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# A poroplastic model of structural reorganisation in porous media of biomechanical interest

Dedicated to Prof. David Steigmann in recognition of his contributions

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1 **Abstract** We present a poroplastic model of structural reorganisation in a  
2 binary mixture comprising a solid and a fluid phase. The solid phase is the  
3 macroscopic representation of a deformable porous medium, which exemplifies  
4 the matrix of a biological system (consisting e.g. of cells, extra-cellular matrix,  
5 collagen fibres). The fluid occupies the interstices of the porous medium and  
6 is allowed to move throughout it. The system reorganises its internal structure  
7 in response to mechanical stimuli. Such structural reorganisation, referred to  
8 as *remodelling*, is described in terms of “plastic” distortions, whose evolution  
9 is assumed to obey a phenomenological flow rule driven by stress. We study  
10 the influence of remodelling on the mechanical and hydraulic behaviour of  
11 the system, showing how the plastic distortions modulate the flow pattern of  
12 the fluid, and the distributions of pressure and stress inside it. To accomplish  
13 this task, we solve a highly non-linear set of model equations by elaborating  
14 a previously developed numerical procedure, which is implemented in a non-  
15 commercial Finite Element solver.

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16 **Keywords** Mixture Theories · Porous Media · Poroplasticity · Remodelling ·  
17 Soft Tissues

## 18 **1 Introduction**

19 The remodelling, or structural reorganisation, of a biological system might  
20 be defined as the result of an *ensemble* of processes that concur to adapt its  
21 structure and material properties to both internal and external stimuli. In  
22 addition to remodelling, a biological system may also experience growth, i.e.,  
23 it may gain or lose mass. Growth can be appositional or volumetric. In the  
24 first case, new material is either removed or laid over the pre-existing one [12].  
25 In the second case, instead, the variation of mass can be diverted either in a  
26 change of volume or in a change of density of the system [31, 62, 63].

27 In principle, a comprehensive study of growth and remodelling requires an  
28 interdisciplinary approach, in which genetic aspects and molecular processes  
29 as well as intracellular and intercellular activities are accounted for. Further-  
30 more, a thorough analysis of the functioning of biological systems calls for  
31 multi-scale and multi-level mathematical models, which should couple chem-  
32 ical, electrical, and mechanical phenomena. Despite these intricacies, some  
33 essential features of the evolution of biological systems can be captured by  
34 purely mechanical theories of growth and remodelling. For a given biological  
35 system, the starring characters of such theories are the parameters describing  
36 its kinematics and structural evolution, and the generalised forces conjugate  
37 with the selected parameters. Within a purely mechanical approach, several  
38 problems of growth and remodelling can be studied. Growth, in general, con-  
39 tributes to change the properties and internal structure of the tissue in which it  
40 occurs. A relevant aspect of this phenomenology is that grown tissues usually  
41 feature residual stresses, which means that, even though a grown tissue finds  
42 itself in an unloaded configuration, it is not necessarily in a stress-free state.  
43 For example, this is true for arteries [52]. Thus, as suggested in [90], growth  
44 can be thought of as the process that brings the tissue from a zero-stress state  
45 to a state in which residual stresses may be present even in the absence of  
46 external loading. As is well-known, the stress-free state of a body (which is  
47 also referred to as “the natural state”) is not a true configuration. Rather,  
48 it is a collection of relaxed body pieces, which cannot be attained by simply  
49 deforming the body. Consequently, growth cannot be described just in terms  
50 of deformation, deformation gradients, and the related measures of stress. In  
51 fact, one has to introduce also the concept of incompatible distortions in order  
52 to account for the transformation connecting the natural state of a tissue with  
53 the unloaded—yet not stress-free—configuration chosen as reference. The dis-  
54 tortions due to growth are generally non-integrable and incompatible. They  
55 are said to be non-integrable when they cannot be expressed as deformation  
56 gradients, and are said to be incompatible when they lead to the loss of flat-  
57 ness of the body manifold [72]. Moreover, distortions are both formally and  
58 conceptually distinct from deformations, which describe the global change of

59 shape of the tissue. To account for distortions, Rodriguez et al. [90] invoked  
60 the Bilby-Kröner-Lee (BKL) decomposition of the deformation gradient tensor,  
61 thereby separating the “elastic” part of the overall deformation from the  
62 “anelastic” one, which is related to growth and remodelling, and need not be  
63 compatible.

64 The use of the BKL decomposition permits to exploit several analogies with  
65 the theory of Elastoplasticity. For example, in a general model of growth, the  
66 anelastic distortions associated with the structural reorganisation of a tissue  
67 were described in terms of material inhomogeneities in [31], while the concept  
68 of “evolving natural configurations” [89] was used for modelling tumour growth  
69 both in monophasic and in multiphasic materials [1, 86].

70 In this paper, we focus on remodelling only. This can be done because there  
71 exist remodelling processes that do not lead to variations of mass. Moreover,  
72 there exist cases in which remodelling takes place over time-scales that are  
73 well-separated from those characterising growth, and can be thus decoupled  
74 from the growth-driven evolution of the system under study. For example,  
75 these conditions are met in cellular aggregates and in tumour spheroids, when  
76 their remodelling consists of the reorganisation of the adhesion bonds among  
77 the cells [42]. This kind of remodelling can be described by hypothesising  
78 that the considered biological systems exhibit elastoplastic behaviour [4] (or,  
79 in some cases, elasto-visco-plastic behaviour [84]), and assuming that plastic  
80 distortions arise when the stress in the system exceeds a given threshold [42].  
81 The closeness of the present setting with the classical theory of Elastoplasticity  
82 (cf. e.g. [65, 71]) makes it rather natural to employ the BKL decomposition in  
83 order to define a stress-free state for the system, and separate the plastic  
84 distortions due to remodelling from the elastic part of the overall deformation  
85 gradient. Moreover, as is the case in Elastoplasticity, also in this framework  
86 the plastic distortions are generally non-integrable and incompatible. For this  
87 reason, when referring to the distortions associated with the evolution of the  
88 internal structure of a body (i.e., with the process of remodelling), we shall  
89 use the adjectives “anelastic” and “plastic” interchangeably.

90 Following [3, 45], we consider a biphasic mixture comprising a solid and  
91 a fluid phase, and take it as an exemplification of a biological system. We  
92 study the evolution of the mixture in response to external loads by prescribing  
93 that, under suitable conditions, a remodelling process of the solid phase  
94 occurs. To accomplish this task, we formulate a finite-deformation poroplastic  
95 model of the mixture, which is able to determine the deformation of the solid  
96 phase, the velocity of the fluid phase, and the plastic distortions associated  
97 with the occurrence of remodelling. To solve the problem, we elaborate a computational  
98 algorithm that aims at generalising a well-established numerical  
99 procedure—the Return Mapping Algorithm (RMA)—to a class of anelastic  
100 models not necessarily complying with all the hypotheses on which the RMA  
101 is based [93]. Besides testing the proposed algorithm, our main purpose is to  
102 evaluate the influence of remodelling on the fluid velocity, overall deformation,  
103 and distributions of pressure and constitutive stress in the system under study.

We remark that mixture theory has been largely employed in modelling biomechanical problems of different kind. These range from tumour growth [2], in which the considered mixture consists of a fluid and one or more cell populations, to bone reconstruction [60], in which the mixture consists of the bone itself and some grafted bio-resorbable material.

The issue of structural reorganisation is largely investigated in the context of bone biomechanics, and has recently contributed also to raise questions in the field of optimal control theory. For example, optimal control procedures have been elaborated in [5] for assessing the structural efficiency of adaptive materials, and in [6] for studying the adaptation of bones under mechanical stimuli. They have also been used for modelling the trabecular architecture of bones [7], and for evaluating the bone density distribution [8]. Another problem connected with bone remodelling, which concerns the interaction of bone with resorbable biomaterial, has been addressed in [9–11].

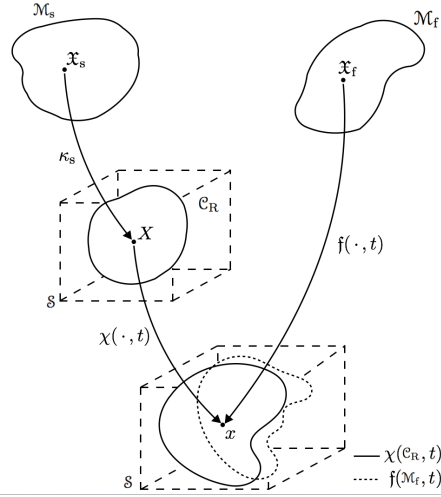
The paper is organised as follows. In Sect. 2, we review some aspects of the theory of biphasic mixtures, and introduce the BKL decomposition in the framework of porous media. In Sect. 3, we establish the constitutive framework, study dissipation, and determine the evolution law for remodelling. In Sect. 4, we describe the numerical procedure elaborated for solving the model equations. In Sect. 5, we present and discuss the obtained results. Finally, in section 6, we outline some critical remarks on the employed model, and propose some plans for future research.

## 2 Theoretical background

For our purposes, we consider a binary system comprising a porous solid medium, also referred to as “matrix” hereafter, and a fluid. The region of space occupied by the system as a whole can be partitioned into two complementary sub-regions. One of these regions is occupied by the solid particles constituting the matrix, while the other one, which is generated by the voids of the matrix, is assumed to be filled with the fluid. If the latter sub-region is connected in topological sense, it is termed “pore space”, and the fluid can circulate throughout it. In the following, the matrix and the fluid shall also be called “phases”. At a sufficiently coarse scale of observation, the system can be viewed as a biphasic mixture, which means that both the matrix and the fluid are admitted to co-exist at each point of space occupied by the system. Note that there may be cases in which the connectedness of the void region is not granted. This happens, e.g., in solid bodies with micro-periodic non-connected inclusions filled with fluid [26].

### 2.1 Kinematics of biphasic mixtures

We base the forthcoming description of the kinematics of biphasic mixtures on the theory developed in [87, 88], and recently summarised in [94]. We denote by



**Fig. 1** Graphical representation of the kinematics of biphasic mixtures. Picture redrawn and adapted from [88]

144  $\mathcal{M}_s$  and  $\mathcal{M}_f$  the two smooth three-dimensional material manifolds associated  
 145 with the matrix and the fluid, respectively. The manifold  $\mathcal{M}_s$  is embedded  
 146 into the three-dimensional Euclidean point space  $\mathcal{S}$  by means of the smooth  
 147 localisation function  $\kappa_s : \mathcal{M}_s \rightarrow \mathcal{S}$ , such that, for every solid particle  $\mathfrak{X}_s \in \mathcal{M}_s$ ,  
 148 there exists a reference placement  $X = \kappa_s(\mathfrak{X}_s) \in \mathcal{S}$ . The set  $\mathcal{C}_R = \kappa_s(\mathcal{M}_s)$  is  
 149 chosen as the reference configuration for the mixture. The motion of the solid  
 150 constituent is the one-parameter family of smooth mappings  $\chi(\cdot, t) : \mathcal{C}_R \rightarrow \mathcal{S}$ ,  
 151 where  $t \in \mathcal{J} \subseteq \mathbb{R}$  is time and  $\mathcal{J}$  is the interval of time over which the mixture  
 152 is observed. The set  $\chi(\mathcal{C}_R, t) \subset \mathcal{S}$  defines the current configuration of the solid  
 153 phase. Each point  $x \in \chi(\mathcal{C}_R, t) \subset \mathcal{S}$  is such that  $x = \chi(X, t)$ , with  $X \in \mathcal{C}_R$   
 154 and  $t \in \mathcal{J}$ . Similarly, the motion of the fluid is defined by  $f(\cdot, t) : \mathcal{M}_f \rightarrow \mathcal{S}$ ,  
 155 with  $t \in \mathcal{J}$ . The map  $f(\cdot, t)$  places a fluid particle  $\mathfrak{X}_f \in \mathcal{M}_f$  in the spatial point  
 156  $x = f(\mathfrak{X}_f, t) \in f(\mathcal{M}_f, t)$ , where  $f(\mathcal{M}_f, t)$  is the region of  $\mathcal{S}$  occupied by the fluid at  
 157 time  $t$ . Therefore, at the same instant of time, the biphasic mixture occupies  
 158 the set  $\mathcal{C}_t = \chi(\kappa_s(\mathcal{M}_s), t) \cap f(\mathcal{M}_f, t) \subset \mathcal{S}$ . By construction, solid and fluid  
 159 particles co-exist at each point  $x \in \mathcal{C}_t$ . A schematic picture of the kinematics  
 160 of biphasic mixtures is reported in Fig. 1, which has been adapted from [88].

161 We introduce the tangent spaces attached at  $x \in \mathcal{S}$  and  $X \in \mathcal{C}_R$ , i.e.,  $T_x \mathcal{S}$  and  
 162  $T_X \mathcal{C}_R$ , the tangent bundles  $T\mathcal{S} := \sqcup_{x \in \mathcal{S}} T_x \mathcal{S}$  and  $T\mathcal{C}_R = \sqcup_{X \in \mathcal{C}_R} T_X \mathcal{C}_R$  (where  
 163  $\sqcup$  stands for “disjoint union” of sets), and their dual spaces  $T^* \mathcal{S}$  and  $T^* \mathcal{C}_R$ ,  
 164 termed cotangent bundles. In addition, for any pair of natural numbers  $r \geq 0$

165 and  $s \geq 0$ , we define the spaces [33]

$$[T\mathcal{S}]_s^r = \underbrace{T\mathcal{S} \otimes \dots \otimes T\mathcal{S}}_{r \text{ times}} \otimes \underbrace{T^*\mathcal{S} \otimes \dots \otimes T^*\mathcal{S}}_{s \text{ times}}, \quad (1a)$$

$$[T\mathcal{C}_R]_s^r = \underbrace{T\mathcal{C}_R \otimes \dots \otimes T\mathcal{C}_R}_{r \text{ times}} \otimes \underbrace{T^*\mathcal{C}_R \otimes \dots \otimes T^*\mathcal{C}_R}_{s \text{ times}}. \quad (1b)$$

166 When  $r$  is zero, we simply write  $[T\mathcal{S}]_s^0$  and  $[T\mathcal{C}_R]_s^0$ . Analogously, when  $s$  is zero,  
 167 we adopt the notation  $[T\mathcal{S}]_0^r$  and  $[T\mathcal{C}_R]_0^r$ . For the sake of generality, we adopt  
 168 the covariant formalism [68]. In the forthcoming calculations, we employ the  
 169 spatial metric tensor,  $\mathbf{g} \in [T\mathcal{S}]_2^0$ , and the “material” metric tensor,  $\mathbf{G} \in [T\mathcal{C}_R]_2^0$ .  
 170 Moreover, we adhere to the convention that the gradient of scalar fields returns  
 171 a covector, and that all the stress tensors introduced in the following have  
 172 contravariant components in their component representation.

173 The velocity of a solid particle,  $\mathfrak{X}_s$ , passing through  $x = \chi(\kappa_s(\mathfrak{X}_s), t)$  at  
 174 time  $t$  is denoted by  $\mathbf{v}_s(x, t) = \dot{\chi}(\kappa_s(\mathfrak{X}_s), t) = \dot{\chi}(X, t) \in T_x\mathcal{S}$ . Analogously,  
 175  $\mathbf{v}_f(x, t) = \dot{\mathfrak{f}}(\mathfrak{X}_f, t) \in T_x\mathcal{S}$  is the spatial velocity of a fluid particle,  $\mathfrak{X}_f$ , passing  
 176 through  $x = \mathfrak{f}(\mathfrak{X}_f, t)$ . The velocity of the fluid relative to the solid is given by  
 177  $\mathbf{v}_{fs}(x, t) = \mathbf{v}_f(x, t) - \mathbf{v}_s(x, t)$ , with  $x = \chi(\kappa_s(\mathfrak{X}_s), t) = \mathfrak{f}(\mathfrak{X}_f, t)$ . Moreover, the  
 178 acceleration of the  $\alpha$ th phase is defined by  $\mathbf{a}_\alpha(x, t) = D_\alpha \mathbf{v}_\alpha(x, t)$ , where  $D_\alpha$   
 179 is the substantial derivative operator with respect to  $\mathbf{v}_\alpha$ , i.e.,

$$D_\alpha \mathbf{v}_\alpha = \partial_t \mathbf{v}_\alpha + (\text{grad } \mathbf{v}_\alpha) \mathbf{v}_\alpha, \quad \alpha = s, f. \quad (2)$$

180 To express the velocity of the  $\alpha$ th phase as a function of the points  $X \in \mathcal{C}_R$ , we  
 181 perform the composition  $\mathbf{u}_\alpha(\cdot, t) = \mathbf{v}_\alpha(\cdot, t) \circ \chi(\cdot, t)$ . However, from now on  
 182 we shall omit the explicit dependence on time in the composition of functions,  
 183 so that, for example, the velocity field  $\mathbf{u}_\alpha : \mathcal{C}_R \times \mathcal{J} \rightarrow T\mathcal{S}$  shall be simply denoted  
 184 by  $\mathbf{u}_\alpha = \mathbf{v}_\alpha \circ \chi$ .

185 The tangent map of  $\chi(\cdot, t)$  at  $X \in \mathcal{C}_R$  defines the deformation gradient  
 186 tensor of the solid motion,  $\mathbf{F}(X, t) = T\chi(X, t) : T_X\mathcal{C}_R \rightarrow T_x\mathcal{S}$ . In order for  
 187  $\chi$  to be admissible, the condition  $J = \det(\mathbf{F}) > 0$  must be respected at all  
 188 points and all times. The transpose, inverse, and transpose inverse of  $\mathbf{F}$  are  
 189 defined as  $\mathbf{F}^T : T^*\mathcal{S} \rightarrow T^*\mathcal{C}_R$ ,  $\mathbf{F}^{-1} : T\mathcal{S} \rightarrow T\mathcal{C}_R$ , and  $\mathbf{F}^{-T} : T^*\mathcal{C}_R \rightarrow T^*\mathcal{S}$ ,  
 190 respectively. It also holds that  $\mathbf{G}^{-1} \mathbf{F}^T \mathbf{g} : T\mathcal{S} \rightarrow T\mathcal{C}_R$ . This combination of  
 191 tensors shall be used in the definition of the Mandel stress tensors (cf. Sect. 3).  
 192 The symmetric, positive definite, second-order tensor  $\mathbf{C} = \mathbf{F}^T \mathbf{g} \mathbf{F} \in [T\mathcal{C}_R]_2^0$   
 193 is the Cauchy-Green deformation tensor induced by  $\mathbf{F}$ . For  $\alpha = s, f$ , we also  
 194 introduce the spatial velocity gradient  $\boldsymbol{\ell}_\alpha = (\text{grad } \mathbf{v}_\alpha) \circ \chi \in T\mathcal{S} \otimes T^*\mathcal{S}$ , which  
 195 is related to the “material” velocity gradient,  $\text{Grad } \mathbf{u}_\alpha \in T\mathcal{S} \otimes T^*\mathcal{C}_R$ , through  
 196  $\text{Grad } \mathbf{u}_s = \dot{\mathbf{F}} = \boldsymbol{\ell}_s \mathbf{F}$  and  $\text{Grad } \mathbf{u}_f := \boldsymbol{\ell}_f \mathbf{F}$ , respectively.

## 197 2.2 The Bilby-Kröner-Lee (BKL) decomposition for porous media

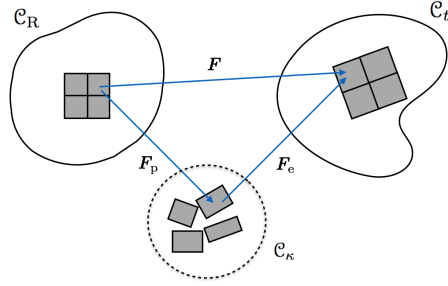
198 In this work, we study the remodelling that occurs in a biological system when,  
 199 under appropriate loading conditions, the system is compelled to reorganise its

internal structure in order to sustain the applied loads. In the case of cellular aggregates [42], this type of remodelling manifests itself through the structural transformation of the actin network in the cells, and the reorganisation of the adhesion bonds among the cells. In soft tissues, such as articular cartilage, the considered remodelling might represent the rearrangement of the cross-links of the fibre network forming the extracellular matrix. These processes can be described in terms of “plastic distortions” [42]. It is important to recall that this kind of remodelling is characterised by time scales (in fact, those induced by the mechanical loads) that allow to decouple it from the structural evolution ascribable to growth. Growth, indeed, is due to the mitosis and apoptosis of the cells, which typically occur over sufficiently larger time scales. For this reason, we disregard growth-related aspects of remodelling in the present framework.

To describe the deformation and the plastic distortions that accompany the remodelling of the biological system under study, we introduce, in addition to  $\mathbf{F}$ , the “tensor of plastic distortions”  $\mathbf{F}_p$ . This leads us to the Bilby-Kröner-Lee (BKL) multiplicative decomposition  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_p$ , where  $\mathbf{F}_e$  is sometimes referred to as the “accommodating part” of the overall deformation gradient tensor. To sketch the conceptual meaning of  $\mathbf{F}_p$ , we refer to [71]. Hence, we consider a body that is brought from its reference, unloaded configuration,  $\mathcal{C}_R$ , to the current configuration,  $\mathcal{C}_t$ , by the action of applied loads. If this evolution is accompanied by a structural reorganisation, the body cannot be brought back to  $\mathcal{C}_R$  by removing the external loads. Rather, even though all external loads were removed, the system would occupy a configuration different from  $\mathcal{C}_t$  and  $\mathcal{C}_R$ , in which residual stresses and residual strains may be present. To eliminate these, one should ideally tear the body to small disjoint pieces (i.e., neighbourhoods of material points), and let each of them individually attain a stress-free state (in doing this ideal tearing, time is kept fixed). The collection of all these stress-free body pieces, determined through the ideal tearing process, is said to be the “natural state” of the body at time  $t$ . The plastic distortion  $\mathbf{F}_p$  is the distortion that has to be applied to the material neighbourhoods of the points in  $\mathcal{C}_R$  to obtain the body pieces collected in the natural state. If the material shows elastic behaviour from its natural state, the accommodating distortion,  $\mathbf{F}_e$ , is the elastic distortion that has to be applied to the body elements in the natural state to retrieve the global configuration  $\mathcal{C}_t$ . The BKL decomposition can be understood as a combination of “tangent bundle maps” [83], so that one can introduce a time-dependent intermediate map  $\chi_\kappa(\cdot, t) : \mathcal{C}_R \rightarrow \mathcal{S}$ , which constitutes the base map for  $\mathbf{F}_p$ . However, the existence of  $\chi_\kappa(\cdot, t)$  does not necessarily imply that  $\mathbf{F}_p$  is the tangent map of  $\chi_\kappa(\cdot, t)$ . In fact, in general,  $\mathbf{F}_p$  is neither compatible nor integrable, i.e., there exists no deformation whose tangent map equals  $\mathbf{F}_p$ . In the following, we call the set  $\chi_\kappa(\mathcal{C}_R, t) = \mathcal{C}_\kappa \subset \mathcal{S}$  “intermediate configuration” (cf. Fig. 2), and associate it with the “natural state” of the solid phase.

Due to the BKL decomposition, the velocity gradient of the solid phase,  $\ell_s = \dot{\mathbf{F}} \mathbf{F}^{-1}$ , can be written as

$$\ell_s = \ell_e + \ell_p = \ell_e + \mathbf{F}_e \mathbf{L}_p \mathbf{F}_e^{-1}, \quad (3)$$



**Fig. 2** Schematic representation of the BKL decomposition of the deformation gradient tensor  $\mathbf{F}$

244 where  $\ell_e = \dot{\mathbf{F}}_e \mathbf{F}_e^{-1}$  is the rate of elastic distortions, while  $\ell_p = \mathbf{F}_e \mathbf{L}_p \mathbf{F}_e^{-1}$   
 245 and  $\mathbf{L}_p = \dot{\mathbf{F}}_p \mathbf{F}_p^{-1}$  are the rates of anelastic distortions associated with the  
 246 current configuration and the natural state of the system, respectively. The  
 247 BKL decomposition implies the identity  $J = J_e J_p$ , where  $J_e := \det(\mathbf{F}_e) > 0$   
 248 and  $J_p := \det(\mathbf{F}_p) > 0$  are referred to as the elastic and anelastic volumetric  
 249 ratios, respectively. The rates  $\dot{J}_e$  and  $\dot{J}_p$  can be computed as  $\dot{J}_e = J_e \text{tr}(\ell_e)$  and  
 250  $\dot{J}_p = J_p \text{tr}(\mathbf{L}_p) = J_p \text{tr}(\ell_p)$ . In the following, we shall assume that the anelastic  
 251 distortions related to remodelling are volume-preserving, thereby leading to  
 252 the constraint  $J_p = \det(\mathbf{F}_p) = 1$ , which implies  $\text{tr}(\ell_p) = 0$  and  $\text{tr}(\mathbf{L}_p) = 0$ .

253 For future use, we introduce  $\boldsymbol{\eta}$ , i.e., the metric tensor associated with the  
 254 intermediate configuration  $\mathcal{C}_k$ , and  $\mathbf{B}_p := \mathbf{C}_p^{-1}$ , where  $\mathbf{C}_p = \mathbf{F}_p^T \boldsymbol{\eta} \mathbf{F}_p$  is the  
 255 anelastic Cauchy-Green deformation tensor. Moreover, we shall exploit the  
 256 kinematic identity

$$\frac{1}{2} \dot{\mathbf{B}}_p = -\mathbf{F}_p^{-1} (\boldsymbol{\eta}^{-1} \mathbf{D}_p \boldsymbol{\eta}^{-1}) \mathbf{F}_p^{-T}, \quad (4)$$

257 where  $\mathbf{D}_p := \text{sym}(\boldsymbol{\eta} \mathbf{L}_p)$ .

### 258 2.3 Dynamics of biphasic mixtures

259 In the absence of external body forces, growth, and mass exchange processes  
 260 between the solid and the fluid phase, the local forms of the mass and linear  
 261 momentum balance laws for the  $\alpha$ th phase of the biphasic mixture ( $\alpha = s, f$ )  
 262 can be written as

$$\partial_t(\phi_\alpha \varrho_\alpha) + \text{div}(\phi_\alpha \varrho_\alpha \mathbf{v}_\alpha) = 0, \quad \text{in } \mathcal{C}_t \times \mathcal{J}, \quad (5a)$$

$$\phi_\alpha \varrho_\alpha \mathbf{a}_\alpha = \text{div} \boldsymbol{\sigma}_\alpha + \mathbf{m}_\alpha, \quad \text{in } \mathcal{C}_t \times \mathcal{J}, \quad (5b)$$

$$\mathbf{m}_s + \mathbf{m}_f = \mathbf{0}, \quad \text{in } \mathcal{C}_t \times \mathcal{J}. \quad (5c)$$

263 In (5),  $\phi_\alpha$  and  $\varrho_\alpha$  denote, respectively, the volumetric fraction and the true  
 264 mass density of the  $\alpha$ th phase,  $\boldsymbol{\sigma}_\alpha$  is the Cauchy stress tensor, and  $\mathbf{m}_\alpha$  is  
 265 the rate of linear momentum exchange between the  $\alpha$ th phase and the other

one. Equation (5c) expresses that the mixture is closed with respect to linear momentum.

If we assume that the pore space of the matrix is completely filled with the fluid, the volumetric fractions  $\phi_s$  and  $\phi_f$  are constrained by the saturation condition  $\phi_s + \phi_f = 1$ , which has to be respected at all times and at all points of the mixture. In the following,  $\varrho_s$  and  $\varrho_f$  shall be assumed to be given constants. Moreover, it will be hypothesised that the inertial terms  $\phi_\alpha \varrho_\alpha \mathbf{a}_\alpha$  are negligible. This latter hypothesis leads to a quasi-static formulation of the problem, in which the only sources of time evolution for the system are provided by time-varying boundary conditions, a (slowly) time-evolving deformation, and the presence of the time-dependent reorganisation of the tissue's internal structure. By accounting for (5c), and summing (5b) over  $\alpha = s, f$ , we obtain

$$\operatorname{div}(\boldsymbol{\sigma}_s + \boldsymbol{\sigma}_f) = \mathbf{0}, \quad \text{in } \mathcal{C}_t \times \mathcal{J}, \quad (6a)$$

$$\operatorname{div} \boldsymbol{\sigma}_f + \mathbf{m}_f = \mathbf{0}, \quad \text{in } \mathcal{C}_t \times \mathcal{J}. \quad (6b)$$

Transforming (5a) by the backward Piola-transformation induced by the solid motion  $\chi(\cdot, t)$ , and writing the transformed equation once for  $\alpha = s$  and once for  $\alpha = f$ , it is possible to show that, after some manipulations, the mass balance laws for the solid and the fluid phase reduce to [44, 45]

$$\phi_s(\chi_s(X, t), t) = \frac{\phi_{sR}(X)}{J(X, t)}, \quad \text{in } \mathcal{C}_R \times \mathcal{J}, \quad (7a)$$

$$\dot{J} + \operatorname{Div} [(J - \phi_{sR}) \mathbf{F}^{-1} \mathbf{u}_{fs}] = 0, \quad \text{in } \mathcal{C}_R \times \mathcal{J}, \quad (7b)$$

with  $\mathbf{u}_{fs} = \mathbf{u}_f - \mathbf{u}_s = \mathbf{v}_{fs} \circ \chi$ . In (7a),  $\phi_{sR}(X)$  represents the volumetric fraction of the solid phase in the reference configuration. Since  $\phi_{sR}$  does not depend on time in the present framework, it can be chosen as a referential value for  $\phi_s$ . We remark that, in the presence of growth, or in the case of non-isochoric plastic distortions, the condition  $J_p = 1$  does not necessarily apply, which means that  $\phi_{sR}$  is not time-independent in general. Rather, in the presence of density-preserving growth [62, 63], it can only be inferred that the volumetric fraction associated with the intermediate configuration, i.e.,  $\phi_{sn}(X) := J_e(X, t) \phi_s(\chi(X, t), t)$ , is constant in time.

### 3 Constitutive framework and remodelling law

If the solid phase exhibits hyperelastic material behaviour from its natural state, and if the fluid phase can be regarded as macroscopically inviscid, then admissible expressions of the Cauchy stresses  $\boldsymbol{\sigma}_s$  and  $\boldsymbol{\sigma}_f$  are given by

$$\boldsymbol{\sigma}_s = -\phi_s p \mathbf{g}^{-1} + \boldsymbol{\sigma}_{sc}, \quad (8a)$$

$$\boldsymbol{\sigma}_f = -\phi_f p \mathbf{g}^{-1}, \quad (8b)$$

295 where  $\boldsymbol{\sigma}_{\text{sc}}$  is the constitutive part of the Cauchy stress associated with the  
296 solid phase, i.e.,

$$\boldsymbol{\sigma}_{\text{sc}} \circ \chi = \frac{1}{J_e} \mathbf{F}_e \left( 2 \frac{\partial \hat{W}_{\text{s}\kappa}}{\partial \mathbf{C}_e}(\mathbf{C}_e) \right) \mathbf{F}_e^{\text{T}}. \quad (9)$$

297 In (9),  $\hat{W}_{\text{s}\kappa}$  is the strain energy density function of the solid phase, expressed  
298 per unit volume of the intermediate configuration, and  $\mathbf{C}_e = \mathbf{F}_p^{-\text{T}} \mathbf{C} \mathbf{F}_p^{-1}$  is  
299 the elastic Cauchy-Green deformation tensor. In this work, we assume that  
300  $\hat{W}_{\text{s}\kappa}$  is of the Holmes-Mow type [57], i.e.,

$$\hat{W}_{\text{s}\kappa}(\mathbf{C}_e) = \alpha_0 \{ \exp(\Psi(\mathbf{C}_e)) - 1 \}, \quad (10a)$$

$$\Psi(\mathbf{C}_e) = \alpha_1 [\hat{I}_1(\mathbf{C}_e) - 3] + \alpha_2 [\hat{I}_2(\mathbf{C}_e) - 3] - \beta \log[\hat{I}_3(\mathbf{C}_e)], \quad (10b)$$

301 where  $\alpha_0$ ,  $\alpha_1$ ,  $\alpha_2$ , and  $\beta$  are model parameters, while  $\hat{I}_1$ ,  $\hat{I}_2$ , and  $\hat{I}_3$  are the  
302 invariants of  $\mathbf{C}_e$ , i.e.,

$$I_1 = \hat{I}_1(\mathbf{C}_e) = \text{tr}(\boldsymbol{\eta}^{-1} \mathbf{C}_e) = \text{tr}(\mathbf{B}_p \mathbf{C}), \quad (11a)$$

$$I_2 = \hat{I}_2(\mathbf{C}_e) = \frac{1}{2} \{ [I_1(\mathbf{C}_e)]^2 - \text{tr}[(\boldsymbol{\eta}^{-1} \mathbf{C}_e)^2] \} = \frac{1}{2} \{ I_1^2 - \text{tr}[(\mathbf{B}_p \mathbf{C})^2] \}, \quad (11b)$$

$$I_3 = \hat{I}_3(\mathbf{C}_e) = \det(\mathbf{C}_e) = J_e^2 = J^2, \quad (11c)$$

303 where the last equality in (11c) is due to the hypothesis of isochoric plastic  
304 distortions, i.e.,  $J_p = 1$ . The strain energy density function  $\hat{W}_{\text{s}\kappa}$  describes  
305 a material exhibiting isotropic elastic properties with respect to the natural  
306 state. In general, if the model of the considered tissue is inhomogeneous, the  
307 parameters  $\alpha_0$ ,  $\alpha_1$ ,  $\alpha_2$ , and  $\beta$  depend on material points. Sometimes, however,  
308 for computational simplicity, or because of lack of experimental data, it is as-  
309 sumed that only one of these parameters is variable. For instance, in modelling  
310 articular cartilage [46],  $\alpha_0$  was expressed by fitting experimental data taken  
311 from the literature as a third-order polynomial function of the axial coordinate  
312 parameterising the depth of a cylindrical specimen of tissue, whereas all the  
313 other material parameters were assumed to be constant.

314 The expressions of  $\boldsymbol{\sigma}_s$  and  $\boldsymbol{\sigma}_f$  reported in (8a) and (8b) can be found in  
315 many works based on Mixture Theory (cf. e.g. [25,81,91]). Here, they have  
316 been adapted from [15,56] to the case of an incompressible, single-constituent  
317 fluid phase, as previously done in [34,35,45,94]. By substituting (8a) and (8b)  
318 into (6a) and (6b), the momentum balance laws for the mixture as a whole  
319 and for the fluid phase become [56]

$$\text{div}(-p \mathbf{g}^{-1} + \boldsymbol{\sigma}_{\text{sc}}) = \mathbf{0}, \quad \text{in } \mathcal{C}_t \times \mathcal{J}, \quad (12a)$$

$$- \mathbf{g}^{-1} (\phi_f \text{grad } p) + (\mathbf{m}_f - p \mathbf{g}^{-1} \text{grad } \phi_f) = \mathbf{0}, \quad \text{in } \mathcal{C}_t \times \mathcal{J}. \quad (12b)$$

320 To determine the material form of the momentum balance law for the mixture  
321 as whole, i.e., the material counterpart of (12a), we introduce the first Piola-  
322 Kirchhoff stress tensors

$$\mathbf{P}_s = J \boldsymbol{\sigma}_s \mathbf{F}^{-\text{T}} = -\phi_{\text{sR}} p \mathbf{g}^{-1} \mathbf{F}^{-\text{T}} + \mathbf{P}_{\text{sc}}, \quad (13a)$$

$$\mathbf{P}_f = J \boldsymbol{\sigma}_f \mathbf{F}^{-\text{T}} = -(J - \phi_{\text{sR}}) p \mathbf{g}^{-1} \mathbf{F}^{-\text{T}}, \quad (13b)$$

323 where  $\mathbf{P}_{\text{sc}} = J\boldsymbol{\sigma}_{\text{sc}}\mathbf{F}^{-\text{T}}$  is the constitutive part of  $\mathbf{P}_{\text{s}}$ , and perform the Piola-  
324 transformation of (12a), i.e.,

$$\text{Div} \left( -Jp\mathbf{g}^{-1}\mathbf{F}^{-\text{T}} + \mathbf{P}_{\text{sc}} \right) = \mathbf{0}, \quad \text{in } \mathcal{C}_{\text{R}} \times \mathcal{J}. \quad (14)$$

325 By adopting the strain energy density function  $\hat{W}_{\text{sr}\kappa}$  specified in (10a) and (10b),  
326  $\mathbf{P}_{\text{sc}}$  can be expressed constitutively as a function of  $\mathbf{F}$  and  $\mathbf{B}_{\text{p}}$ , i.e.,

$$\begin{aligned} \mathbf{P}_{\text{sc}} = \hat{\mathbf{P}}_{\text{sc}}(\mathbf{F}, \mathbf{B}_{\text{p}}) &= 2b_1\mathbf{F}\mathbf{B}_{\text{p}} + 2b_2(I_1\mathbf{F}\mathbf{B}_{\text{p}} - \mathbf{F}\mathbf{B}_{\text{p}}\mathbf{C}\mathbf{B}_{\text{p}}) \\ &\quad + 2b_3I_3\mathbf{g}^{-1}\mathbf{F}^{-\text{T}}, \end{aligned} \quad (15)$$

327 where  $b_i = \hat{b}_i(\mathbf{F}, \mathbf{B}_{\text{p}}) = \frac{\partial \hat{W}_{\text{sr}\kappa}}{\partial I_i}(\mathbf{F}, \mathbf{B}_{\text{p}})$ ,  $i = 1, 2, 3$ , are constitutive functions of  
328 the invariants of  $\mathbf{C}_{\text{e}}$ , and can be thus written as functions of  $\mathbf{F}$  and  $\mathbf{B}_{\text{p}}$ . Hence,  
329 the overall stress  $\mathbf{P} := \mathbf{P}_{\text{s}} + \mathbf{P}_{\text{f}}$ , which has to be substituted into (14), reads

$$\mathbf{P} = \hat{\mathbf{P}}(p, \mathbf{F}, \mathbf{B}_{\text{p}}) = -Jp\mathbf{g}^{-1}\mathbf{F}^{-\text{T}} + \hat{\mathbf{P}}_{\text{sc}}(\mathbf{F}, \mathbf{B}_{\text{p}}). \quad (16)$$

330 For future use, we also introduce the Kirchhoff stress tensor  $\boldsymbol{\tau} = \mathbf{P}\mathbf{F}^{\text{T}}$  and  
331 the Mandel stress tensor  $\boldsymbol{\Sigma} = \mathbf{G}^{-1}\mathbf{F}^{\text{T}}\mathbf{g}\boldsymbol{\tau}\mathbf{F}^{-\text{T}}$ , i.e.,

$$\boldsymbol{\tau} = \hat{\boldsymbol{\tau}}(p, \mathbf{F}, \mathbf{B}_{\text{p}}) = -Jp\mathbf{g}^{-1} + \hat{\boldsymbol{\tau}}_{\text{sc}}(\mathbf{F}, \mathbf{B}_{\text{p}}), \quad (17a)$$

$$\boldsymbol{\Sigma} = \hat{\boldsymbol{\Sigma}}(p, \mathbf{F}, \mathbf{B}_{\text{p}}) = -Jp\mathbf{G}^{-1} + \hat{\boldsymbol{\Sigma}}_{\text{sc}}(\mathbf{F}, \mathbf{B}_{\text{p}}). \quad (17b)$$

332 In (17a) and (17b), the constitutive parts of  $\boldsymbol{\tau}$  and  $\boldsymbol{\Sigma}$ , given by  $\boldsymbol{\tau}_{\text{sc}} = \hat{\boldsymbol{\tau}}_{\text{sc}}(\mathbf{F}, \mathbf{B}_{\text{p}})$   
333 and  $\boldsymbol{\Sigma}_{\text{sc}} = \hat{\boldsymbol{\Sigma}}_{\text{sc}}(\mathbf{F}, \mathbf{B}_{\text{p}})$ , respectively, read

$$\begin{aligned} \boldsymbol{\tau}_{\text{sc}} = \hat{\boldsymbol{\tau}}_{\text{sc}}(\mathbf{F}, \mathbf{B}_{\text{p}}) &= (2b_1 + 2b_2I_1)\mathbf{F}\mathbf{B}_{\text{p}}\mathbf{F}^{\text{T}} - 2b_2\mathbf{F}\mathbf{B}_{\text{p}}\mathbf{C}\mathbf{B}_{\text{p}}\mathbf{F}^{\text{T}} \\ &\quad + 2b_3I_3\mathbf{g}^{-1}, \end{aligned} \quad (18a)$$

$$\begin{aligned} \boldsymbol{\Sigma}_{\text{sc}} = \hat{\boldsymbol{\Sigma}}_{\text{sc}}(\mathbf{F}, \mathbf{B}_{\text{p}}) &= (2b_1 + 2b_2I_1)\mathbf{G}^{-1}\mathbf{C}\mathbf{B}_{\text{p}} - 2b_2\mathbf{G}^{-1}\mathbf{C}\mathbf{B}_{\text{p}}\mathbf{C}\mathbf{B}_{\text{p}} \\ &\quad + 2b_3I_3\mathbf{G}^{-1}. \end{aligned} \quad (18b)$$

334 We remark that, although  $\boldsymbol{\Sigma}_{\text{sc}}$  is not symmetric in general, the assumption of  
335 isotropic hyperelastic response of the solid phase, which leads to (18b), implies  
336 the symmetry conditions [69]

$$\mathbf{B}_{\text{p}}\mathbf{G}\boldsymbol{\Sigma}_{\text{sc}} = (\mathbf{B}_{\text{p}}\mathbf{G}\boldsymbol{\Sigma}_{\text{sc}})^{\text{T}}, \quad (19a)$$

$$\mathbf{G}\boldsymbol{\Sigma}_{\text{sc}}\mathbf{B}_{\text{p}}^{-1} = (\mathbf{G}\boldsymbol{\Sigma}_{\text{sc}}\mathbf{B}_{\text{p}}^{-1})^{\text{T}}. \quad (19b)$$

### 337 3.1 Dissipation inequality

338 The local form of the dissipation inequality characterising the biphasic system  
339 under investigation can be written as follows

$$D_{\text{m}} = -\{\mathbf{m}_{\text{f}} - p\mathbf{g}^{-1}\text{grad } \phi_{\text{f}}\} \cdot \mathbf{v}_{\text{fs}} + \boldsymbol{\sigma}_{\text{sc}} : \mathbf{g}\boldsymbol{\ell}_{\text{p}} \geq 0, \quad (20)$$

where  $D_m$  is the dissipation function of the mixture as a whole, written per unit volume of  $\mathcal{C}_t$  (cf. [15, 44, 56] for details). Note that, more rigorously, and for consistency with (3), we should write  $\sigma_{sc} : \mathbf{g}(\ell_p \circ \chi^{-1})$  in (20). However, for the sake of a lighter notation, we shall omit the composition of maps in the forthcoming calculations. Since  $\ell_p$  has vanishing trace, only the deviatoric part of  $\sigma_{sc}$  contributes to the dissipation  $D_m$ . Following [15, 56, 94],  $\mathbf{m}_f$  can be written as

$$\mathbf{m}_f = \mathbf{m}_{fd} + p \mathbf{g}^{-1} \text{grad } \phi_f, \quad (21)$$

where  $\mathbf{m}_{fd}$  and  $p \mathbf{g}^{-1} \text{grad } \phi_f$  represent, respectively, the dissipative and the non-dissipative contribution to  $\mathbf{m}_f$ . By substituting (21) into (12b), we obtain

$$\mathbf{m}_{fd} = \mathbf{g}^{-1} (\phi_f \text{grad } p). \quad (22)$$

Moreover, the dissipation inequality (20) takes on the form

$$D_m = D_{\text{flow}} + D_{\text{rem}} = \underbrace{-\mathbf{m}_{fd} \cdot \mathbf{v}_{fs}}_{D_{\text{flow}}} + \underbrace{\sigma_{sc} : \mathbf{g} \ell_p}_{D_{\text{rem}}} \geq 0. \quad (23)$$

The inequality (23) states that, within a purely mechanical framework, the only two sources of dissipation for the considered biphasic system are given by  $D_{\text{flow}} := -\mathbf{m}_{fd} \cdot \mathbf{v}_{fs}$ , i.e., the power expended by the dissipative force  $\mathbf{m}_{fd}$ , which is power-conjugate with the relative velocity  $\mathbf{v}_{fs}$ , and by  $D_{\text{rem}} := \sigma_{sc} : \mathbf{g} \ell_p$ , i.e., the power expended to trigger the evolution of the internal structure of the solid phase.

### 3.1.1 Darcy's law

We assume that the dissipative force  $\mathbf{m}_{fd}$  can be expressed constitutively as a linear function of the filtration velocity  $\mathbf{q} := \phi_f \mathbf{v}_{fs}$  [14], i.e.,

$$\mathbf{m}_{fd} = -\mathbf{g}^{-1} \mathbf{r} \phi_f \mathbf{v}_{fs}, \quad (24)$$

where  $\mathbf{r} \in [TS]_2^0$  is the tensor describing the resistivity of the porous medium to fluid flow [56]. By substituting (24) into (23), the following expression of dissipation is obtained

$$D_m = D_{\text{flow}} + D_{\text{rem}} = \underbrace{\text{sym}(\mathbf{r}) : \phi_f (\mathbf{v}_{fs} \otimes \mathbf{v}_{fs})}_{D_{\text{flow}}} + \underbrace{\sigma_{sc} : \mathbf{g} \ell_p}_{D_{\text{rem}}} \geq 0, \quad (25)$$

where  $\text{sym}(\mathbf{r})$  is the symmetric part of  $\mathbf{r}$ . A direct consequence of (25) is that, if  $\text{sym}(\mathbf{r})$  is positive semi-definite,  $D_{\text{flow}}$  is always non-negative, i.e.,  $D_{\text{flow}} \geq 0$ , for any possible realisation of the relative velocity  $\mathbf{v}_{fs}$ . Typically, the resistivity tensor  $\mathbf{r}$  is assumed to be symmetric and positive definite, so that  $\mathbf{m}_{fd}$  vanishes if, and only if,  $\mathbf{v}_{fs}$  is null. Under these assumptions, and the further hypothesis that  $\mathbf{m}_{fd}$  is linear in  $\mathbf{v}_{fs}$ , the standard form of Darcy's law is obtained. Indeed, by substituting (24) into (22), and solving for  $\phi_f \mathbf{v}_{fs}$ , the filtration velocity is found to be

$$\mathbf{q} = \phi_f \mathbf{v}_{fs} = -\mathbf{k} \text{grad } p, \quad (26)$$

370 where  $\mathbf{k} = \phi_f \mathbf{r}^{-1} \in [TS]_0^2$  is the hydraulic conductivity tensor of the system.

371 By performing a Piola transformation of (26), we obtain the material form  
372 of Darcy's law:

$$J\mathbf{F}^{-1}(\phi_f \mathbf{u}_{fs}) = (J - \phi_{sR})\mathbf{F}^{-1}\mathbf{u}_{fs} = -\mathbf{K}\text{Grad } p. \quad (27)$$

373 The material second-order tensor  $\mathbf{K} = J\mathbf{F}^{-1}\mathbf{k}\mathbf{F}^{-T}$  is the material hydraulic  
374 conductivity tensor, and is determined by means of the Piola transformation  
375 of  $\mathbf{k}$  with respect to the solid motion. Substituting (27) into (7b) yields

$$\dot{J} - \text{Div}[\mathbf{K}\text{Grad } p] = 0. \quad (28)$$

376 The constitutive law defining the hydraulic conductivity,  $\mathbf{k}$ , should comply  
377 with the material symmetries of the considered tissue (e.g., isotropy, trans-  
378 verse isotropy, or orthotropy). Recently, a review on several constitutive laws  
379 expressing  $\mathbf{k}$  as a function of the tissue deformation has been given in [13]. An  
380 expression of  $\mathbf{k}$  suitable for articular cartilage was determined by employing  
381 upscaling arguments in the small deformation regime [36, 37], and subsequently  
382 adapted to the finite-deformation framework in [34, 35, 94]. If the hydraulic re-  
383 sponse of the mixture is isotropic, and the hypothesis is made that the isotropy  
384 of the hydraulic conductivity does not change with the deformation, the ten-  
385 sors  $\mathbf{k}$  and  $\mathbf{K}$  can be expressed constitutively as

$$\mathbf{k} = \hat{\mathbf{k}}(J) = \hat{k}_0(J)\mathbf{g}^{-1}, \quad (29a)$$

$$\mathbf{K} = \hat{\mathbf{K}}(\mathbf{F}) = J\hat{k}_0(J)\mathbf{C}^{-1}, \quad (29b)$$

386 where the scalar hydraulic conductivity function  $\hat{k}_0$  is given by

$$\hat{k}_0(J) = k_{0R} \left( \frac{J - \phi_{sR}}{1 - \phi_{sR}} \right)^{m_0} \exp \left[ \frac{m_1}{2}(J^2 - 1) \right], \quad (30)$$

387 and  $m_0$  and  $m_1$  are material parameters [57]. According to (29a) and (30),  
388 the deformation influences the hydraulic conductivity through the volumetric  
389 ratio  $J$  only. When the condition  $J = 1$  is met, the identity  $\hat{k}_0(1) = k_{0R}$  is  
390 obtained, which means that the scalar hydraulic conductivity becomes equal  
391 to the referential one,  $k_{0R}$ . In general,  $k_{0R}$ ,  $m_0$ , and  $m_1$  depend on material  
392 points. The constitutive choice of the hydraulic conductivity tensor permits to  
393 express  $D_{\text{flow}}$  constitutively as

$$D_{\text{flow}} = \hat{D}_{\text{flow}}(J, \text{grad } p) = \hat{\mathbf{k}}(J) : \text{grad } p \otimes \text{grad } p \geq 0, \quad (31)$$

394 with  $\hat{D}_{\text{flow}}$  being quadratic in  $\text{grad } p$ , and highly non-linear in  $J$ .

### 3.1.2 Law of remodelling

In this section, we introduce the fundamental hypotheses that lead to the law of remodelling adopted in our work. We recall that, as announced in Sect. 2.2, we are considering a type of structural evolution that can be interpreted in terms of isochoric plastic distortions. Therefore, the first hypothesis is that  $\mathbf{F}_p$  is restricted by the constraint  $J_p = \det(\mathbf{F}_p) = 1$ . This implies that the rates of plastic distortions,  $\ell_p$  or  $\mathbf{L}_p$ , are deviatoric, i.e.,  $\text{tr}(\ell_p) = 0$ , and  $\text{tr}(\mathbf{L}_p) = 0$ . A direct consequence of these facts is that the part of the dissipation function related to remodelling, i.e.,  $D_{\text{rem}}$ , can be written as

$$D_{\text{rem}} = \sigma_{\text{sc}} : \mathbf{g}\ell_p = J_e^{-1} \boldsymbol{\Sigma}_{\text{sc}\kappa} : \boldsymbol{\eta}\mathbf{L}_p \geq 0. \quad (32)$$

The tensor  $\boldsymbol{\Sigma}_{\text{sc}\kappa} := J_e \boldsymbol{\eta}^{-1} \mathbf{F}_e^{\text{T}} \mathbf{g} \sigma_{\text{sc}} \mathbf{F}_e^{-\text{T}}$  is the constitutive part of the solid phase Mandel stress tensor as computed with respect to the natural state, and represents the measure of stress power-conjugate to  $\mathbf{L}_p$ . The prescription on the non-negativeness of  $D_{\text{rem}}$  is due to the Principle of Maximum Dissipation (cf. e.g. [53]), which is based on the requirement that the overall dissipation function,  $D_m$ , be non-negative for all possible realisations of the generalised velocities  $\phi_f \mathbf{v}_{\text{fs}}$  and  $\ell_p$  (or  $\mathbf{L}_p$ ). Thus, in the case in which the fluid filtration velocity is null, which implies  $D_{\text{flow}} = 0$ , it must hold that  $D_m = D_{\text{rem}} \geq 0$ . We remark that the expression of  $D_{\text{rem}}$  given in (32) appears quite naturally in all the theories of anelastic processes constructed on the BKL decomposition (cf. e.g. [22, 65, 69] for the case of finite strain Elastoplasticity, and [31, 40, 42, 45, 48, 49, 61, 64] for the case of growth and remodelling of biological tissues), and stems from the hypothesis that the strain energy density of the solid phase,  $\hat{W}_{\text{sc}}$ , can be written as a constitutive function of the elastic part of the overall deformation alone,  $\mathbf{C}_e$ , as done in (9). By relating  $\mathbf{F}_p$  with the production of material inhomogeneities in uniform bodies [31], a rationale for this constitutive hypothesis is obtained by invoking the Principle of Material Uniformity [29–31, 83].

The second hypothesis is that the solid phase exhibits isotropic elastic behaviour from its natural state. Since this property implies the symmetry of  $\boldsymbol{\Sigma}_{\text{sc}\kappa}$ ,  $D_{\text{rem}}$  can be rewritten as  $D_{\text{rem}} = J_e^{-1} \boldsymbol{\Sigma}_{\text{sc}\kappa} : \mathbf{D}_p$ . Hence, by exploiting the kinematic identity (4), recalling the relation  $J_p \boldsymbol{\Sigma}_{\text{sc}\kappa} = \boldsymbol{\eta}^{-1} \mathbf{F}_p^{-\text{T}} \mathbf{G} \boldsymbol{\Sigma}_{\text{sc}} \mathbf{F}_p^{\text{T}}$  that links  $\boldsymbol{\Sigma}_{\text{sc}\kappa}$  with  $\boldsymbol{\Sigma}_{\text{sc}}$ , and accounting for the symmetry condition (19b), it is possible to show that  $D_{\text{rem}}$  admits the equivalent form

$$D_{\text{rem}} = -\frac{1}{2J} (\mathbf{G} \boldsymbol{\Sigma}_{\text{sc}} \mathbf{B}_p^{-1}) : \dot{\mathbf{B}}_p. \quad (33)$$

Moreover, since the condition  $J_p = 1$  can be rephrased as  $\mathbf{B}_p^{-1} : \dot{\mathbf{B}}_p = 0$ , only the deviatoric part of  $\boldsymbol{\Sigma}_{\text{sc}}$  contributes to  $D_{\text{rem}}$ . Consequently,  $D_{\text{rem}}$  becomes

$$D_{\text{rem}} = -\frac{1}{2J} [\mathbf{G} \text{dev}(\boldsymbol{\Sigma}_{\text{sc}}) \mathbf{B}_p^{-1}] : \dot{\mathbf{B}}_p \geq 0, \quad (34)$$

with

$$\text{dev}(\boldsymbol{\Sigma}_{\text{sc}}) = \boldsymbol{\Sigma}_{\text{sc}} - \frac{1}{3} \text{tr}[\mathbf{G} \boldsymbol{\Sigma}_{\text{sc}}] \mathbf{G}^{-1}. \quad (35)$$

431 Note that a direct consequence of the hypothesis of isotropy is that the plastic  
 432 flow rule can be expressed in terms of  $\mathbf{B}_p$ , rather than  $\mathbf{F}_p$ . If, on the one hand,  
 433 this leads to a loss of information, on the other hand, computations become  
 434 much lighter.

435 The third hypothesis concerns the type of remodelling addressed in this  
 436 paper. As anticipated in Sect. 2.2, we assume that the considered system re-  
 437 models when the stress induced by external loading exceeds a characteristic  
 438 threshold, thereby triggering the onset of plastic distortions. Hence, as we  
 439 would do in the theory of Elastoplasticity, we search for an evolution law for  
 440 the remodelling variable  $\mathbf{B}_p$  in the form of a generalised plastic “flow rule”. To  
 441 this end, following [42], we imitate the theory of associative, rate-independent  
 442 Elastoplasticity [22,93], and articulate in two steps the determination of the  
 443 flow rule. In the first step, we postulate that a yield surface exists in the space  
 444 of the deviatoric Kirchhoff stress tensors, which can be defined by the equation

$$f(\boldsymbol{\tau}_{sc}) := \varphi(\boldsymbol{\tau}_{sc}) - \sqrt{(2/3)}\tau_y = 0. \quad (36)$$

445 Here,  $\tau_y > 0$  is a scalar measure of stress playing the role of the “yield stress”  
 446 of the considered material, and  $\varphi(\boldsymbol{\tau}_{sc})$  is given by

$$\varphi(\boldsymbol{\tau}_{sc}) := \|\text{dev}(\boldsymbol{\tau}_{sc})\| = \sqrt{\text{tr}[(\mathbf{g}\text{dev}(\boldsymbol{\tau}_{sc}))^2]}. \quad (37)$$

447 In the second step, we require that the plastic flow is orthogonal to the yield  
 448 surface. This leads to the “normality rule” [92,93]

$$\mathcal{L}_{\mathbf{v}_s} \mathbf{b}_e = -2\gamma_p \mathbf{n} \mathbf{g} \mathbf{b}_e, \quad (38)$$

449 where  $\mathcal{L}_{\mathbf{v}_s} \mathbf{b}_e$  is the Lie derivative of  $\mathbf{b}_e = \mathbf{F}_e \boldsymbol{\eta}^{-1} \mathbf{F}_e^T = \mathbf{F} \mathbf{B}_p \mathbf{F}^T$  (i.e., the  
 450 elastic left Cauchy deformation tensor) with respect to the velocity of the solid  
 451 phase,  $\mathbf{v}_s$ ,  $\gamma_p \geq 0$  is a non-negative function of stress, and  $\mathbf{n}$  is the normalised  
 452 Kirchhoff stress tensor, orthogonal to the yield surface, defined by

$$\mathbf{n}^b := \mathbf{g} \mathbf{n} \mathbf{g} = \frac{\partial f}{\partial \boldsymbol{\tau}_{sc}}(\boldsymbol{\tau}_{sc}), \quad \mathbf{n} := \frac{\text{dev}(\boldsymbol{\tau}_{sc})}{\|\text{dev}(\boldsymbol{\tau}_{sc})\|}. \quad (39)$$

453 Finally, by exploiting the identity  $\mathcal{L}_{\mathbf{v}_s} \mathbf{b}_e = \mathbf{F} \dot{\mathbf{B}}_p \mathbf{F}^T$ , and rewriting (38) as an  
 454 evolution law for  $\mathbf{B}_p$ , we obtain the equivalent flow rule

$$\dot{\mathbf{B}}_p = -2\gamma_p \frac{\mathbf{B}_p \mathbf{G} \text{dev}(\boldsymbol{\Sigma}_{sc})}{\|\text{dev}(\boldsymbol{\tau}_{sc})\|}, \quad (40)$$

455 where, coherently with [42], we set

$$\gamma_p := \lambda \left[ \|\text{dev}(\boldsymbol{\tau}_{sc})\| - \sqrt{(2/3)}\tau_y \right]_+ = \lambda [f(\boldsymbol{\tau}_{sc})]_+. \quad (41)$$

456 In (38), (40), and (41),  $\gamma_p$  is a non-negative “plastic” multiplier,  $\lambda$  is a strictly  
 457 positive model parameter, and the operator  $[\cdot]_+$  is such that, for any real  
 458 number  $A$ ,  $[A]_+ = A$ , if  $A > 0$ , and  $[A]_+ = 0$  otherwise. The physical units

of  $\gamma_p$  and  $\lambda$  are  $[\gamma_p] = \text{s}^{-1}$  and  $[\lambda] = (\text{s} \cdot \text{MPa})^{-1}$ , respectively. Equation (40), which represents a stress-driven evolution law for the plastic variable  $\mathbf{B}_p$ , is the remodelling law sought for. We remark that it complies with the prescription  $D_{\text{rem}} \geq 0$ . Indeed, by substituting (40) into (34), we obtain

$$D_{\text{rem}} = \hat{D}_{\text{rem}}(\mathbf{F}, \mathbf{B}_p) = \frac{\gamma_p}{J} \|\text{dev}(\boldsymbol{\tau}_{\text{sc}})\| \geq 0. \quad (42)$$

If, for a given choice of the model parameters,  $\mathbf{F}$  and  $\mathbf{B}_p$  are such that the condition  $f(\boldsymbol{\tau}_{\text{sc}}) \leq 0$  applies (which amounts to say that the Frobenius norm of  $\boldsymbol{\tau}_{\text{sc}}$  is such that  $\|\text{dev}(\boldsymbol{\tau}_{\text{sc}})\| \leq \sqrt{(2/3)}\tau_y$ ), then it holds that  $[f(\boldsymbol{\tau}_{\text{sc}})]_+ = 0$  and, consequently, the plastic multiplier  $\gamma_p$  vanishes identically, thereby implying that  $D_{\text{rem}} = 0$ . In this situation, no remodelling occurs, and the material deforms while preserving its internal structure. However, when  $\|\text{dev}(\boldsymbol{\tau}_{\text{sc}})\|$  exceeds the threshold stress  $\sqrt{(2/3)}\tau_y$  (i.e., when  $f(\boldsymbol{\tau}_{\text{sc}}) > 0$ ), remodelling takes place, and  $\mathbf{B}_p$  evolves as prescribed by (40). In this case,  $D_{\text{rem}}$  becomes

$$D_{\text{rem}} = \frac{\lambda[f(\boldsymbol{\tau}_{\text{sc}})]_+}{J} \|\text{dev}(\boldsymbol{\tau}_{\text{sc}})\| > 0. \quad (43)$$

A relevant difference between the model presented so far and the standard model of associative, rate-independent  $J_2$ -plasticity is that  $\gamma_p$  does not stem from any optimality condition of the Karush-Kuhn-Tucker type [22, 93]. Rather,  $\gamma_p$  is defined phenomenologically, and, in the biological context analysed in [42], it expresses the fact that a cellular aggregate, in which the stress exceeds a prescribed threshold value, reorganises its internal structure by breaking the adhesion bonds connecting the cells. Note also that no hardening is considered in this biological problem.

### 3.2 Summary of the mathematical model

The mathematical model presented in this paper is grounded on the mass balance law (28), the balance law of linear momentum (14), and on the flow rule (40). Thus, in summary, we have to solve the following set of equations:

$$\dot{J} - \text{Div} \left[ \hat{\mathbf{K}}(\mathbf{F}) \text{Grad} p \right] = 0, \quad (44a)$$

$$\text{Div} \left( -Jp \mathbf{g}^{-1} \mathbf{F}^{-T} + \hat{\mathbf{P}}_{\text{sc}}(\mathbf{F}, \mathbf{B}_p) \right) = \mathbf{0}, \quad (44b)$$

$$\dot{\mathbf{B}}_p + \hat{\mathbf{R}}(\mathbf{F}, \mathbf{B}_p) = \mathbf{0}, \quad (44c)$$

in which  $\hat{\mathbf{K}}(\mathbf{F})$  and  $\hat{\mathbf{P}}_{\text{sc}}(\mathbf{F}, \mathbf{B}_p)$  are defined in (29b) and (15), respectively,  $\hat{\mathbf{R}}(\mathbf{F}, \mathbf{B}_p)$  stands for

$$\mathbf{R} \equiv \hat{\mathbf{R}}(\mathbf{F}, \mathbf{B}_p) := 2\gamma_p \frac{\mathbf{B}_p \mathbf{G} \text{dev}(\boldsymbol{\Sigma}_{\text{sc}})}{\|\text{dev}(\boldsymbol{\tau}_{\text{sc}})\|}, \quad (45)$$

and  $\gamma_p$  is specified in (41). The model equations (44a)–(44c) are equivalent to a set of ten scalar equations in the ten unknowns represented by the three

487 components of the solid phase motion,  $\chi$ , pressure,  $p$ , and the six independent  
 488 components of the symmetric second-order tensor  $\mathbf{B}_p$ . The model is thus  
 489 closed. Moreover, it is completed by the following boundary conditions:

$$\chi = \chi_b, \quad \text{on } \Gamma_D^X, \quad (46a)$$

$$\left(-Jp\mathbf{g}^{-1}\mathbf{F}^{-T} + \hat{\mathbf{P}}_{sc}(\mathbf{F}, \mathbf{B}_p)\right) \cdot \mathbf{N} = \mathbf{f}_R, \quad \text{on } \Gamma_N^X, \quad (46b)$$

$$p = p_b, \quad \text{on } \Gamma_D^p, \quad (46c)$$

$$\left(-\hat{\mathbf{K}}(\mathbf{F})\text{Grad}p\right) \cdot \mathbf{N} = Q_b, \quad \text{on } \Gamma_N^p. \quad (46d)$$

490 In (46),  $\mathbf{N}$  is the unit vector normal to  $\partial\mathcal{C}_R$ , i.e., the boundary of the reference  
 491 configuration  $\mathcal{C}_R$ , and the sets  $\Gamma_D^X$  and  $\Gamma_D^p$  are the Dirichlet-portions of  $\partial\mathcal{C}_R$ , on  
 492 which the deformation,  $\chi$ , and the pressure,  $p$ , are equal to the prescribed data  
 493  $\chi_b$  and  $p_b$ , respectively. Analogously,  $\Gamma_N^X$  and  $\Gamma_N^p$  are the Neumann-portions  
 494 of  $\partial\mathcal{C}_R$ , on which the contact force,  $\mathbf{f}_R$ , and the fluid flux,  $Q_b$ , are supplied,  
 495 respectively. It holds that  $\partial\mathcal{C}_R = \Gamma_D^X \sqcup \Gamma_N^X = \Gamma_D^p \sqcup \Gamma_N^p$ . The values assigned to  
 496  $\chi_b$ ,  $p_b$ ,  $\mathbf{f}_R$ , and  $Q_b$  are problem-dependent and should be discussed on a case-  
 497 by-case basis. In the following, however, we shall restrict our formulation to  
 498 a problem obeying Neumann-zero boundary conditions on  $\Gamma_N^X$  and  $\Gamma_N^p$ , which  
 499 implies  $Q_b = 0$  and  $\mathbf{f}_R = \mathbf{0}$ . Finally, initial conditions are needed because (44a)  
 500 and (44c) feature the time derivatives of the volumetric ratio,  $J$ , and of the  
 501 anelastic deformation tensor  $\mathbf{B}_p$ , respectively. Here, we assume the following  
 502 initial conditions:

$$J(X, t_0) = 1, \quad \forall X \in \mathcal{C}_R, \quad (47a)$$

$$\mathbf{B}_p(X, t_0) = \mathbf{G}^{-1}, \quad \forall X \in \mathcal{C}_R. \quad (47b)$$

503 For the sake of simplicity, we assume now that the tissue is homogeneous.  
 504 Thus, all the elements of the sets of parameters  $\{\alpha_0, \alpha_1, \alpha_2, \beta\}$  and  $\{m_0, m_1\}$ ,  
 505 which characterise, respectively, the strain energy density function,  $\hat{W}_{s\kappa}$ , and  
 506 the hydraulic conductivity,  $\mathbf{k}$ , are regarded as constants. We assume that also  
 507  $\phi_{sR}$  and  $k_{0R}$  are constants. Clearly, the hypotheses of homogeneity and isotropy  
 508 provide a poor approximation of real tissues. Nonetheless, they are useful  
 509 hypotheses at this stage, since they help to better visualise the influence of  
 510 remodelling on the mechanical and fluid dynamic properties of the specimen.

## 511 4 Numerics

512 The numerical procedure elaborated in this paper to solve (44a)–(44c) is based  
 513 on the Finite Element Method. Therefore, it is necessary to start with the weak  
 514 formulation of (44a) and (44b). Although there exist numerical strategies that  
 515 perform finite element discretisations also for the plastic flow rule [32], we  
 516 prefer here to keep (44c) in local form. This is legitimate since it involves  
 517 no partial derivative with respect to space coordinates. Before proceeding,  
 518 we notice that the particular choice of the constitutive law expressing the

519 hydraulic conductivity,  $\mathbf{K} = \hat{\mathbf{K}}(\mathbf{F})$ , is such that the mass balance law (44a)  
 520 does not contain  $\mathbf{B}_p$ . This weakens the coupling among the model equations,  
 521 as will be discussed in Sect. 6. Our procedure, however, can be generalised to  
 522 the cases in which constitutive laws of the type  $\mathbf{K} = \hat{\mathbf{K}}(\mathbf{F}, \mathbf{B}_p)$  are employed.

#### 523 4.1 Weak Formulation

524 The weak form of (44a) and (44b) is given by

$$\mathfrak{F}_p(p, \chi, \tilde{p}) := - \int_{\mathcal{C}_R} \left\{ (\text{Grad } \tilde{p}) \left[ \hat{\mathbf{K}}(\mathbf{F}) \text{Grad } p \right] + \tilde{p} j \right\} = 0, \quad (48a)$$

$$\mathfrak{F}_\chi(p, \chi, \mathbf{B}_p, \tilde{\mathbf{u}}) := \int_{\mathcal{C}_R} \hat{\mathbf{P}}(p, \mathbf{F}, \mathbf{B}_p) : \mathbf{g} \text{Grad } \tilde{\mathbf{u}} = 0, \quad (48b)$$

525 where  $\hat{\mathbf{P}}(p, \mathbf{F}, \mathbf{B}_p)$  is defined in (16), and we introduced the space of test  
 526 functions

$$\tilde{\mathcal{P}} \times \tilde{\mathcal{V}} := \{(\tilde{p}, \tilde{\mathbf{u}}) \in H_0^1(\mathcal{C}_R) \times \mathbf{H}_0^1(\mathcal{C}_R) : \tilde{p}|_{\Gamma_D^p} = 0, \tilde{\mathbf{u}}|_{\Gamma_D^x} = \mathbf{0}\}. \quad (49)$$

527 In (48) and (49),  $\tilde{p}$  and  $\tilde{\mathbf{u}}$  denote the test pressure and the test velocity. As such,  
 528 both fields satisfy homogeneous Dirichlet boundary conditions. The functional  
 529 spaces  $H_0^1(\mathcal{C}_R)$  and  $\mathbf{H}_0^1(\mathcal{C}_R)$  are, respectively, the Sobolev spaces of all scalar-  
 530 valued and vector-valued functions vanishing on  $\Gamma_D^p$  and  $\Gamma_D^x$ , square-integrable  
 531 in  $\mathcal{C}_R$ , and whose weak derivatives of order  $m \leq 1$  are all square-integrable in  
 532  $\mathcal{C}_R$  too. We recall that (48a) and (48b) are obtained by multiplying (44a) by  
 533  $\tilde{p}$  and (44b) by  $\tilde{\mathbf{u}}$ , and applying Gauss' Theorem [59]. By construction, the  
 534 functionals  $\mathfrak{F}_p$  and  $\mathfrak{F}_\chi$  are linear in  $\tilde{p}$  and  $\tilde{\mathbf{u}}$ , respectively. For the sake of a  
 535 lighter notation, we omit the explicit dependence of  $\mathfrak{F}_p$  and  $\mathfrak{F}_\chi$  on the test  
 536 fields  $\tilde{p}$  and  $\tilde{\mathbf{u}}$  in the forthcoming discussion. This dependence is, however,  
 537 understood. Finally, we notice that the use of Darcy's law to describe the fluid  
 538 flow implies that both  $\mathfrak{F}_p$  and  $\mathfrak{F}_\chi$  are affine with respect to the pressure  $p$ .

#### 539 4.2 Time-discrete setting

540 The time-discrete version of (48a), (48b) and (44c) is obtained by performing  
 541 an implicit Euler finite difference scheme. To this end, we discretise the time  
 542 interval  $\mathcal{J}$ , over which the system is observed, into  $N$  disjoint subintervals  
 543  $[t_{n-1}, t_n]$ , with  $n \geq 1$ ,  $n \in \mathbb{N}$ , and replace the derivatives  $\dot{J}$  and  $\dot{\mathbf{B}}_p$  with the  
 544 expressions  $(J_n - J_{n-1})/\Delta t_n$  and  $(\mathbf{B}_{pn} - \mathbf{B}_{p(n-1)})/\Delta t_n$ , where  $\Delta t_n$  is the size  
 545 of the  $n$ th time-step. From here on, given an arbitrary function  $f$  of space and  
 546 time, the notation  $f_n(X)$ , or simply  $f_n$ , stands for  $f(X, t_n)$ , for all values of  
 547  $n$ .

548 The initial instant of time  $t_0$  corresponds to the reference, undeformed  
 549 configuration,  $\mathcal{C}_R$ , in which  $J_0 = J(X, t_0) = 1$ . We also set  $\mathbf{B}_{p0} = \mathbf{B}_p(X, t_0) =$   
 550  $\mathbf{G}^{-1}$ , thereby implying that no plastic distortion is associated with the initial

551 state of the solid phase. At the  $n$ th instant of time,  $n \geq 1$ , the time-discrete  
552 model equations become:

$$\mathfrak{F}_p(p_n, \chi_n) := - \int_{\mathcal{C}_R} \left\{ (\text{Grad } \tilde{p}) [\mathbf{K}_n \text{Grad } p_n] + \tilde{p} \frac{J_n - J_{n-1}}{\Delta t_n} \right\} = 0, \quad (50a)$$

$$\mathfrak{F}_\chi(p_n, \chi_n, \mathbf{B}_{pn}) := \int_{\mathcal{C}_R} \hat{\mathbf{P}}(p_n, \mathbf{F}_n, \mathbf{B}_{pn}) : \mathbf{g} \text{Grad } \tilde{\mathbf{u}} = 0, \quad (50b)$$

$$\mathfrak{G}(\chi_n, \mathbf{B}_{pn}) := \mathbf{B}_{pn} - \mathbf{B}_{p(n-1)} + \Delta t_n \hat{\mathbf{R}}(\mathbf{F}_n, \mathbf{B}_{pn}) = \mathbf{0}. \quad (50c)$$

553 The notation  $\mathfrak{G}(\chi_n, \mathbf{B}_{pn})$  means that the time-discrete form of the plastic  
554 flow rule is a function of  $\mathbf{B}_{pn}$ , and a “functional” of the deformation,  $\chi_n$ ,  
555 through the deformation gradient tensor,  $\mathbf{F}_n$  [19]. Equations (50a)–(50c) are  
556 now solved sequentially for varying  $n \geq 1$ . Apart from (50c), the formulation  
557 of (50a) and (50b) largely follows the procedure presented in [44].

### 558 4.3 Linearisation and Finite Element Discretisation

559 In this section, we demonstrate in detail the computational procedure, adapted  
560 from [47], that is used to solve numerically (50a)–(50c). We search for solutions  
561 to the problem (50a)–(50c) by means of a linearisation algorithm based on the  
562 Newton method, and articulated in two stages. In addition to  $n$ , we introduce  
563 two other subscripts: For each  $n$ ,  $l \in \mathbb{N}$  and  $k \in \mathbb{N}$  count the iterations with  
564 respect to  $\mathbf{B}_{pn}$  and the pair  $(p_n, \chi_n)$ , respectively. Consequently, we construct  
565 the sequences

$$\chi_{n,k} = \chi_{n,k-1} + \mathbf{h}_{n,k}, \quad (51a)$$

$$p_{n,k} = p_{n,k-1} + \pi_{n,k}, \quad (51b)$$

$$\mathbf{B}_{pn,l} = \mathbf{B}_{pn,l-1} + \mathbf{\Phi}_{pn,l}, \quad k, l \geq 1 \quad (51c)$$

566 with  $\mathbf{h}_{n,k}$ ,  $\pi_{n,k}$ , and  $\mathbf{\Phi}_{pn,l}$  being the increments associated with  $\chi_n$ ,  $p_n$ , and  
567  $\mathbf{B}_{pn}$ , respectively. Moreover, for conciseness, we adopt the notation

$$w_n := (p_n, \chi_n), \quad (52a)$$

$$\Theta_{n,l-1} := (\chi_n, \mathbf{B}_{pn,l-1}), \quad (52b)$$

$$A_{n,l-1} := (p_n, \chi_n, \mathbf{B}_{pn,l-1}). \quad (52c)$$

568 In the first stage of the algorithm, we linearise (50b) and (50c) with respect to  
569  $\mathbf{B}_{pn}$  only. The linearisation is done in a neighbourhood of  $A_{n,l-1}$  and  $\Theta_{n,l-1}$ ,  
570 respectively, and leads to the approximated expressions

$$\mathfrak{F}_\chi^{(1)}(A_{n,l-1}, \mathbf{\Phi}_{n,l}) := \mathfrak{F}_\chi(A_{n,l-1}) + D_{\mathbf{B}_p} \mathfrak{F}_\chi(A_{n,l-1})[\mathbf{\Phi}_{n,l}], \quad (53a)$$

$$\mathfrak{G}^{(1)}(\Theta_{n,l-1}, \mathbf{\Phi}_{n,l}) := \mathfrak{G}(\Theta_{n,l-1}) + \mathbb{Y}(\Theta_{n,l-1}) : \mathbf{\Phi}_{n,l}, \quad (53b)$$

571 where  $D_{\mathbf{B}_p} \mathfrak{F}_\chi(A_{n,l-1})[\boldsymbol{\Phi}_{n,l}]$  denotes the Gâteaux derivative of  $\mathfrak{F}_\chi$ , computed at  
 572  $A_{n,l-1}$  along the direction of the plastic increment  $\boldsymbol{\Phi}_{n,l}$  [19, 93], and  $\mathbb{Y}(\Theta_{n,l-1})$   
 573 is the fourth-order tensor defined by

$$\mathbb{Y}(\Theta_{n,l-1}) = \frac{\partial \mathcal{G}}{\partial \mathbf{B}_{pn}}(\Theta_{n,l-1}) \in [T\mathbb{C}_R]_2^2. \quad (54)$$

574 Note that  $\mathbb{Y}(\Theta_{n,l-1})$  is pair-symmetric, i.e., in its component representation,  
 575 it holds that

$$[\mathbb{Y}(\Theta_{n,l-1})]_{CD}^{AB} = [\mathbb{Y}(\Theta_{n,l-1})]_{CD}^{BA}, \quad (55a)$$

$$[\mathbb{Y}(\Theta_{n,l-1})]_{CD}^{AB} = [\mathbb{Y}(\Theta_{n,l-1})]_{DC}^{AB}. \quad (55b)$$

576 We remark that also  $\mathfrak{F}_p$  has to be linearised in the same fashion as  $\mathfrak{F}_\chi$  in  
 577 the cases in which it involves  $\mathbf{B}_{pn}$  among its arguments (e.g., if the hydraulic  
 578 conductivity depends on  $\mathbf{B}_p$ ).

579 We now set  $\mathcal{G}^{(1)}(\Theta_{n,l-1}, \boldsymbol{\Phi}_{n,l}) = \mathbf{0}$ , and solve for  $\boldsymbol{\Phi}_{n,l}$ , thereby obtaining

$$\boldsymbol{\Phi}_{n,l} \equiv \hat{\boldsymbol{\Phi}}(\Theta_{n,l-1}) = -[\mathbb{Y}(\Theta_{n,l-1})]^{-1} : \mathcal{G}(\Theta_{n,l-1}). \quad (56)$$

580 In this way, the increment  $\boldsymbol{\Phi}_{n,l}$  is written as a function of  $\chi_n$ . This allows to  
 581 eliminate statically  $\boldsymbol{\Phi}_{n,l}$  from  $\mathfrak{F}_\chi^{(1)}$ . Indeed, by substituting the right-hand-side  
 582 of (56) into (53a), we obtain the new functional

$$\tilde{\mathfrak{F}}_\chi^{(2)}(A_{n,l-1}) = \mathfrak{F}_\chi(A_{n,l-1}) - \mathcal{L}(A_{n,l-1}), \quad (57)$$

583 where the auxiliary quantity  $\mathcal{L}(A_{n,l-1})$  reads

$$\mathcal{L}(A_{n,l-1}) := D_{\mathbf{B}_p} \mathfrak{F}_\chi(A_{n,l-1}) [[\mathbb{Y}(\Theta_{n,l-1})]^{-1} : \mathcal{G}(\Theta_{n,l-1})]. \quad (58)$$

584 In (56), the inversion of  $\mathbb{Y}$  is performed as follows. Let  $d$  denote the dimension  
 585 of the tangent space  $T_X \mathbb{C}_R$  at  $X \in \mathbb{C}_R$  (e.g.,  $d = \dim(T_X \mathbb{C}_R) = 3$ ). Since  $\mathbb{Y}$  is  
 586 a fourth-order tensor, it has  $d^4$  components in the representation

$$\mathbb{Y} = Y^{AB}_{CD} \mathbf{E}_A \otimes \mathbf{E}_B \otimes \mathbf{E}^C \otimes \mathbf{E}^D,$$

587 where  $\{\mathbf{E}_M\}_{M=1}^d \subset T_X \mathbb{C}_R$  and  $\{\mathbf{E}^M\}_{M=1}^d \subset T_X^* \mathbb{C}_R$  are bases of  $T_X \mathbb{C}_R$  and  
 588  $T_X^* \mathbb{C}_R$ , respectively. Now, the set of  $d^4$  scalars  $[Y^{AB}_{CD}]$  is identified with a  
 589  $d^2 \times d^2$  matrix  $[\mathbf{Y}]$ , which can be inverted by means of standard methods. In  
 590 this work, we used a LU-decomposition. We remark that, due to the spatial dis-  
 591 cretisation of the Finite Element Method, this LU-decomposition needs to be  
 592 performed at every integration point of an element. The inverse matrix  $[\mathbf{Y}^{-1}]$   
 593 is thus a representation of the fourth-order tensor  $\mathbb{Y}^{-1}$ . Note that, in (56), the  
 594 second-order tensor  $\mathcal{G}$  is represented as a vector-like column array  $\mathbf{G}$  with  $d^2$   
 595 entries.

596 Now, for fixed  $\mathbf{B}_{pn,l-1}$ , with  $l \geq 1$ , we determine pressure and deformation  
 597 by solving iteratively the sub-problem

$$\tilde{\mathfrak{F}}_\chi^{(2)}(A_{n,l-1}) = 0, \quad (59a)$$

$$\tilde{\mathfrak{F}}_p(w_n) = 0. \quad (59b)$$

598 At the  $k$ th iteration,  $k \geq 1$ , we define

$$w_{n,k} := (p_{n,k}, \chi_{n,k}), \quad (60a)$$

$$\Theta_{n,k,l-1} := (\chi_{n,k}, \mathbf{B}_{pn,l-1}), \quad (60b)$$

$$\Lambda_{n,k,l-1} := (p_{n,k}, \chi_{n,k}, \mathbf{B}_{pn,l-1}), \quad (60c)$$

599 and apply a Newton method to (59a) and (59b). That is, we solve the linearised  
600 set of equations

$$a(\mathbf{h}_{n,k}, \tilde{\mathbf{u}}) - b(\tilde{\mathbf{u}}, \pi_{n,k}) = -\tilde{\mathfrak{F}}_{\chi}^{(2)}(\Lambda_{n,k-1,l-1}), \quad (61a)$$

$$-c(\mathbf{h}_{n,k}, \tilde{p}) - d(\pi_{n,k}, \tilde{p}) = -\tilde{\mathfrak{F}}_p(w_{n,k-1}). \quad (61b)$$

601 In (61a) and (61b), the bilinear forms  $a(\cdot, \cdot)$ ,  $b(\cdot, \cdot)$ ,  $c(\cdot, \cdot)$  and  $d(\cdot, \cdot)$  are  
602 defined by means of the Gâteaux derivatives of the functionals  $\tilde{\mathfrak{F}}_{\chi}^{(2)}$  and  $\tilde{\mathfrak{F}}_p$ :

$$\begin{aligned} a(\mathbf{h}_{n,k}, \tilde{\mathbf{u}}) &= D_{\chi} \tilde{\mathfrak{F}}_{\chi}^{(2)}(\Lambda_{n,k-1,l-1})[\mathbf{h}_{n,k}] \\ &= \int_{\mathbb{C}_{\mathbb{R}}} \mathbf{g} \text{Grad } \tilde{\mathbf{u}} : \mathbb{A}_{n,k-1,l-1}^{(2)} : \mathbf{H}_{n,k}, \end{aligned} \quad (62a)$$

$$\begin{aligned} -b(\tilde{\mathbf{u}}, \pi_{n,k}) &= D_p \tilde{\mathfrak{F}}_{\chi}^{(2)}(\Lambda_{n,k-1,l-1})[\pi_{n,k}] \\ &= \int_{\mathbb{C}_{\mathbb{R}}} \left\{ -J_{n,k-1} \pi_{n,k} \mathbf{F}_{n,k-1}^{-\text{T}} : \text{Grad } \tilde{\mathbf{u}} \right\}, \end{aligned} \quad (62b)$$

$$\begin{aligned} -c(\mathbf{h}_{n,k}, \tilde{p}) &= D_{\chi} \tilde{\mathfrak{F}}_p(w_{n,k-1})[\mathbf{h}_{n,k}] \\ &= - \int_{\mathbb{C}_{\mathbb{R}}} \{ (\text{Grad } \tilde{p}) [(\mathbb{K}_{n,k-1} : \mathbf{H}_{n,k}) \text{Grad } p_{n,k-1}] \} \\ &\quad - \frac{b(\mathbf{h}_{n,k}, \tilde{p})}{\Delta t_n}, \end{aligned} \quad (62c)$$

$$\begin{aligned} -d(\pi_{n,k}, \tilde{p}) &= D_p \tilde{\mathfrak{F}}_p(w_{n,k-1})[\pi_{n,k}] \\ &= - \int_{\mathbb{C}_{\mathbb{R}}} (\text{Grad } \tilde{p}) [\mathbf{K}_{n,k-1} \text{Grad } \pi_{n,k}], \end{aligned} \quad (62d)$$

603 where  $\mathbf{H}_{n,k} = \text{Grad } \mathbf{h}_{n,k}$ , and the tensors  $\mathbb{A}_{n,k-1,l-1}^{(2)}$ ,  $\mathbb{K}_{n,k-1}$  and  $\mathbf{K}_{n,k-1}$  are  
604 given by

$$\begin{aligned} \mathbb{A}_{n,k-1,l-1}^{(2)} &= -J_{n,k-1} p_{n,k-1} \mathbf{g}^{-1} \mathbf{F}_{n,k-1}^{-\text{T}} \otimes \mathbf{F}_{n,k-1}^{-\text{T}} \\ &\quad + J_{n,k-1} p_{n,k-1} \mathbf{g}^{-1} \mathbf{F}_{n,k-1}^{-\text{T}} \overline{\otimes} \mathbf{F}_{n,k-1}^{-1} \\ &\quad + (\mathbb{A}_{n,k-1,l-1}^{\text{sc}} - \mathbb{L}_{n,k-1,l-1}), \end{aligned} \quad (63a)$$

$$\mathbb{K}_{n,k-1} = \frac{\partial \hat{\mathbf{K}}}{\partial \mathbf{F}}(\mathbf{F}_{n,k-1}), \quad (63b)$$

$$\mathbf{K}_{n,k-1} = \hat{\mathbf{K}}(\mathbf{F}_{n,k-1}). \quad (63c)$$

605 In particular,  $\mathbb{A}_{n,k-1,l-1}^{\text{sc}}$  is the constitutive acoustic tensor computed at the  
606  $l$ th iteration in  $\mathbf{B}_{pn}$  and at the  $k$ th iteration in  $\chi_n$ , i.e.,

$$\mathbb{A}_{n,k-1,l-1}^{\text{sc}} = \frac{\partial \hat{\mathbf{P}}^{\text{sc}}}{\partial \mathbf{F}}(\mathbf{F}_{n,k-1}, \mathbf{B}_{pn,l-1}), \quad (64)$$

607 while  $\mathbb{L}_{n,k-1,l-1}$  is a fictitious acoustic tensor, introduced by the algorithm,  
 608 and induced by the Gâteaux derivative of the functional  $\mathfrak{L}$  with respect to the  
 609 deformation (cf. (58)), i.e.,

$$D_{\chi}\mathfrak{L}(\Lambda_{n,k-1,l-1})[\mathbf{h}_{n,k}] := \int_{\mathbb{C}_R} \mathbf{g} \text{Grad } \tilde{\mathbf{u}} : \mathbb{L}_{n,k-1,l-1} : \mathbf{H}_{n,k}. \quad (65)$$

610 It is important to remark that the effective acoustic tensor

$$(\mathbb{A}_{n,k-1,l-1}^{\text{sc}} - \mathbb{L}_{n,k-1,l-1})$$

611 should be positive definite (cf., for example, [17, 18]). Although our numerical  
 612 simulations produced reasonable results, we have not formulated theorems yet,  
 613 which predict when this condition fails to be satisfied.

614 The algorithm proposed in this paper requires a linearisation with respect  
 615 to  $\chi_n$  and one with respect to  $\mathbf{B}_{pn}$ . Therefore, compared with the classical  
 616 RMA [93], an additional linearisation iteration is performed. This increases the  
 617 computational effort, but makes our algorithm more flexible and suitable for  
 618 various types of remodelling laws, which could also be much more complicated  
 619 than the one given in (41).

620 The numerical method presented in this work has been implemented in  
 621 UG4, a novel version of the software framework UG (“Unstructured Grids”)[96].  
 622 Its algebra, discretisation and grid libraries as well as its massive-parallel  
 623 solvers for coupled partial differential equations served as a basis for com-  
 624 puting a benchmark problem of the presented poroplastic model.

## 625 5 Results

626 The unconfined compression test is a very common experimental procedure  
 627 that is performed to determine the mechanical and fluid dynamic properties  
 628 of hydrated soft tissues, such as articular cartilage [51]. In this benchmark, a  
 629 sample of tissue is inserted between two rigid and impermeable parallel plates,  
 630 and compressed according to some prescribed loading protocol, which can be  
 631 either in force- or in displacement-control. During compression, the parts of  
 632 the specimen’s boundary that are not in contact with the plates, and through  
 633 which the interstitial fluid can escape, can expand freely.

634 For our simulations, we consider a cylindrical specimen of biphasic material  
 635 characterised by initial height  $H_0 = 1$  mm and initial radius  $R_0 = 1.5$  mm.  
 636 The lower boundary of the specimen is clamped at the lower plate of the  
 637 experimental apparatus and kept fixed. The upper boundary, instead, is in  
 638 contact with the moving plate and is assumed to expand without friction in  
 639 axial-symmetric way. Finally, the lateral boundary of the specimen is traction-  
 640 free and permeable to fluid flow. The above description of the experiment can  
 641 be translated into mathematical formulae as follows: let  $\Gamma_l$ ,  $\Gamma_L$ , and  $\Gamma_u$  be  
 642 the lower, lateral, and upper boundaries of the specimen in its undeformed

643 configuration (which is taken coincident with the reference configuration), i.e.,  
 644  $\partial\mathcal{C}_R = \Gamma_1 \cup \Gamma_L \cup \Gamma_u$ . Then, for all  $t \in \mathcal{J} \equiv [0, T]$ , we prescribe:

$$\text{on } \Gamma_1, \quad \begin{cases} \chi(X, t) = \chi(X, 0) = X \in \Gamma_1, \\ (-\mathbf{K}\text{Grad } p) \cdot \mathbf{N} = 0, \end{cases} \quad (66a)$$

$$\text{on } \Gamma_L, \quad \begin{cases} (-Jp \mathbf{g}^{-1} \mathbf{F}^{-T} + \mathbf{P}_{sc}) \cdot \mathbf{N} = \mathbf{0}, \\ p = 0, \end{cases} \quad (66b)$$

$$\text{on } \Gamma_u, \quad \begin{cases} \chi^z(X, t) = \chi_b^z(t) [\Theta(t) - \Theta(t - T_0)] + \chi_b^z(T_0) \Theta(t - T_0), \\ (-\mathbf{K}\text{Grad } p) \cdot \mathbf{N} = 0, \end{cases} \quad (66c)$$

645 where  $\Theta$  is the Heaviside function (here defined such that  $\Theta(\xi) = 1$ , if  $\xi \geq 0$ ,  
 646 and  $\Theta(\xi) = 0$ , if  $\xi < 0$ ),  $\chi_b^z$  is the axial compressive deformation imposed at  
 647 the upper boundary, i.e.,

$$\chi_b^z(t) = H_0/2 - u_T \frac{t}{T_0}, \quad (67)$$

648  $T_0 \in (0, T)$  is the final instant of time of the loading ramp, and  $u_T > 0$  is  
 649 the target displacement (here, we took  $u_T = 0.15$  mm). To observe how the  
 650 perturbed system relaxes towards a stationary state, we took  $T_0 = 30$  s and  
 651  $T = 80$  s. Looking at (66a)–(66c), it is clear that the Dirichlet- and Neumann-  
 652 like subsets of  $\partial\mathcal{C}_R$  associated with the deformation,  $\Gamma_D^X$  and  $\Gamma_N^X$ , are given by  
 653  $\Gamma_D^X = \Gamma_1 \cup \Gamma_u$  and  $\Gamma_N^X = \Gamma_L$ , respectively, while it holds that  $\Gamma_D^p = \Gamma_L$  and  
 654  $\Gamma_N^p = \Gamma_1 \cup \Gamma_u$  for the pressure.

655 The material parameters used for this test are reported in Table 1. The  
 656 elastic coefficients for the Holmes-Mow strain energy density given in (10a)  
 657 are selected in such a way that  $\beta = \alpha_1 + 2\alpha_2 = 1$  [57].

**Table 1** material parameters

elastic coefficient	$\alpha_0$	0.125 N/mm <sup>2</sup>
elastic coefficient	$\alpha_1$	0.78
elastic coefficient	$\alpha_2$	0.11
elastic coefficient	$\beta$	1.0
referential hydraulic conductivity	$k_{0R}$	$3.7729 \cdot 10^{-3} \text{ mm}^4 / (\text{N} \cdot \text{s})$
material parameter	$m_0$	0.0848
material parameter	$m_1$	4.638
referential solidity	$\phi_{sR}$	0.2
initial yields stress	$\tau_y$	0.002 N/mm <sup>2</sup>
coefficient in the plastic flow rule	$\lambda$	0.5 mm <sup>2</sup> / (N · s)

658 The results of our numerical tests, reported in Figs. 3-5, are plotted on a  
 659 section of the specimen containing the symmetry axis.

660 To highlight the influence of the plastic distortions on the mechanical and  
 661 fluid dynamic response of the specimen, we compared the radial and axial  
 662 components of the fluid filtration velocity, the pressure distribution, and the  
 663 first invariant of the constitutive part of the Mandel stress tensor obtained in

664 the poroelastic case (left column of Fig. 3) with those obtained in the presence  
 665 of plastic distortions, i.e., in the poroplastic case (right column of Fig. 3). To  
 666 characterise the poroelastic case, we simulated an unconfined compression test  
 667 in which the yields stress,  $\tau_y$ , which determines the onset of plastic flow, was  
 668 set equal to a value that is never reached by the stress in the tissue. More  
 669 precisely, we chose  $\tau_y = 2000.00 \text{ N/mm}^2$ , a value that led to a purely elastic  
 670 material response. In this situation, it holds that  $\mathbf{B}_p = \mathbf{G}^{-1}$  at all times of the  
 671 observation time interval. In the poroplastic case, instead,  $\mathbf{B}_p$  is determined  
 672 by means of the numerical procedure reported in Sect. 4. Figures 5(a) and 5(b)  
 673 show the first and second invariant of  $\mathbf{B}_p$  at the end of the loading ramp, i.e.,  
 674 at  $t = 30 \text{ s}$ . We found that the higher values of these invariants are attained  
 675 at the points of the specimen's boundary corresponding to the intersection of  
 676  $\Gamma_1$  and  $\Gamma_L$ .

677 Before commenting our results, we recall that, in this paper, remodelling  
 678 is entirely described in terms of “plastic” distortions. Thus, plotting figures in  
 679 which plastic distortions are switched off actually means to show the results  
 680 in which no remodelling occurs.

681 Looking at Fig. 3, we notice that the influence of remodelling manifests  
 682 itself through the modulation of the fluid filtration velocity, the change of  
 683 the pore pressure distribution, and the lowering of the constitutive part of  
 684 the stress of the solid phase. In particular, Figs. 3(a) and 3(b) show that the  
 685 magnitudes of the radial and the axial filtration velocity, computed according  
 686 to Darcy's law, i.e.,  $\phi_f \mathbf{v}_{fs} = -\mathbf{k} \text{grad } p$ , decrease in the poroplastic case. In  
 687 Figs. 3(a) and 3(b), the arrows represent the local direction of the flow. The  
 688 decrease of the magnitude of the filtration velocity characterising the poro-  
 689 plastic case is related to the decrease of the pore pressure (see Fig. 3(c)) and  
 690 the decrease of stress, which in Fig. 3(d) is accounted for by the first invariant  
 691 of  $\boldsymbol{\Sigma}_{sc}$ . The values of the first invariant of  $\boldsymbol{\Sigma}_{sc}$  are computed with respect to  
 692 the reference configuration. However, in a visualisation post-process, the refer-  
 693 ence configuration is deformed by the motion map, so that the previously  
 694 computed values of the first invariant are visualised in the deformed config-  
 695 uration. We remark that the onset and evolution of plastic distortions affect  
 696 both quantitatively and qualitatively the time trend of pressure. Indeed, in  
 697 Fig. 4, where pressure is evaluated at the midpoint of the lower boundary of  
 698 the specimen, one can see that at least three facts distinguish the evolution of  
 699 pressure in the poroplastic case (i.e., when remodelling occurs) from that per-  
 700 taining the poroelastic one. Firstly, the maximum value of pressure attained  
 701 in the absence of remodelling is much higher than the maximum reached in  
 702 the presence of remodelling (it should be noticed, however, that in both cases  
 703 the maxima are attained at the end of the loading ramp, i.e., at  $t = 30 \text{ s}$ ).  
 704 Secondly, the rate with which the pressure tends towards the stationary state  
 705 is much higher in the poroplastic case than in the poroelastic one. Thirdly,  
 706 pressure seems to be a convex function of time over the interval  $[0, T_0]$  in the  
 707 poroelastic case, and to become concave in the poroplastic case. A possible  
 708 explanation for the change of the pressure's behaviour could be given by the  
 709 following argument: The pressure and the deformation are determined by the

710 coupled equations representing the mass and linear momentum balance laws,  
 711 i.e., (44a) and (44b). In particular, the balance of momentum (44b) relates the  
 712 pressure,  $p$ , to the constitutive part of the first Piola-Kirchhoff stress tensor,  
 713  $\mathbf{P}_{\text{sc}}$ . The plastic distortions, represented by  $\mathbf{B}_p$ , have a direct influence on  $\mathbf{P}_{\text{sc}}$ ,  
 714 since it holds that  $\mathbf{P}_{\text{sc}} = \hat{\mathbf{P}}_{\text{sc}}(\mathbf{F}, \mathbf{B}_p)$ , and, through the balance of momentum,  
 715 they also have an indirect influence on  $p$ .

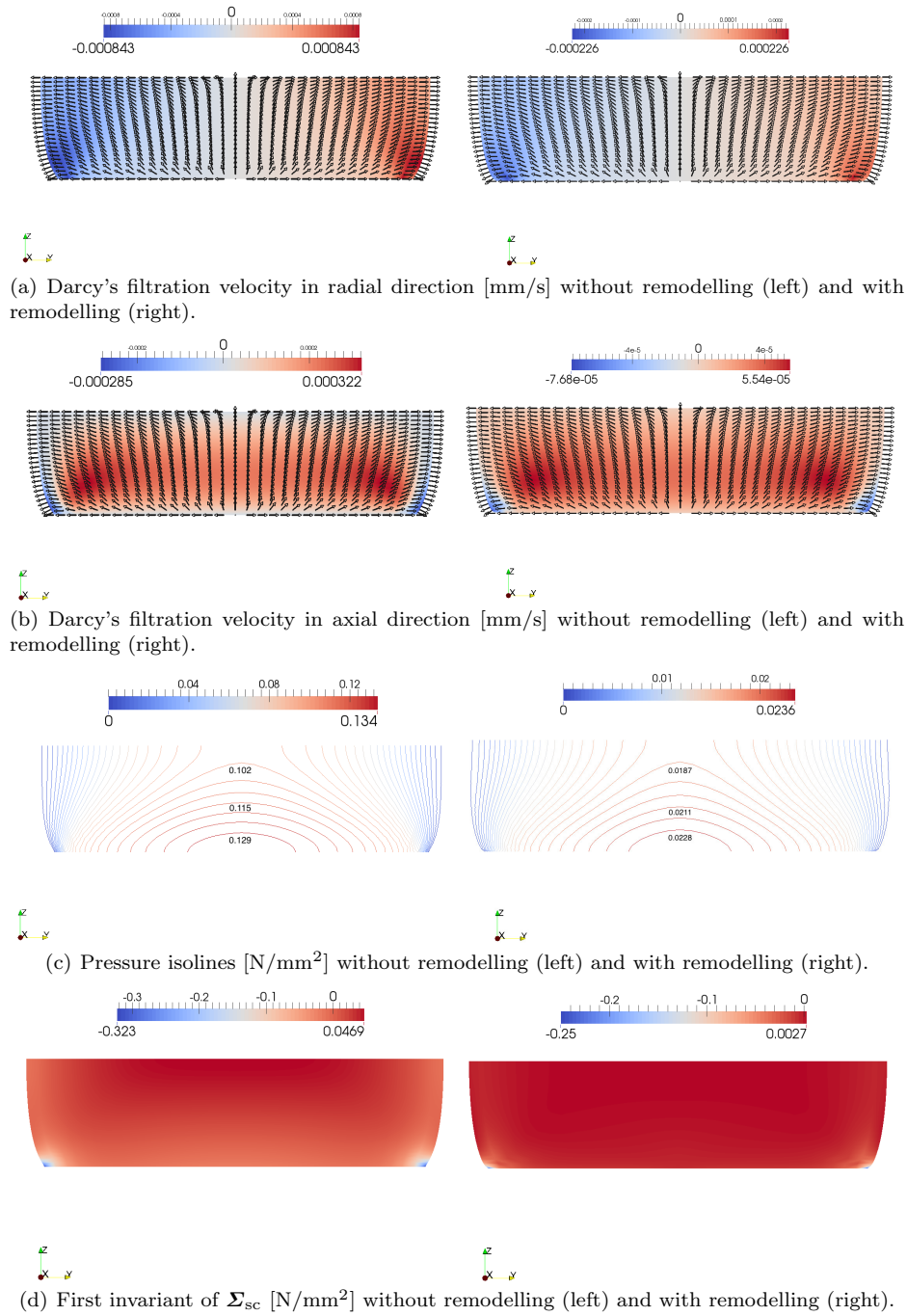
716 The coarsest computational grid consists of 144 prismatic elements. A regu-  
 717 lar refinement is performed three times, so that the finest grid provides 163300  
 718 degrees of freedom for the deformation and the pressure. Both the deforma-  
 719 tion,  $\chi$ , and the pressure,  $p$ , are approximated by linear *ansatz* functions. As  
 720 a numerical solver a Newton-method is applied. Within the Newton iteration,  
 721 a Bi-CGSTAB-method, preconditioned by an ILU-decomposition, solves the  
 722 linearised sub-problems. The non-linear convergence is ensured by the appli-  
 723 cation of a line-search method.

## 724 6 Conclusions

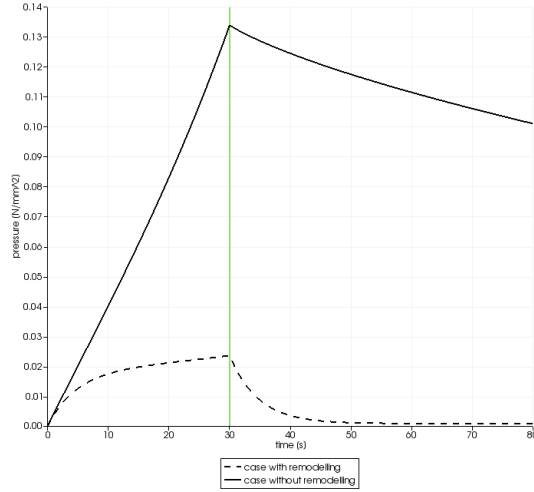
725 In this work, we considered a biphasic, solid-fluid mixture as an idealisation of  
 726 a biological system, and studied its mechanical and fluid dynamic behaviour  
 727 by simulating an unconfined compression test. Our fundamental hypothesis  
 728 was that the mechanical loads applied to the mixture, besides leading to a  
 729 global change of shape, also induce a structural reorganisation of the solid  
 730 phase, which manifests itself through plastic distortions. In order to describe  
 731 this physical picture, we used the poroplastic model reported in (44a)–(44c),  
 732 which results into a set of coupled, and highly non-linear equations. We recall  
 733 that all our calculations have been run under the assumptions that the solid  
 734 phase exhibits hyperelastic response and that the fluid obeys Darcy’s law.

735 Equations (44a)–(44c) were solved numerically by applying a numerical  
 736 procedure recently developed for monophasic continua [47], and adapted to the  
 737 biphasic framework in this paper. The results of our simulations, performed  
 738 with our own code, and implemented in the non-commercial software UG [96],  
 739 are reported in Sect. 5. It is shown that the plastic distortions, described by  $\mathbf{B}_p$ ,  
 740 influence the overall deformation, the stress distribution in the medium, and  
 741 the fluid filtration velocity. This influence can be observed by comparing the  
 742 results obtained in the poroplastic case with those pertaining the poroelastic  
 743 one (see Figs. 3(a)–5(b)). We found that the reorganisation of the medium’s  
 744 internal structure has repercussions on the magnitudes of both the axial and  
 745 the radial component of the Darcy’s filtration velocity, which are smaller in the  
 746 poroplastic case than in the poroelastic one, and has the effect of decreasing  
 747 the fluid pressure as well as the magnitude of the constitutive stress in the  
 748 tissue.

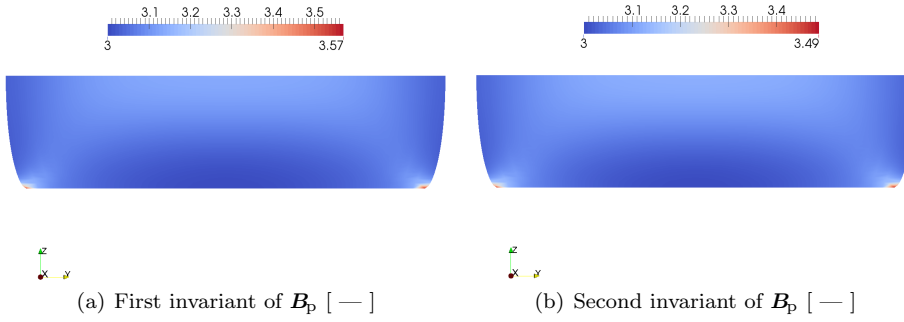
749 Our results could contribute to estimate the mechanical conditions leading  
 750 to the onset of remodelling, and seem to suggest some possible consequences  
 751 of the structural reorganisation of hydrated soft tissues. Moreover, they may  
 752 provide indications about the mechanical conditions regulating the health of



**Fig. 3** Comparison of the results of the unconfined compression test in the absence (left column) and in the presence (right column) of remodelling. All quantities are plotted in the deformed configuration of the sample at time  $t = 30$  s



**Fig. 4** Comparison of the time evolution of pressure, evaluated at the midpoint of the lower boundary of the sample, between the case without remodelling and the case with remodelling



**Fig. 5** First and second invariant of  $\mathbf{B}_p$  in the presence of remodelling. The invariants are computed at time  $t = 30$  s with respect to the reference configuration. However, in a visualisation post-process, the reference configuration is deformed by the motion map, and the invariants are visualised in the deformed configuration.

753 tissues. For example, in the case of articular cartilage, the health of the tissue  
 754 depends on the mechanical environment in which chondrocytes live (the  
 755 chondrocytes are the cells that synthesise extracellular matrix, cf. [54,55,73,  
 756 74] and references therein). Finally, the mathematical model presented in this  
 757 paper could be generalised to include also growth [45] and damage [23,24,41].  
 758 Indeed, both processes have many features in common with remodelling, and  
 759 can be described by extending the BKL decomposition as follows [71]

$$\mathbf{F} = \mathbf{F}_e \mathbf{F}_g \mathbf{F}_p \mathbf{F}_d, \quad (68)$$

760 where  $\mathbf{F}_g$  and  $\mathbf{F}_d$  denote the tensors of anelastic distortions related to growth  
 761 and damage, respectively. A mathematical model based on (68) would require

the introduction of two other independent evolution laws, one for  $\mathbf{F}_g$  and one for  $\mathbf{F}_d$ , which would call for further adapting the numerical procedure presented in Sect. 4. This is part of our current investigations. We remark that the order with which the tensors  $\mathbf{F}_g$  and  $\mathbf{F}_d$  appear in (68) is not unique, but it becomes irrelevant if  $\mathbf{F}_g$  and  $\mathbf{F}_d$  are assumed to be purely volumetric [43].

Equations (44a) and (44b) are rather standard and constitute the starting point for both poroelastic and poroplastic models of solid-fluid mixtures. They are obtained on the basis of a series of assumptions that require: the intrinsic incompressibility of both the solid and the fluid phase, the fluid phase to be macroscopically inviscid (this implying that it can only sustain hydrostatic stresses), the hyperelastic behaviour of the solid phase, and the validity of Darcy’s law. The latter involves the hydraulic conductivity tensor, which was specified in this work by means of the simplest constitutive law for isotropic materials (cf. (29a)). Although many of these hypotheses are physically sound, it could be interesting to investigate the consequences of relaxing some of them. This could lead to more general models that, on the one hand, would stimulate the development of more flexible and efficient computational algorithms, and, on the other hand, might capture some physical aspects (such as, e.g., the pore scale interactions between the solid and the fluid), which are often neglected in the standard theory.

From the computational point of view, assuming that  $\mathbf{k}$  is proportional to  $\mathbf{g}^{-1}$  introduces the great advantage of weakening the coupling among (44a)–(44c). Indeed, since  $\mathbf{B}_p$  does not feature in the mass balance law (44a), the linearisation of the functional  $\mathfrak{F}_p$  needs to be performed only with respect to pressure and deformation (we recall that, actually,  $\mathfrak{F}_p$  is affine in  $p$ , and that the linearisation of  $\mathfrak{F}_p$  with respect to the pressure is done to get the set of equations (61a) and (61b), whose algebraic form leads to a “generalised saddle-point problem” [16, 44]). In other circumstances, however,  $\mathfrak{F}_p$  has to be linearised according to the same procedure as  $\mathfrak{F}_\chi$ . This happens, for instance, if the hydraulic conductivity is isotropic, but its constitutive expression is of the type

$$\mathbf{k} = \hat{\mathbf{k}}(\mathbf{F}, \mathbf{B}_p) = k_0 \mathbf{g}^{-1} + k_1 \mathbf{F} \mathbf{B}_p \mathbf{F}^T. \quad (69)$$

In this case, indeed, the coupling among the model equations is due to both  $\chi$  and  $\mathbf{B}_p$ .

The evolution of the plastic distortions depends strongly on the physics of the anelastic phenomenon that has to be described, and, even when the “same” phenomenon is investigated, it can vary considerably depending on the accuracy of the mathematical model, on the strength of the coupling between the rate of anelastic distortions and the other variables, and on the intrinsic features of the anelastic process (which could be either rate-dependent or rate-independent, either associative or non-associative). In this paper, we chose to describe the evolution of  $\mathbf{B}_p$  by means of (40) because this plastic flow rule has already been successfully employed in [42] to model the reorganisation of cellular aggregates. Equation (40), however, can be generalised to include a great variety of physical situations.

The major limitation of our model is that it is isotropic and homogeneous. Indeed, both the strain energy density,  $\tilde{W}_{sk}$ , and the hydraulic conductivity,  $\mathbf{k}$ , are isotropic (see (10a) and (29a)), and all the parameters appearing in their constitutive expressions, including the referential volumetric fraction of the solid phase,  $\phi_{sR}$ , are set equal to constants. If, on the one hand, the model could be acceptable for studying the structural evolution of tumour tissues, which are often assumed to be elastically and hydraulically isotropic [1, 42, 84], it fails to be accurate for tissues, such as articular cartilage, in which the presence of reinforcing collagen fibres induces anisotropy [35, 78, 79, 94], and the constitutive laws are strongly dependent on material points. In these cases, whereas the balance laws (44a) and (44b) only need to account for the contribution of the fibres to the strain energy density and hydraulic conductivity, the plastic flow rule (44c) should be reformulated. Some of our plans for the future include the specification of the numerical techniques put forward in [39, 75] to anisotropic and inhomogeneous porous media.

One of the projects of our future research is to extend the theoretical and computational framework outlined in this paper to models accounting for phase transitions [28], to theories that describe the reorganisation of the internal structure of a body by augmenting its kinematics [21, 27, 45, 50], and to the more general context of biomechanical models of growth and remodelling that involve, among plasticity [80], damage [81], and pre-stress effects [82], also higher order gradients of the deformation [66, 67].

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## References

1. D. Ambrosi, F. Mollica, On the mechanics of a growing tumor, *Int. J. Eng. Sci.*, 40, 1297–1316 (2002)
2. D. Ambrosi, L. Preziosi, On the closure of mass balance models for tumour growth, *Mathematical Models and Methods in Applied Sciences*, 12(5), 737–754 (2002)
3. D. Ambrosi, L. Preziosi, G. Vitale, The insight of mixtures theory for growth and remodeling, *Z. Angew. Math. Phys.*, 61, 177–191 (2010)
4. D. Ambrosi, L. Preziosi, G. Vitale, The interplay between stress and growth in solid tumors, *Mechanics Research Communications*, 42, 87–91 (2012)
5. U. Andreaus, M. Colloca, D. Iacoviello, M. Pignataro, Optimal-tuning PID control of adaptive materials for structural efficiency, *Struct. Multidiscip. O.*, 43(1), 43–59 (2011)
6. U. Andreaus, M. Colloca, D. Iacoviello, An optimal control procedure for bone adaptation under mechanical stimulus, *Control Eng. Pract.*, 20, 575–583 (2012)
7. U. Andreaus, M. Colloca, D. Iacoviello, Modelling of trabecular architecture as result of an optimal control procedure. In: *Biomedical Imaging and Computational Modeling in Biomechanics*, Iacoviello D. and Andreaus U. (Eds.), Chapter II, pp. 19–37. Springer, Dordrecht, (2012)
8. U. Andreaus, M. Colloca, D. Iacoviello, Optimal bone density distributions: Numerical analysis of the osteocyte spatial influence in bone remodeling, *Comput. Methods Programs Biomed.*, 113(1), 80–91 (2014)

- 852 9. U. Andreaus, I. Giorgio, T. Lekszycki, A 2-D continuum model of a mixture of bone tissue  
853 and bio-resorbable material for simulating mass density redistribution under load slowly  
854 variable in time, *J. Appl. Math. Mech.*, 8, 1–23 (2013), DOI:10.1002/zamm.201200182
- 855 10. U. Andreaus, I. Giorgio, A. Madeo, The influence of different loads on the remodel-  
856 ing process of a bone and bioresorbable material mixture with voids, *Continuum Mech.*  
857 *Thermodyn.*, Published online: 11 December 2014, DOI: 10.1007/s00161-014-0397-y
- 858 11. U. Andreaus, I. Giorgio, A. Madeo, Modeling of the interaction between bone tissue  
859 and resorbable biomaterial as linear elastic materials with voids, *Z. Angew. Math. Phys.*,  
860 66(1), 209–237 (2015)
- 861 12. G.A. Ateshian, On the Theory of Reactive Mixtures for Modeling Biological Growth,  
862 *Biomech. Model. Mechanobiol.*, 6(6), 423–445 (2007)
- 863 13. G.A. Ateshian, J.A. Weiss, Anisotropic hydraulic permeability under finite deformation,  
864 *Journal of Biomedical Engineering*, 132, 111004-111004-7 (2010)
- 865 14. J. Bear, *Dynamics of Fluids in Porous Media*, Dover Publications Inc., New York (1972)
- 866 15. L.S. Bennethum, M.A. Murad, J.H. Cushman, Macroscale thermodynamics and the  
867 chemical potential for swelling porous media, *Transport in Porous Media*, 39, 187–225  
868 (2000)
- 869 16. A. Benzi, G.H. Golub, J. Liesen, Numerical solution of saddle point problems, *Acta*  
870 *Numerica*, 14, 1–137 (2005)
- 871 17. D. Bigoni, D. Zaccaria, Strong ellipticity of comparison solids in elastoplasticity with  
872 volumetric non-associativity, *Int. J. Solid Structures*, 29(17), 2123–2136 (1992)
- 873 18. D. Bigoni, D. Zaccaria, On the eigenvalues of the acoustic tensor in elastoplasticity,  
874 *European Journal of Mechanics A/Solids*, 13(5), 621–638 (1994)
- 875 19. J. Bonet, R.D. Wood, *Nonlinear Continuum Mechanics for Finite Element Analysis*,  
876 Cambridge University Press, Cambridge, New York (2008)
- 877 20. R.M. Bowen, Compressible porous media models by use of the theory of mixtures,  
878 *International Journal of Engineering Science*, 20, 697–735, (1982)
- 879 21. P. Cermelli, E. Fried, S. Sellers, Configurational stress, yield and flow in rate-  
880 independent plasticity, *Proc R Soc A*, 457, 1447–1467 (2001)
- 881 22. S. Cleja-Tigoiu, G.A. Maugin, Eshelby’s stress tensors in finite elastoplasticity, *Acta*  
882 *Mech.*, 139, 231–249 (2000)
- 883 23. L. Contrafatto, M. Cuomo. A new thermodynamically consistent continuum model for  
884 hardening plasticity coupled with damage, *Int J Sol Struct*, 39, 6241–6271 (2002)
- 885 24. M. Cuomo, L. Contrafatto, Stress rate formulation for elastoplastic models with internal  
886 variables based on augmented Lagrangian regularisation, *Int J Sol Struct*, 37, 3935–3964  
887 (2000)
- 888 25. G. Del Bufalo, L. Placidi, M. Porfiri, A mixture theory framework for modeling the  
889 mechanical actuation of ionic polymer metal composites, *Smart Materials and Structures*,  
890 17(4), 1–17, ISSN: 0964-1726, doi: 10.1088/0964-1726/17/4/045010
- 891 26. F. dell’Isola, L. Rosa, C. Woźniak, Dynamics of solids with micro periodic nonconnected  
892 fluid inclusions, *Archive of Applied Mechanics*, 67, 215–228 (1997)
- 893 27. A. DiCarlo, S. Quiligotti, Growth and balance, *Mechanics Research Communications*,  
894 29, 449–456 (2002)
- 895 28. V. Emerenyevev, W. Pietraszkiewicz, The nonlinear theory of elastic shells with phase  
896 transitions, *Journal of Elasticity*, 85(2), 67–86 (2004)
- 897 29. M. Epstein, *The Geometric Language of Continuum Mechanics*. Cambridge: Cambridge  
898 University Press (2010)
- 899 30. M. Epstein, G.A. Maugin, The energy-momentum tensor and material uniformity in  
900 finite elasticity, *Acta Mechanica*, 83, 127–133 (1990)
- 901 31. M. Epstein, G.A. Maugin, Thermomechanics of volumetric growth in uniform bodies,  
902 *Int. J. Plasticity*, 16, 951–978 (2000)
- 903 32. R.A. Eve, B.D. Reddy, The variational formulation and solution of problems of finite-  
904 strain elastoplasticity based on the use of a dissipation function. *Int. J. Numer. Meth.*  
905 *Engng.*, 37, 1673–1695 (1994)
- 906 33. S. Federico, Covariant Formulation of the Tensor Algebra of Non-Linear Elasticity. *Int*  
907 *J Nonlin Mech*, 47, 273–284 (2012)
- 908 34. S. Federico, Porous Materials with Statistically Oriented Reinforcing Fibres. In: *Non-*  
909 *linear Mechanics of Soft Fibrous Materials (CISM Courses and Lectures No. 559, Interna-*  
910 *tional Centre for Mechanical Sciences)*, L. Dorfmann and R.W. Ogden (Eds.), Springer,  
911 Berling, pp. 49–120 (2015)

- 912 35. S. Federico, A. Grillo, Elasticity and permeability of porous fibre-reinforced materials  
913 under large deformations, *Mechanics of Materials*, 44, 58–71 (2012)
- 914 36. S. Federico, W. Herzog, On the permeability of fibre-reinforced porous materials, *Int.*  
915 *J. Solids Struc.*, 45, 2160–2172 (2008)
- 916 37. S. Federico, W. Herzog, On the Anisotropy and Inhomogeneity of Permeability in Ar-  
917 ticular Cartilage, *Biomech Model Mechanobiol*, 7, 367–378 (2008)
- 918 38. L. Fusi, A. Farina, D. Ambrosi, Mathematical modelling of a solid-liquid mixture with  
919 mass exchange between constituents, *Math. Mech. Solids*, 11, 575–595 (2006).
- 920 39. G. Gabriel, K.-J. Bathe. Some Computational Issues in Large Strain Elasto-Plastic  
921 Analysis, *Computers and Structures*, 56(2/3), 249–267 (1995)
- 922 40. J.-F. Ganghoffer, On Eshelby tensors in the context of the thermodynamics of open  
923 systems: Application to volumetric growth, 48(12), 2081–2098 (2010)
- 924 41. T.C. Gasser, An irreversible constitutive model for fibrous soft biological tissue: A 3-D  
925 microfiber approach with demonstrative application to abdominal aortic aneurysms. *Acta*  
926 *Biomater*, 7, 2457–2466 (2011)
- 927 42. C. Giverso, L. Preziosi, Modelling the compression and reorganization of cell aggregates.  
928 *Math. Med. Biol.*, 29, 181–204 (2012)
- 929 43. C. Giverso, M. Scianna, A. Grillo, Growing avascular tumours as elasto-plastic bodies  
930 by the theory of evolving natural configurations, *Mechanics Research Communications*,  
931 <http://dx.doi.org/10.1016/j.mechrescom.2015.04.004>. In press.
- 932 44. A. Grillo, C. Giverso, M. Favino, R. Krause, M. Lampe, G. Wittum, Mass Transport in  
933 Porous Media with Variable Mass. In: *Numerical Analysis of Heat and Mass Transfer in*  
934 *Porous Media, Advanced Structured Materials 27*, Delgado JMPQ et al. (Eds.), Springer,  
935 Berlin, Heidelberg (2012).
- 936 45. A. Grillo, S. Federico, G. Wittum, Growth, mass transfer and remodeling in fiber-  
937 reinforced, multi-constituent materials, *Int J Nonlin Mech*, 47, 388–401 (2012)
- 938 46. A. Grillo, A. Guaily, C. Giverso, S. Federico, Non-linear model for compression tests  
939 on articular cartilage, *Journal of Biomechanical Engineering*, doi: 10.1115/1.4030310. In  
940 press.
- 941 47. A. Grillo, R.-A. Prohl, G. Wittum, A generalised algorithm for anelastic processes in  
942 elastoplasticity and biomechanics. *Mathematics and Mechanics of Solids*. (Accepted)
- 943 48. A. Grillo, S. Federico, G. Wittum, G. Giaquinta, S. Imatani, M.V. Mićunović, Evolution  
944 of a fiber-reinforced mixture, *Nuovo Cimento C*, 32(1), 97–119 (2009).
- 945 49. A. Grillo, G. Wittum, G. Giaquinta, M.V. Mićunović, A multiscale analysis of growth  
946 and diffusion dynamics in biological mixtures, *International Journal of Engineering Sci-*  
947 *ence*, 47, 261–283 (2009)
- 948 50. A. Grillo, G. Wittum, A. Tomic, S. Federico, Remodelling in statistically oriented fibre-  
949 reinforced materials and biological tissues, *Mathematics and Mechanics of Solids*, DOI:  
950 10.1177/1081286513515265.
- 951 51. F. Guilak, A. Ratcliffe, V.C. Mow, Chondrocyte deformation and local tissue strain-  
952 ing articular cartilage: A confocal microscopy study, *Journal of Orthopaedic Research*,  
953 13(3):410-421 (1995).
- 954 52. A. Guillou, R.W. Ogden, Growth in soft biological tissue and residual stress develop-  
955 ment, in: G.A. Holzapfel, R.W. Ogden (Eds.), *Mechanics of Biological Tissue*, Springer,  
956 Berlin, Heidelberg, 2006.
- 957 53. K. Hackl, F.D. Fischer, On the relation between the principle of maximum dissipation  
958 and inelastic evolution given by dissipation potentials, *Proc. R. Soc. A*, 464, 117–132  
959 (2008).
- 960 54. S.-K. Han, S. Federico, A. Grillo, G. Giaquinta, W. Herzog, The mechanical behaviour  
961 of chondrocytes predicted with a micro-structural model of articular cartilage, *Biomech.*  
962 *Model. Mechanobiol.*, 6(3), 139–150 (2007)
- 963 55. S.-K. Han, S. Federico, W. Herzog, A Depth-Dependent Model of the Pericel-  
964 lular Microenvironment of Chondrocytes in Articular Cartilage, *Computer Meth-*  
965 *ods in Biomechanics and Biomedical Engineering*, 14 (7), 657–664 (2010), DOI:  
966 10.1080/10255842.2010.493512
- 967 56. M.S. Hassanizadeh, Derivation of basic equations of mass transport in porous media.  
968 Part II. Generalized Darcy’s and Fick’s Laws, *Advances in Water Resources*, 9, 208–222  
969 (1986)

- 970 57. M.H. Holmes, V.C. Mow, The nonlinear characteristics of soft gels and hydrated con-  
971 nective tissues in ultrafiltration, *Journal of Biomechanics*, 23(11), 1145–1156 (1990).
- 972 58. G.A. Holzapfel, T.C. Gasser, R.W. Ogden, A new constitutive framework for arterial  
973 wall mechanics and a comparative study of material models, *Journal of Elasticity*, 61, 1–48  
974 (2000)
- 975 59. T.J.R. Hughes, *The finite element method: Linear static and dynamic finite element*  
976 *analysis*, Dover, New York, (2000)
- 977 60. T. Lekszycki, F. dell’Isola, A mixture model with evolving mass densities for describing  
978 synthesis and resorption phenomena in bones reconstructed with bio-resorbable materials,  
979 *ZAMM – Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte*  
980 *Mathematik und Mechanik*, 6(92), 426–444 (2012)
- 981 61. Y. Liu, H. Zhang, Y. Zheng, S. Zhang, B. Chen, A nonlinear finite element model of  
982 the stress analysis of soft solids with a growing mass, *Int. J. Solids Struct*, 51(17), 24,  
983 2964–2978 (2014)
- 984 62. B. Loret, F.M.F. Simões, A framework for deformation, generalized diffusion, mass  
985 transfer and growth in multi-species multi-phase biological tissues, *European Journal of*  
986 *Mechanics A/Solids* 24, 757–781 (2005)
- 987 63. V.A. Lubarda, A. Hoger, On the mechanics of solids with a growing mass, *International*  
988 *Journal of Solids and Structures* 39, 4627–4664 (2002)
- 989 64. V.A. Lubarda, Constitutive theories based on the multiplicative decomposition of defor-  
990 mation gradient: Thermoelasticity, elastoplasticity, and biomechanics, *Appl. Mech. Rev.*,  
991 57(2), 95–108 (2004)
- 992 65. J. Lubliner, *Plasticity Theory*, Dover Publications, Inc., Mineola, New York (2008)
- 993 66. A. Madeo, T. Lekszycki, F. dell’Isola, A continuum model for the biomechanical inter-  
994 actions between living tissue and bio-resorbable graft after bone reconstructive surgery,  
995 *CR Mecanique*, 339, 625–640 (2011)
- 996 67. A. Madeo, F. dell’Isola, F. Darve, A continuum model for deformable, second gradient  
997 porous media partially saturated with compressible fluids. *J Mech Phys Sol*, 61, 2196–2211  
998 (2013)
- 999 68. J.E. Marsden, T.J.R. Hughes, *Mathematical Foundations of Elasticity*, Dover Publica-  
1000 tions Inc., New York (1983)
- 1001 69. G.A. Maugin, M. Epstein, Geometrical Material Structure of Elastoplasticity, *Internation-*  
1002 *al Journal of Plasticity*, 14(1-3), 109–115 (1998)
- 1003 70. A. Menzel, A fibre reorientation model for orthotropic multiplicative growth. Configu-  
1004 rational driving stresses, kinematics-based reorientation and algorithmic aspects. *Biome-*  
1005 *chanics and Modeling Mechanobiology*, 6(5), 303–320 (2007)
- 1006 71. M.V. Mićunović, *Thermomechanics of viscoplasticity—fundamentals and applications*.  
1007 Springer, Heidelberg (2009)
- 1008 72. Minozzi, M., Nardinocchi, P., Teresi, L., Varano, V.: Growth-induced compatible strains.  
1009 *Math. Mech. Solids* (2015). doi: 10.1177/1081286515570510.
- 1010 73. E.K. Moo, W. Herzog, S.-K. Han, N.A. Abu Osman, B. Pinguan-Murphy, S. Federico,  
1011 *Mechanical Behaviour of In-Situ Chondrocytes Subjected to Different Loading Rates: A*  
1012 *Finite Element Study, Biomechanics and Modeling in Mechanobiology*, 11(7), 983–993  
1013 (2012) DOI: 10.1007/s10237-011-0367-2
- 1014 74. E.K. Moo, S.-K. Han, S. Federico, A. Jinha, S. Sibole, N.A. Abu Osman, B. Pinguan-  
1015 Murphy, W. Herzog, Extracellular Matrix Integrity Affects the Mechanics of In-Situ Chon-  
1016 drocytes under Compression, *Journal of Biomechanics*, 47(5), 1004–1013 (2014) DOI:  
1017 10.1016/j.jbiomech.2014.01.003
- 1018 75. F.J. Montáns, K.-J. Bathe, Computational issues in large strain elasto-plasticity: an al-  
1019 gorithm for mixed hardening and plastic spin. *International Journal of Numerical Methods*  
1020 *in Engineering*, 63, 159–196 (2005)
- 1021 76. B. Nedjar, On finite strain poroplasticity with reversible and irreversible porosity laws.  
1022 Formulation and computational aspects, *Mechanics of Materials*, 68, 237–252 (2014)
- 1023 77. T. Olsson, A. Klarbring, Residual stresses in soft tissue as a consequence of growth  
1024 and remodelling: Application to an arterial geometry, *European Journal of Mechanics*  
1025 *A/Solids*, 27, 959–974 (2008)
- 1026 78. D.M. Pierce, T. Ricken, G.A. Holzapfel, A hyperelastic biphasic fibre-reinforced model  
1027 of articular cartilage considering distributed collagen fibre orientations: continuum basis,

- 1028 computational aspects and applications, *Comput. Methods Biomech. Biomed. Eng.*, 16,  
1029 1344–1361 (2013)
- 1030 79. D.M. Pierce, T. Ricken, G.A. Holzapfel, Modeling sample/patient-specific structural  
1031 and diffusional responses of cartilage using DT-MRI. *Int J Numer Methods Biomed Eng*,  
1032 29, 807–821 (2013)
- 1033 80. L. Placidi, A variational approach for a nonlinear one-dimensional damage-elasto-plastic  
1034 second-gradient continuum model, *Continuum Mechanics and Thermodynamics*, ISSN:  
1035 0935-1175, doi: 10.1007/s00161-014-0405-2 (In press)
- 1036 81. L. Placidi, A variational approach for a nonlinear 1-dimensional second gradient contin-  
1037 uum damage model, *Continuum Mechanics and Thermodynamics*, 1–16, ISSN: 0935-1175,  
1038 doi: 10.1007/s00161-14-0338-9 (2014)
- 1039 82. L. Placidi, F. dell’Isola, N. Ianiro, G. Sciarra, Variational formulation of pre-stressed  
1040 solid-fluid mixture theory, with an application to wave phenomena, *European Journal of*  
1041 *Mechanics, A/Solids*, 27(4), 582–606 (2008)
- 1042 83. S. Preston, E. Elżanowski, Material uniformity and the concept of the stress space. In:  
1043 *Continuous Media with Microstructure*, 91–101, Springer, Heidelberg (2010)
- 1044 84. L. Preziosi, D. Ambrosi, C. Verdier, An elasto-visco-plastic model of cell aggregates,  
1045 *Journal of Theoretical Biology*, 262(1), 35–47 (2010)
- 1046 85. L. Preziosi, A. Farina, On Darcy’s law for growing porous media, *International Journal*  
1047 *of Non-Linear Mechanics*, 37(3), 485–491 (2002)
- 1048 86. L. Preziosi, G. Vitale, A multiphase model of tumour ad tissue growth including cell ad-  
1049 hesions and plastic reorganization, *Mathematical Models and Methods in Applied Science*,  
1050 21, 1901–1932 (2011)
- 1051 87. S. Quiligotti, On bulk growth mechanics of solid-fluid mixtures: kinematics and invari-  
1052 ance requirements, *Theor. Appl. Mech. TEOPM7*, 28, 1–11 (2002)
- 1053 88. S. Quiligotti, G.A. Maugin, F. dell’Isola, An Eshelbian approach to the nonlinear me-  
1054 chanics of constrained solid-fluid mixtures. *Acta Mechanica*, 160, 45–60, (2003)
- 1055 89. K.R. Rajagopal, Multiple configurations in continuum mechanics, *Rep. Inst. Comput.*  
1056 *Appl. Mech*, 6 (1995)
- 1057 90. E.K. Rodriguez, A. Hoger, A.D. McCulloch, Stress-dependent finite growth in soft elas-  
1058 tic tissues, *Journal of Biomechanics*, 27, 455–467 (1994)
- 1059 91. G. Sciarra, F. dell’Isola, K. Hutter, Dilatational and compacting behavior around a  
1060 cylindrical cavern leached out in a solid-fluid elastic rock salt, *International Journal of*  
1061 *Geomechanics*, 5(3), 233–243 (2005)
- 1062 92. J.C. Simo, A framework for finite strain elastoplasticity based on maximum plastic dis-  
1063 sipation and the multiplicative decomposition: Part I. *Continuum Formulation*, *Comput.*  
1064 *Mech. Appl. M.*, 66, 199–219 (1988).
- 1065 93. J.C. Simo, T.J.R. Hughes, *Computational Plasticity*, Springer, New York (1988)
- 1066 94. A. Tomic, A. Grillo, S. Federico, Poroelastic materials reinforced by statistically oriented  
1067 fibres—numerical implementation and application to articular cartilage. *IMA Journal of*  
1068 *Applied Mathematics*, 79(5), 1017–1059 (2014) DOI:10.1093/imamat/hxu039.
- 1069 95. C. Truesdell, W. Noll, *The Non-Linear Field Theories of Mechanics*, *Handuch der*  
1070 *Physik*, III/3, Berlin: Ed. S. Flugge. Springer-Verlag.
- 1071 96. A. Vogel, S. Reiter, M. Rupp, A. Nägel, G. Wittum, UG4 - A Novel Flexible Software  
1072 System for Simulating PDE Based Models on High Performance Computers, *Computing*  
1073 *and Visualization in Science*, DOI: 10.1007/s00791-014-0232-9 (2013)