct.*, **38**, 1274–1284.

2 Li, X.S. (2012) Effects of inclusions on very high cycle fatigue properties of high strength steels. *Int. Mater. Review*, **57 (2)**, 92–114.

3 Chapetti, M.D., Tagawa, T. and Miyata, T. (2003) Ultra-long cycle fatigue of high-strength carbon steels part I: review and analysis of the mechanism of failure. *Mater. Sci. Engng. A*, **356**, 227–235.

4 Chapetti, M.D., Tagawa, T. and Miyata, T. (2003) Ultra-long cycle fatigue of high-strength carbon steels part II: estimation of fatigue limit for failure from internal inclusions. *Mater. Sci. Engng. A*, **356**, 236–244.

5 Murakami, Y. (2002) *Metal Fatigue: Effects Of Small Defects And Nonmetallic Inclusions*. Elsevier Ltd, Oxford, UK.

6 Murakami, Y., Nomoto, T., Ueda, T. and Murakami, Y. (2000) On the mechanism of fatigue failure in the superlong life regime (N>107 cycles). Part I: influence of hydrogen trapped by inclusions. *Fatigue Fract. Engng. Mater. Struct.*, **38**, 893–902.

7 Sakai, T., Sato, Y. and Oguma, N. (2002) Characterisitc S-N properties of high-carbon-chromium-bearing steel under axial loading in long-life fatigue. *Fatigue Fract. Engng. Mater. Struct.*, **25**, 765–773.

8 Shiozawa, K., Lu, L. and Ishihara, S. (2001) S–N curve characteristics and subsurface crack initiation behaviour in ultra-long life fatigue of a high carbon-chromium bearing steel. *Fatigue Fract. Engng. Mater. Struct.*, **24**, 781–790.

9 Li, Y.D., Zhang, L.L., Fei, Y.H., Liu, X.Y. and Li, M.X. (2016) On the formation mechanisms of fine granular area (FGA) on the fracture surface for high strength steels in the VHCF regime. *Int. J. Fatigue*, **82**, 402–410.

10 Hong, Y., Liu, X., Lei, Z. and Sun, C. (In press) The Formation Mechanism of Characteristic Region at Crack Initiation for Very-High-Cycle Fatigue of High-Strength Steels. *Int. J. Fatigue*, doi: http://dx.doi.org/10.1016/j.ijfatigue.2015.11.029.

11 Zhao, A., Xie, J., Sun, C., Lei, Z. and Hong, Y. (2011) Prediction of threshold value for FGA formation. *Mater. Sci. Engng. A*, **528**, 6872–6877.

12 Tanaka, K. and Akiniwa, Y. (2002) Fatigue crack propagation behavior derived from S-N data in very high cycle regime. *Fatigue Fract. Engng. Mater. Struct.*, **25**, 775–784.

13 Murakami Y, Nomoto, T. and Ueda, T. (1999) Factors influencing the mechanism of superlong fatigue failure in steels. *Fatigue Fract. Engng. Mater. Struct.*, **22**, 581–590.

14 Karsch, T., Bomas, H., Zoch, H.-W. and Mändl S. (2014) Influence of hydrogen content and microstructure on the fatigue behaviour of steel SAE 52100 in the VHCF regime. *Int. J. Fatigue*, **60**, 74–89.

15 Qian, G., Zhou, C. and Hong Y. (2010) Experimental and theoretical investigation of environmental media on very-high-cycle fatigue behavior for a structural steel. *Acta Materialia*, **59**, 1321–1327.

16 Furuya, Y., Hirukawa, H. and Hayakawa, M. (2010) Gigacycle Fatigue Properties of Hydrogen-Charged JIS-SCM440 Low-Alloy Steel Under Ultrasonic Fatigue Testing. *Metal. Mater. Trans. A.*, **41A**, 2248–2256.

17 Liu, Y.B., Yang, Z.G., Li, Y.D., Chen, S.M., Li, S.X., Hui, W.J. and Weng, Y.Q. (2008) On the formation of GBF of high-strength steels in the very high cycle fatigue regime. *Mater. Sci. Engng. A*, **497**, 408–415.

18 Gangloff, R.P. and Somerday, B.P., Ed. (2012) *Gaseous Hydrogen Embrittlement of Materials in Energy Technologies, Volume1: The Problem, Its Characterization and Effects on Particular Alloy Classes*. Woodhead publishing, Oxford, UK.

19 Gangloff, R.P. and Somerday, B.P., Ed. (2012) *Gaseous Hydrogen Embrittlement of Materials in Energy Technologies, Volume 2: Mechanisms, modeling and future developments*. Woodhead publishing, Oxford, UK.

20 Rajabipour, A. and Melchers, R.E. (2015) Application of Paris' law for estimation of hydrogen-assisted fatigue crack growth. *Int. J. Fatigue*, **80**, 357–363.

21 Xing, X., Chen, W. and Zhang, H. (2015) Prediction of crack propagation under cyclic loading based on hydrogen diffusion. *Mater. Lett.,* **152**, 86–89.

22 Drexler, E.S., Slifka, A.J., Amaro, R.L., Barbosa, N., Lauria, D.S., Hayden, L.E. and Stalheim, D.G. (2014) Fatigue crack growth rates of API X70 pipeline steel in a pressurized hydrogen gas environment. *Fatigue Fract. Engng. Mater. Struct.*, **37**, 517–525.

23 Murakami, Y., Kanezaki, T. and Sofronis, P. (2013) Hydrogen embrittlement of high strength steels: Determination of the threshold stress intensity for small cracks nucleating at nonmetallic inclusions. *Engng. Fract. Mech.*, **97**, 227–243.

24 Murakami, Y., Matsunaga, H., Abyazi, A. and Fukushima, Y. (2013) Defect size dependence on threshold stress intensity for high-strength steel with internal hydrogen. *Fatigue Fract. Engng. Mater. Struct.*, **36**, 836–850.

25 Sun, Z., Moriconi, C., Benoit, G., Halm, D. and Henaff, G. (2013) Fatigue Crack Growth under High Pressure of Gaseous Hydrogen in a 15-5PH Martensitic Stainless Steel: Influence of Pressure and Loading Frequency. *Metal. Mater. Trans. A.,* **44A**, 1320–1330.

26 Matsuoka, S., Tanaka, H., Homma, N. and Murakami, Y. (2011) Influence of hydrogen and frequency on fatigue crack growth behavior of Cr-Mo steel. *Int. J. Fract*., **168**, 101–112.

27 Hołobut, P. (2010) Fatigue crack growth model for a thin steel plate containing hydrogen. *Int. J. Fatigue*, **32**, 1895–1903.

28 Thomas, R.L.S., Scully, J.R. and Gangloff, R.P. (2003) Internal Hydrogen Embrittlement of Ultrahigh-Strength AERMET 100 Steel. *Metal. Mater. Trans. A.,* **34A**, 327–344.

29 Wei, P.W. and Gangloff, R.P. (1987) Environmentally assisted crack growth in structural alloys: perspective and new directions. Invited paper for the ASTM 20th National Symposium on Fracture Mechanics, Bethlehem.

30 Liu, Y.B., Yang, Z.G., Li, Y.D., Chen, S.M., Li, S.X., Hui, W.J. and Weng, Y.Q. (2009) Dependence of fatigue strength on inclusion size for high-strength steels in very high cycle fatigue regime. *Mater. Sci. Engng. A*, **517**, 180–184.

31 Liu, Y.B., Li, Y.D., Li, S.X., Yang, Z.G., Chen, S.M., Hui, W.J. and Weng, Y.Q. (2010) Prediction of the S–N curves of high-strength steels in the very high cycle fatigue regime. *Int. J. Fatigue*, **32**, 1351–1357.

32 Li, W., Sakai, T., Li, Q., Lu, L.T. and Wang, P. (2010) Reliability evaluation on very high cycle fatigue property of GCr15 bearing steel. *Int. J. Fatigue*, **32**, 1096–1107.

33 Hong , Y., Lei, Z., Sun, C. and Zhao A. (2014) Propensities of crack interior initiation and early growth for very-high-cycle fatigue of high strength steels. *Int. J. Fatigue*, **58**, 144–151.

34 Beretta, S. and Carboni, M., (2006) Experiments and stochastic model for propagation lifetime of railway axles. *Engng. Fract. Mech.*, **73**, 2627–2641.

35 Paolino, D.S. and Cavatorta, M.P. (2013) Sigmoidal crack growth rate curve: statistical modeling and applications. *Fatigue Fract. Engng. Mater. Struct.,* **36**, 316–326.

36 Paolino, D.S., Chiandussi, G. and Rossetto, M. (2013) A unified statistical model for S-N fatigue curves: probabilistic definition. *Fatigue Fract. Engng. Mater. Struct*., **36**, 187–201.

37 Sun, C., Liu, X., and Hong, Y. (2015) A two-parameter model to predict fatigue life of high-strength steels in a very high cycle fatigue regime. *Acta Mech. Sin.*, **31 (3)**, 383–391.

38 Schuller, R., Fitzka, M., Irrasch, D., Tran, D., Pennings, B. and Mayer, H. (2015) VHCF properties of nitrided 18Ni maraging steel thin sheets with different Co and Ti content. *Fatigue Fract. Engng. Mater. Struct*., **38**, 518–527.

39 Li, W., Yuan, H., Wang, P. and Sakai, T. (2013) Micro-defect effect on gigacycle fatigue S-N property and very slow crack growth of high strength low alloy steel. *Mater. Sci. Tech.,* **29 (9)** 1101–1110.

40 Paolino, D.S., Tridello, A., Chiandussi, G. and Rossetto, M. (2015) Statistical distributions of Transition Fatigue Strength and Transition Fatigue Life in duplex S–N fatigue curves. *Theor. Appl. Fract. Mec.,* **80 (A)** 31–39.

41 Paolino, D.S., Tridello, A., Chiandussi, G., Rossetto, M. (2014) On specimen design for size effect evaluation in ultrasonic gigacycle fatigue testing. *Fatigue Fract. Engng. Mater. Struct.,* **37**, 570–579.

42 Tridello, A., Paolino, D.S., Chiandussi, G. and Rossetto, M. (2015) VHCF response of AISI H13 steel: assessment of size effects through Gaussian specimens. *Procedia Engng.,* **109**, 121–127.

43 Tridello, A., Paolino, D.S., Chiandussi, G. and Rossetto, M. (2016) Different inclusion contents in H13 steel: Effects on VHCF response of Gaussian specimens. *Key Eng. Mater.,* **665**, 49–52.

44 Furuya, Y. (2011) Notable size effects on very high cycle fatigue properties of high strength steel. *Mater. Sci. Eng. A*, **528**, 5234–5240.