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On the Error Performance Bound of Ordered Statistics Decoding of Linear Block Codes

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***Abstract*—In this paper, a novel simplified statistical approach to evaluate the error performance bound of Ordered Statistics Decoding (OSD) of Linear Block Codes (LBC) is investigated. First, we propose a novel statistic which depicts the number of errors contained in the ordered received noisy codeword. Then, simplified expressions for the probability mass function and cumulative distribution function are derived exploiting the implicit statistical independence property of the samples of the received noisy codeword before reordering. Second, we incorporate the properties of this new statistic to derive the simplified error performance bound of the OSD algorithm for all order$-I$ reprocessing. Finally, with the proposed approach, we obtain computationally simpler error performance bounds of the OSD than those proposed in literature for all length LBCs.**

I. Introduction

Channel coding is an important technique which attempts to minimize data loss as a result of errors introduced in transmission due to imperfect channels, by adding redundancy to the information before transmission. For a Linear Block Code (LBC) $C(n, k, d_{\min})$: codeword length n, information word length k and minimum hamming distance $d_{\min}$ between any two codewords, the brute-force approach to Maximum Likelihood (ML) decoding is generally impossible for non-trivial codes.

Low complexity decoding of LBCs has long been investigated by many coding theorists: a detailed bibliography of the contributions in this area can be found in [1]. Many universal decoding algorithms have been proposed including the highly efficient Viterbi decoding [2], [3]. Although all the LBCs possess the trellis structure which is the backbone of Viterbi decoding, the number of states $(\min\{2^k, 2^{n-k}\})$ becomes too large to practically implement for long length codes. Thus, for long length codes, sub-optimal iterative decoding algorithms became a very good option but in order to obtain a good performance, the code must have some peculiar properties, e.g., Low Density Parity Check Codes (LDPC) [5] (sparse parity check matrix) or Turbo Codes [4] (efficient decomposition into easy-to-decode component codes). Iterative decoding of powerful classical codes such as Bose, Chaudhuri, and Hocquenghem (BCH) code [6] and Reed-Solomon codes [7] is quite sub-optimal with respect to ideal Maximum-Likelihood Decoding (MLD), due to their structures. As a consequence, sub-optimal (near optimal) soft decision decoding based on the ordered statistic of the received noisy codeword has been proposed [8]–[12] proving to be efficient with considerable complexity for LBCs.

Since Fossorier and Lin, in their original contribution [11] presented a novel Ordered Statistics Decoding (OSD) scheme for soft decision decoding of LBCs based on ordered statistics of the received noisy samples (although an algorithm belonging to this set was first proposed by Dorsch [8] and also used by [9] and [10]), OSD is widely being studied in the literature [11]–[15]. Over the years, various new methods [16]–[22] based on reliability information and many modifications [14], [15] on original OSD have been proposed in the literature to minimize the performance-complexity trade-offs. In addition to wide spread application of ordered Statistic in decoding of LBCs, the contributions in [23]–[25], [28] show its use also in decoding of LDPC and convolution codes. In all these optimum and sub-optimum decoding algorithms, a reliability measure of the received symbols has been used to reduce the search space and find the most likely codewords.

The original concept of OSD [11] is basically implemented in two stages, a) determining the Most Reliable Independent (MRI) bits from the Most Reliable Basis (MRB) of the code and b) Order$-I$ reprocessing on MRI using most likely Test Error Patterns (TEPs). Out of these two stages, order$-I$ reprocessing is designed to improve the hard decision decoded codeword progressively until either practically optimum or a desired error performance is achieved. The approach of ML resource test based on the cost function calculated from the soft valued samples of the permuted received sequence is introduced as a stopping criterion after each stage $j, 0 \leq j \leq I$ of order$-I$ reprocessing which indeed proved excellent in reducing the average number of computations. Furthermore, an upper bound on the error performance for order$-I$ reprocessing OSD has been derived based on the noise statistics after reordering in [11], [26].

However, there are two major drawbacks of the performance bound derived in [11]. The first drawback is its complexity of evaluation (requiring $(I + 1)$ dimensional integral for any order$-I$ reprocessing). The second is the tightness of the bound, since this bound has been derived based on the assumption that the events associated with the reordered vector components are statistically independent, although, this does not hold true in practical scenarios. These issues related to computationally complex error performance bound has been revisited in [13], where a computationally simpler and comparatively tighter upper-bound on the error performance has been derived based on the statistical approach proposed by Agrawal & Vardy in [27]. It has been shown that, compared to $(I+1)$ dimensional integral computation in [11], the expression derived in [13] requires only a $2-$dimensional integral for any order$-I$ reprocessing. Although, the reduction of integral dimension from $(I+1)$ to 2 seems quite impressive, computing $2-$dimensional integral $(n-k-1)$ times as referred to [13] is still computationally complex.

In this paper, we propose a novel statistic of the ordered vector components which highlights and makes evidence to OSD property. Furthermore, the proposed statics can be applied to derive a further simplified error performance bound. More importantly, simplified expressions for the Probability Density Function (pdf) and Cumulative Distribution Function (cdf) of the proposed statistic are derived. Subsequently, we incorporate the properties of this statistic to derive the simplified error performance bound for OSD with order$-I$ reprocessing. The computational complexity of the corresponding bound is found to be even simpler (requiring single dimensional integral evaluation) to that of the bounds derived in [11] and [13]. Furthermore, the error performance bound derived in this paper is as tight as the one proposed in [13] and is also derived without any assumption.

The rest of the paper is organized as follows: A basic overview of OSD with order$-I$ reprocessing and the problem definition are presented in Section II. A novel statistic based on the error properties of the ordered noisy vector is proposed and its pdf & cdf are derived in Section III. Based on the statistical properties of the new statistic presented in Section III, a proposed upper bound on the error performance of order$-I$ OSD is derived in Section IV. Simulation results for some well known LBCs are given in Section V and some concluding remarks are provided in Section VI.

II. Overview of OSD and Conventional Reprocessing

Given an LBC $C(n,k)$, with a systematic generator matrix $\mathbf{G}$, at the transmitter side, a $k-$bit information vector, $\overline{\mathbf{v}} = (v_1, v_2, \ldots, v_k),\ v_i \in GF(2)$ is mapped into a codeword $\overline{\mathbf{c}} = \overline{\mathbf{v}} \cdot \mathbf{G} = (c_1, c_2, \ldots, c_n)$ where GF(2) stands for Galois field of order-2.

Under Binary Phase Shift Keying (BPSK), the codeword is mapped into a real-valued vector as

$$\overline{\mathbf{s}} = (s_1, s_2, \ldots, s_n) \qquad s_i \in \{-1, +1\} \subseteq \mathcal{R}, \tag{1}$$

where $c_i = 0 \rightarrow s_i = -1$ and $c_i = 1 \rightarrow s_i = +1$.

The vector $\overline{\mathbf{s}}$ is transmitted over an Additive White gaussian Noise (AWGN) channel. At the receiver side, we observe the received vector,

$$\overline{\mathbf{r}} = (r_1, r_2, \ldots, r_n) \qquad r_i \in \mathcal{R}, \tag{2}$$

where $r_i = s_i + w_i$, w_i is a white Gaussian noise sample with mean zero and variance σ^2.

Given $\overline{\mathbf{r}}$, we want to perform a soft-decision decoding. As already stated in section I, for small-medium block codes (with k upto some hundreds of bits), an effective solution is provided by Ordered Statistics Decoding (OSD) algorithms, like the Most reliable Basis (MRB) algorithm [11]. This algorithm starts by reordering the received vector in the descending order of the absolute values. In this way, the first symbols are characterized with a high reliability, i.e., a large probability of being correct.

Given $\overline{\mathbf{r}}$, by reordering its components in decreasing magnitude $\lambda_i = |r_i|$, we obtain a vector,

$$\overline{\mathbf{r}}^* = (r_1^*, r_2^*, \ldots, r_n^*) \qquad r_i^* \in \mathcal{R}, \tag{3}$$

such that $|r_i^*| > |r_{i+1}^*|$ for $1 \leq i \leq n$. Let us define ρ_1 as the permutation rule applied on $\overline{\mathbf{r}}$ to obtain $\overline{\mathbf{r}}^*$.

The generator matrix $\mathbf{G}$ is also permuted using the same permutation rule ρ_1, to give a new permuted generating matrix

$$\mathbf{G}^* = \rho_1(\mathbf{G}). \tag{4}$$

The matrix $\mathbf{G}^*$ in (4) is then processed using elementary row operations to obtain a systematic form $\mathbf{G}'$. As well known in the OSD literature [11], [13], it may happen that the first k columns of $\mathbf{G}^*$ may not be linearly independent, i.e., the k most reliable components do not correspond to an information set. In this case, it is necessary to slightly change the permutation ρ_1 until an information set is obtained. This introduces a second permutation defined as ρ_2 which needs to be applied both on $\overline{\mathbf{r}}^*$ and $\mathbf{G}^*$ to obtain $\overline{\mathbf{r}}'$ and $\mathbf{G}'$ respectively. Thus, the final permutation relations can be written as,

$$\overline{\mathbf{r}}' = \rho_2(\overline{\mathbf{r}}^*) \Leftrightarrow \rho_2(\rho_1(\overline{\mathbf{r}})), \mathbf{G}' = \rho_2(\mathbf{G}^*) \Leftrightarrow \rho_2(\rho_1(\mathbf{G})), \tag{5}$$

with $|r'_1| \geq |r'_2| \geq \ldots \geq |r'_k|$ and $|r'_{k+1}| \geq |r'_{k+2}| \geq \ldots \geq |r'_n|$.

In the following, we consider ρ_2 as an identity permutation function such that $\rho_2(\overline{\mathbf{x}}) = \overline{\mathbf{x}}$ where $\overline{x}$ is a arbitrary vector. Thus, we suppose $\overline{\mathbf{r}}'$ has exact reliability ordering. We perform a symbol-by-symbol hard decision on $\overline{\mathbf{r}}'$ to obtain the binary vector,

$$\overline{\mathbf{y}}' = (y'_1, y'_2, \ldots, y'_n) \qquad y'_i \in GF(2), \tag{6}$$

where $r'_i < 0 \rightarrow y'_i = 0$ and $r'_i \geq 0 \rightarrow y'_i = 1$.

Next, we take its first k bits of $\overline{\mathbf{y}}'$ to form the candidate information vector,

$$\overline{\mathbf{v}}' = (v'_1, v'_2, \ldots, v'_k) \qquad v'_i \in GF(2). \tag{7}$$

Due to reordering, with high probability, $\overline{\mathbf{v}}'$ contains a few errors because its bits have high reliability. One of the objectives of this paper is to provide computationally efficient expression for the number of errors contained in $\overline{\mathbf{v}}'$.

Exploiting this reliability property, OSD algorithm considers a set of patterns,

$$S = \{\overline{\mathbf{p}} = (p_1, \ldots, p_i, \ldots, p_k) \qquad p_i \in GF(2)\}, \tag{8}$$

with Hamming weight of the TEP $w_H(\overline{\mathbf{p}})$, $0 \leq w_H(\overline{\mathbf{p}}) \leq I$, where I is called the order of the algorithm. Each pattern is added to the candidate information vector given by (7), which is then encoded by the matrix $\mathbf{G}'$ to obtain a reprocessing codeword $\bar{\mathbf{c}}^*$ in the following way,

$$\forall \bar{\mathbf{p}} \in S : \bar{\mathbf{v}}' \rightarrow \bar{\mathbf{v}}^* = \bar{\mathbf{v}}' + \bar{\mathbf{p}} \rightarrow \bar{\mathbf{c}}^* = \bar{\mathbf{v}}^* \cdot \mathbf{G}'. \tag{9}$$

When all patterns are considered, the codeword $\bar{\mathbf{c}}^*$ in (9) at minimum Euclidean distance from the permuted received vector $\bar{\mathbf{r}}'$ can be chosen as the received codeword. Obviously, if we set $I = K$ and we test all the corresponding 2^k patterns, we can surely find the maximum-likelihood codeword, but the algorithm becomes equivalent to an exhaustive decoding, which is impossible for non-trivial codes [11]. Then, a key issue for these algorithms is the choice of the order I and the set S to optimize the complexity-performance trade-off. It has been shown analytically that, for small/medium size codes (e.g., $n \leq 150$, code rate $k/n > 0.5$), an order $I \approx d_{\min}/4$ is able to provide nearly-optimal decoding performance, i.e, very close to that of ideal maximum-likelihood decoding [11].

An approximation of the closed form expression for the upper-bound on the error performance of the order$-I$ reprocessing has been first derived in [11] which requires the computation of an $(I+1)$-dimensional integral for any reprocessing order-I. Later, a relatively simple (requiring $2-$dimensional integral evaluation) and relatively accurate (without any assumption) upper bound on the error performance of OSD for each stage reprocessing is derived in [13] based on the statistics introduced in [27]. In the following section, we present a new statistic on error properties of the permuted binary received vector $\bar{\mathbf{y}}'$ which can be applied to evaluate a simplified expression of the OSD error performance bound.

III. Proposed New Statistic of Ordered Vector Components

Given an LBC $C(n,k)$, without loss of generality, let us consider an all-zero transmitted codeword, $\bar{\mathbf{c}} = (0,\dots,0,\dots,0)$, which after BPSK mapping, corresponds to the transmitted vector, $\bar{\mathbf{s}} = (-1,\dots,-1,\dots,-1)$. At the output of the AWGN channel, we observe the received vector $\bar{\mathbf{r}} = (r_1,\dots,r_i,\dots,r_n)$, with:

$$r_i = -1 + w_i, \tag{10}$$

where w_i is a Gaussian random variable with zero mean and variance σ^2. All w_i noise components are considered to be statistically independent.

Each component r_i in (10) has a pdf given by,

$$f_r(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x+1)^2}{2\sigma^2}}. \tag{11}$$

If we consider the magnitude of the components of $\bar{\mathbf{r}}$ written as $\lambda_i = |r_i|$, the pdf of λ_i is given by,

$$f_{\lambda_i}(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{e^{-\frac{(x+1)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} + \frac{e^{-\frac{(x-1)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} & \text{if } x \geq 0, \end{cases} \tag{12}$$

while its cdf is given by,

$$F_{\lambda_i}(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 - Q\left(\frac{x+1}{\sigma}\right) - Q\left(\frac{x-1}{\sigma}\right) & \text{if } x \geq 0, \end{cases} \tag{13}$$

where $Q(x) \triangleq \int_x^\infty 1/\sqrt{2\pi} exp(-y^2/2)dy$ is the standard normal tail function.

Now, let us focus on the reordered vector $\bar{\mathbf{r}}'$. A first study to estimate the behavior of the components of $\bar{\mathbf{r}}'$ was done in [11], where the pdf of the reordered noise samples conditioned upon the transmitted symbol has been presented.

Given the vector $\bar{\mathbf{r}}$ observed by transmitting the all-zero codeword over an AWGN channel with BPSK modulation and the ordered vector $\bar{\mathbf{r}}'$ obtained by ordering $\bar{\mathbf{r}}$ in decreasing magnitude, we can obtain the pdf of the components of $\bar{\mathbf{r}}'$, and the pdf of the magnitude of the components of $\bar{\mathbf{r}}'$ by using arguments from ordered statistics [26] as follows. The pdf of the i-th component r'_i of $\bar{\mathbf{r}}'$ is given by,

$$\forall x : f_{r'_i}(x) = \frac{n!}{(i-1)!(n-i)!}(1 - F_\lambda(|x|))^{i-1} \cdot (F_\lambda(|x|))^{n-i} f_r(x). \tag{14}$$

In the similar manner, using the pdf of λ_i, the pdf of the magnitude of i-th component $\lambda'_i = |r'_i|$ of $\bar{\mathbf{r}}'$ is given by,

$$\forall x : f_{\lambda'_i}(x) = \frac{n!}{(i-1)!(n-i)!}(1 - F_\lambda(x))^{i-1} \cdot (F_\lambda(x))^{n-i} f_\lambda(x). \tag{15}$$

In the following, we define a new random variable E_L which represents the number of errors contained in the first L positions of the permuted received vector $\bar{\mathbf{r}}'$. Furthermore, we present a computationally efficient expression for the pdf of E_L. Normally, under identity permutation rule ρ_2, order$-I$ OSD does not include the chances of having more than I errors in MRPs of the permuted received vector which constitutes the maximum probability of missing the true codeword by an order$-I$ OSD under ML performance. In this relation, the complementary cdf of E_L at $E_L = I$ actually provides a simplified expression to evaluate the maximum probability of missing a true codeword by an order$-I$ OSD, thus justifying the importance of E_L in evaluating the performance bound of the order$-I$ OSD. Further details are discussed in Section IV. The following theorem provides an exact expression for the pmf of the random variable E_L.

Theorem 1. Given a random variable E_L which represents the number of errors we expect on the first L, $1 \leq L \leq n$, positions of the permuted received vector $\bar{\mathbf{r}}'$, its pmf is given by,

$$p_{E_L}(E_L = j) = \int_0^{+\infty} \binom{l}{j} p(x)^j (1 - p(x))^{l-j} f_{\lambda'_{l+1}}(x) dx, \tag{16}$$

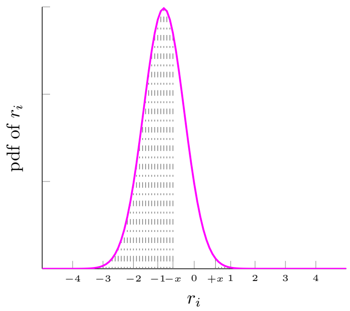

Fig. 1. Probability density function of r_i: $E_b/N_0 = 3.5$ dB.

where,

$$p(x) = \frac{Q\left(\frac{x+1}{\sigma}\right)}{1 + Q\left(\frac{x+1}{\sigma}\right) - Q\left(\frac{-x+1}{\sigma}\right)}, \tag{17}$$

with $f_{\lambda'_{l+1}}(x)$ given by (15) and its cdf is given by,

$$F_{E_L}(E_L = j) = \sum_{i=0}^{j} p_{E_L}(E_L = i). \tag{18}$$

Proof. Fix a value x, and suppose the magnitude of the $(l+1)$-th component of the reordered vector $\bar{\mathbf{r}}'$ is $\lambda'_{l+1} = x$. Then, the vector $\bar{\mathbf{r}}$ contains exactly l components with $|r_i| \geq x$. As can be observed in Fig. 1 (the shaded region corresponds to $|r_i| \geq x$), for each of these components, the probability of having an error is,

$$\begin{aligned} p(x) &= P(r_i > 0|_{|r_i| \geq x}) \\ &= \frac{Q\left(\frac{x+1}{\sigma}\right)}{1 + Q\left(\frac{x+1}{\sigma}\right) - Q\left(\frac{-x+1}{\sigma}\right)}. \end{aligned} \tag{19}$$

Since we are working with the components of $\bar{\mathbf{r}}$ we can use their implicit statistically independent property, which instead does not hold if we try to work with the components of $\bar{\mathbf{r}}'$. As a result, the probability of having j errors among these k components is given by,

$$P(E_L = j|\{\lambda'_{l+1} = x, L = l\}) = \binom{l}{j} p(x)^j (1 - p(x))^{l-j}. \tag{20}$$

The above result is obtained under the condition $\lambda'_{l+1} = x$. By integrating over all x values by using $f_{\lambda'_{l+1}}(x)$, we obtain the final pmf of E_L as in (16) while its cdf at some value $E_L = j$ is obtained as in (18) simply by summing the normalized pmf of E_L for all $E_L : 0 \leq E_L \leq j$. □

IV. OSD Error Performance Based on distribution of E

In this section, a different look at the error performance of order$-I$ OSD is presented based on the distribution of E detailed in Section III. Let $P_{e_{OSD-I}}$ denote the code error performance of the order$-I$ OSD and $P_e(I)$ denote the probability that the correct codeword is not among the candidate codewords supported by the order$-I$ OSD. The upper bound on the order$-I$ OSD performance can be written as an inequality as

$$\begin{aligned} P_{e_{OSD-I}} &\leq P_{e_{ML}} + P_e(I) \\ &\leq P_{e_{ML}} + P\left(\begin{array}{c}\text{More than } I \text{ errors occur} \\ \text{in } 1^{st}k \text{ positions of } \bar{\mathbf{r}}'\end{array}\right) \end{aligned} \tag{21}$$

where $P_{e_{ML}}$ is the MLD code error rate.

In (21), the probability of having more than I errors in the first k ordered received symbols in $\bar{\mathbf{r}}'$ given a identity permutation function ρ_2 can be simply evaluated from the cdf of E_L as,

$$P_e(I)|_{\rho_2(x)=x} = 1 - F_{E_L}(I), \tag{22}$$

where L in this case is equal to k.

In a real scenario, the permutation function ρ_2 may or may not be identity. In fact, the second permutation ρ_2 directly relates with the number of column permutation required to obtain first k columns of $\mathbf{G}^*$ to be linearly independent. Let us consider d is the number of dependent columns before k^{th} independent one and $P(d)$ be the probability associated with d. It has been shown in [11] that the maximum number of dependent columns that can be found before k^{th} independent one for a given generator matrix is given by,

$$d_{\max} = n - k - d_H^{\min} - 1, \tag{23}$$

where $d_H^{\min}$ is the minimum Hamming distance of the considered code.

Thus, $P_e(I)$ can be expressed under all cases of ρ_2 as,

$$\begin{aligned} P_e(I) &= \sum_{d=0}^{d_{\max}} P\left(\begin{array}{c}\text{More than } I \text{ errors occur} \\ \text{in } 1^{st}k + d \text{ positions of } \bar{\mathbf{r}}'\end{array}\right) \cdot P(d) \\ &= \sum_{d=0}^{d_{\max}} P(d)\left(1 - F_{E_{k+d}}(I)\right). \end{aligned} \tag{24}$$

where $F_{E_{k+d}}$ is the cdf of E_L at $L = k + d$. The probability $P(d)$ can be evaluated either from simulation or from the approximated distribution proposed in [11].

As stated earlier, the random variable E denotes the number of errors, thus, can only take integer values. Also the reprocessing order$-I$ is always fairly small ($I = \frac{d_H^{\min}}{4} \leq 5$) for linear block codes with practical $d_H^{\min}$. Thus, (24) can be written in terms of pdf of E_L as,

$$P_e(I) = \sum_{d=0}^{d_{\max}} P(d)\left[1 - \sum_{i=0}^{I} f_{E_{k+d}}(E_{k+d} = i)\right]. \tag{25}$$

where $f_{E_{k+d}}(E_{k+d} = i)$ can be evaluated by a single integral given by (16).

The above discussion shows that, for all permissible values of order-I, the theoretical upper bound can be evaluated. It is quite clear that the computational complexity of this new upper bound requires a single dimensional integral for any order-I compared to $(I + 1)$ and 2 dimensional integral for the one presented in [11] and [13] respectively. Furthermore, this bound is as tight as the one proposed in [13] (see Fig. 3

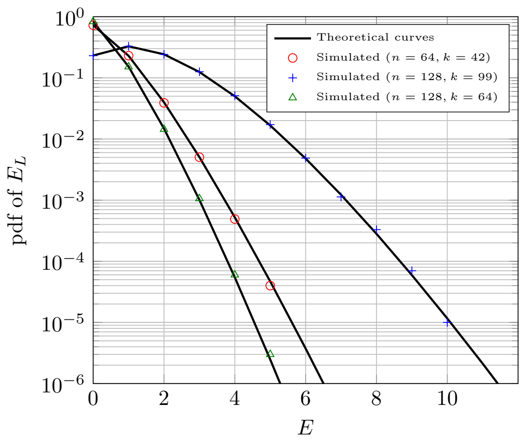

Fig. 2. Probability density function of E: $E_b/N_0 = 3.5$ dB, $L = k$.

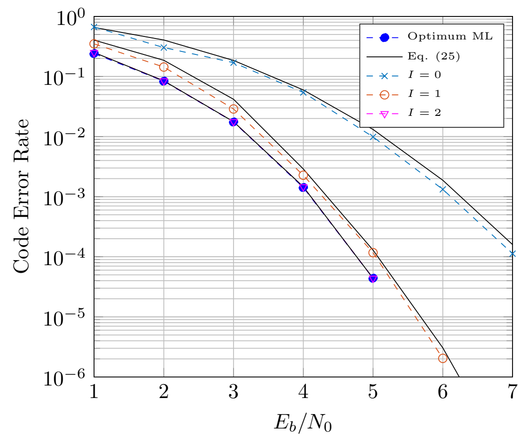

Fig. 3. Code Error Rate of OSD with order$-I$ reprocessing for (64,42,8) Reed Muller code

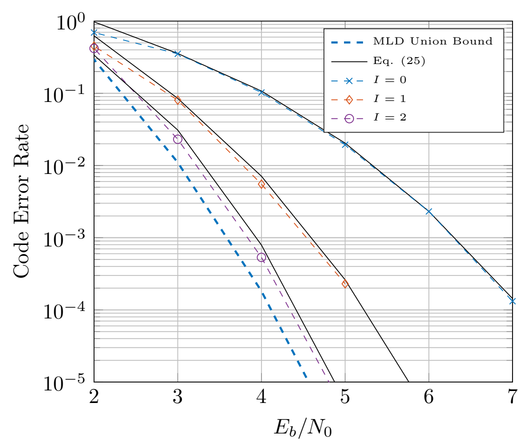

Fig. 4. Code Error Rate of OSD with order$-I$ reprocessing for (128,99,10) extended BCH code

to 5) and is also derived without any assumption.

V. Numerical Results

For the purpose of comparison and performance evaluation, we adapt the same scenario of BPSK transmissions over an AWGN channel as described in Section II and Section III. In order to accommodate the code rates of different LBCs, we adopt the ratio of the energy per bit to noise power spectral density ratio E_b/N_0 with $E_b/N_0 = \frac{E_s}{N_0} \cdot \frac{k}{n}$, where E_s is the signal energy and N_0 is the noise power spectral density.

A. Pmf of E_L

Figs. 2 plots the pmf of random variable E_L for different combinations of n and k. For each (n, k) combinations, the simulation results are plotted and compared with the theoretical result obtained from (16). We observe a perfect matching for all permissible values of E_L which justifies the validity of the expression presented in (16).

B. Error Performance

It is worthwhile to mention clearly that the evaluation of (21) involves the prior evaluation of the MLD code error rate for the code under consideration. Thus, for our simulation purpose, we use the simplest upper bound, i.e., the union bound [29],

$$P_{e_{ML}}(\bar{\mathbf{c}}) \leq \sum_{d=d_H^{min}}^{n} A_d Q\left(\frac{\sqrt{d}}{\sigma}\right). \tag{26}$$

In (26), $d : d \geq d_H^{min}$ represents the Hamming weight of the codeword and A_d is the multiplicity (number of codeword with hamming weight equal to d) of the code. $Q(\cdot)$ is the pair-wise error probability with,

$$Q(x) = \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$$

For simplicity, we consider at most the first four components of the summation which provide a good approximation for medium dimension codes at considerable E_b/N_0. For higher dimensional codes, the total union bound or other tight upper bound alternatives represent more accurate performance measure [29].

Figs. 3 to 5 depict the error performances of the $(64, 42, 8)$ Reed Muller code, the $(128, 99, 10)$ extended BCH (eBCH) code and the $(128, 64, 22)$ eBCH code, respectively. Each plot includes the simulation results and the corresponding upper bounds computed from (21). We observe that for all values of reprocessing order-I, the theoretical upper bounds are tight. The bounds are as tight as those derived in [11] and [13]. However, while the error performance bound of order$-I$ reprocessing OSD based on [11] requires the computation of $(I + 1)-$dimensional integral and [13] requires the computation of a two-dimensional integral, the new upper bounds require the computation of a single dimensional integral for any reprocessing order-I.

VI. Conclusion

In this paper, a simplified but effective error performance bound of OSD with order$-I$ reprocessing has been derived without altering the tightness of the bound. In order to

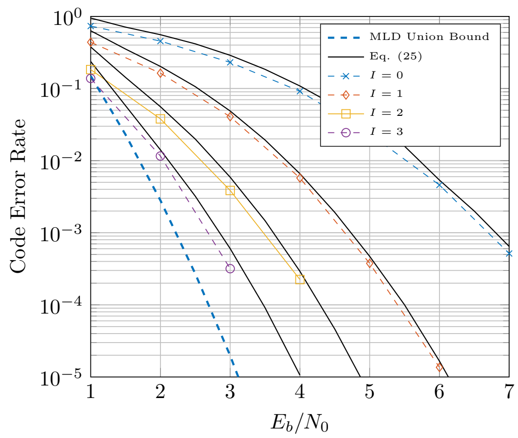

Fig. 5. Code Error Rate of OSD with order$-I$ reprocessing for (128,64,22) extended BCH code

achieve this, a new statistical random variable representing an important error performance property of the ordered received noisy codeword has been proposed. The pmf and cdf of the proposed random variable has also been derived. The example of the derived error performance bound has been applied for different codes.

As an extension to this work, the application of this new statistical approach can be applied to derive further simplified error performance bounds of other ordered statistics based algorithms for LBCs like Chase-type, Generalized GMD and Chase-type decoding.

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