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A POSTERIORI ERROR ESTIMATE FOR A PDE-CONSTRAINED OPTIMIZATION FORMULATION FOR THE FLOW IN DFNs*

STEFANO BERRONE†, ANDREA BORIO†, AND STEFANO SCIALÒ†

Abstract. Flows in fractured media have been modeled with many different approaches in order to get reliable and efficient simulations for many critical applications. The common issues to be tackled are the wide range of scales involved in the phenomenon, the complexity of the domain, and the huge computational cost. In this paper we introduce residual-based "a posteriori" error estimates for a formulation of the flow in the discrete fracture networks based on a constrained optimization approach (see Berrone, Pieraccini, and Scialò [*SIAM J. Sci. Comput.*, 35 (2013), pp. B487–B510], [*SIAM J. Sci. Comput.*, 35 (2013), pp. A908–A935], [*J. Comput. Phys.*, 256 (2014), pp. 838–853]), suitable to overcome all the difficulties related to a good quality mesh generation with conformity requirement.

Key words. "a posteriori" error estimates, discrete fracture networks, single-phase flows, PDE-constrained optimization

AMS subject classifications. 65N15, 65N30, 65N50, 86-08, 86A05

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1. Introduction.

The simulation of flows in underground formations is a complex and challenging task, involving multiple physical phenomena on different scales and intricate computational domains. Among the different models available in the literature, discrete fracture network (DFN) models aim at a simplified representation of the network of fractures in the underground, but still reflecting the fracture pattern of the peculiar geological site under investigation and its key hydrological and geometrical characteristics [1, 12, 15]. A DFN is a stochastically generated network of fracture-resembling planar polygons in a three-dimensional space. Size, orientation, density, and hydrological properties of the fractures, such as the hydraulic transmissivity, are determined using probability distributions based on probing data and laboratory tests on soil specimens. Due to the stochastic nature of the input data, uncertainty quantification methods are then used to describe the flow properties [18, 6, 16]. The quantity of interest is the hydraulic head in the whole system of fractures, given by the sum of the pressure head and elevation. This is ruled by the Darcy law on each fracture with additional constraints of continuity and flux conservation at fracture intersections, called traces. Uncertainty quantification strategies for DFN problems require the repeated computation of the hydraulic head on stochastically generated networks; therefore reliability and efficiency of numerical tools are of paramount importance in this context. The major source of complexity lies in the intricate nature of the domain, characterized by intersecting fractures possibly with extremely narrow angles, almost overlapping parallel traces [10], and, due to the multiscale nature of the problem, the simultaneous presence of very small and very large fractures intersecting each other. As a consequence, the generation of a

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†Dipartimento di Scienze Matematiche, Politecnico di Torino, 10129 Torino, Italy (stefano.berrone@polito.it, andrea.borio@polito.it, stefano.scialo@polito.it).

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conforming mesh suitable for the resolution of the problem with finite element–based discretizations might be very hard or even impossible [16]. A variety of strategies is available in the literature in order to overcome these difficulties, mainly suggesting the use of mortar methods to relax the conformity constraint at fracture intersections [21, 22], sometimes coupled with mixed [13] and domain decomposition methods [23]; these geometrical difficulties may be even more relevant for more complex flow models such as multiphase flows. In other papers the conformity requirement is accomplished through modification of the geometry of the DFN [16, 19]. In the approach developed in [7, 8, 9] the DFN problem is seen as a PDE-constrained optimization problem, in which a cost functional measuring the discontinuity and flux unbalance at fracture intersections is minimized, constrained by the Darcy law on the fractures. In this framework, no mesh conformity is required at fracture intersections, and the solution is obtained through the resolution of small weakly dependent subproblems on the fractures with an iterative solver. Any difficulty related to the generation of the mesh is avoided, and the approach has a natural parallel implementation with good scalability performances [10]. Further, no modification of the geometry of the network is required, and this is particularly important for uncertainty quantification procedures, in which a modification of the disposition of fracture would imply a modification of the probabilistic law at the basis of the generation of the network.

In the present paper, residual-based "a posteriori" error estimates [25, 26, 14, 17, 27, 2] are derived for the optimization formulation of the DFN problem described above, in view of a possible future use within an adaptive algorithm. In deriving the a posteriori error estimates, particular attention is devoted to highlighting the effect of discontinuities of the discrete solution and unbalance of fluxes at fracture intersections that can cross the interior of mesh elements. Indeed, the error estimator proposed herein contains several additional terms with respect to classical residual-based a posteriori error estimates; some of these additional terms exploit known properties of the exact solution. Moreover, part of the work is devoted to estimating the errors generated by the nonconformity of triangles to fracture intersections and to tracking the influence of this nonconformity on the effectivity of the global estimate. In particular, in deriving the lower bounds (Theorem 6.1) we explicitly define a nonconformity coefficient that affects the effectivity index.

The structure of the paper is as follows. In section 2 some useful notation concerning DFNs is introduced; in section 3 the problem and its discrete formulation are stated; in sections 4 and 5 suitable estimators are defined and an upper bound of the error is provided; in section 6 the efficiency of these estimators is proved; and in section 7 some numerical results are described.

2. Nomenclature and main assumptions. In the present work we consider a network of fractures surrounded by an impervious rock matrix, with flow occurring only through fractures and across fracture intersections in the normal direction. Let us denote by Ω the DFN, composed of N intersecting fractures (see Figure 1a). Each fracture F_i, $i \in \mathcal{I} = \{1, \ldots, N\}$, is a planar open polygon, with boundary ∂F_i, and the boundary of Ω is $\partial\Omega = \bigcup_{i\in\mathcal{I}} \partial F_i$. We assume that all the fractures in Ω are connected, i.e., each fracture has at least one intersection with another fracture in the network, and we call these intersections *traces*, each denoted by Γ_m, with $m \in \mathcal{M} = \{1, \ldots, M\}$. For the sake of simplicity we assume that there are no intersections between traces and that each trace is shared by exactly two fractures; so, if $\Gamma_m = \bar{F}_i \cap \bar{F}_j$, there is a bijective correspondence between the index m and the couple of indices (i, j), thus allowing us to define the ordered couple $\mathcal{I}_m = (i, j)$, $i < j$

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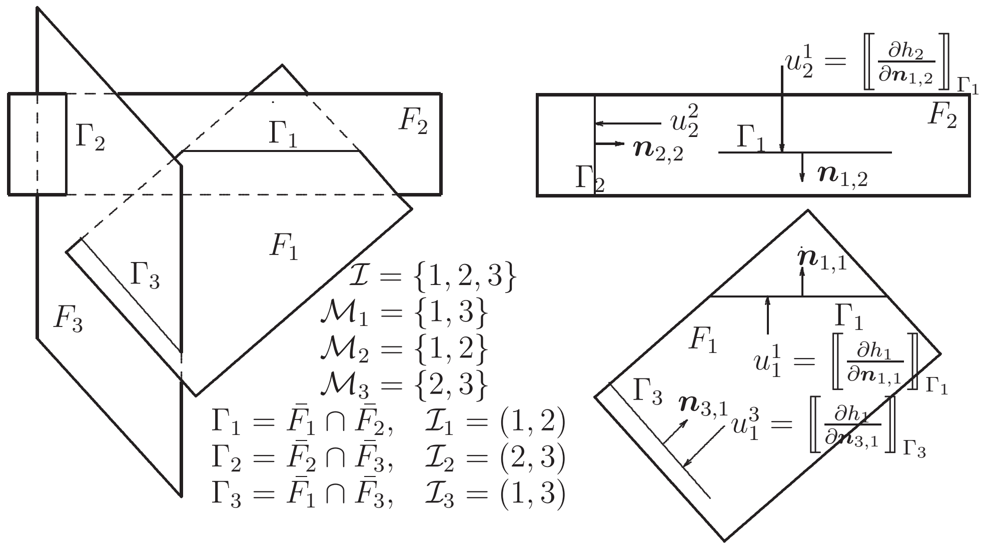

(a) (b)

FIG. 1. (a) *Simple DFN with three rectangular fractures. For each fracture F_i we list the set of the indices of the traces of that fracture $\mathcal{M}_i$, and for each trace Γ_m we list the ordered set of the intersecting fractures along that trace $\mathcal{I}_m = (i, j)$, with $i < j$.* (b) *Detail of normals and fluxes on the fractures F_1 and F_2.*

(see [7] for relaxing these hypotheses). We further introduce, for each fracture F_i, the ordered set $\mathcal{M}_i \subset \mathcal{M}$ (Figure 1a) collecting indices of traces belonging to $\bar{F}_i$ in increasing order, with $M_i = \#\mathcal{M}_i$. $\mathcal{M}_i(k)$ for $k = 1, \ldots, M_i$ indicates the kth index of a trace in $\mathcal{M}_i$. For each $i \in \mathcal{I}$ and each $m \in \mathcal{M}_i$, $\boldsymbol{n}_{m,i}$ is a fixed normal unit vector to the trace Γ_m on F_i (Figure 1b). The reader can refer to Figure 2 for some simple DFNs and to [10, 3, 5] for more complex ones.

We denote by $(\cdot,\cdot)_\omega$ and by $\|\cdot\|_\omega$ the scalar product in $\mathrm{L}^2(\omega)$ and the $\mathrm{L}^2(\omega)$ norm, respectively, and by $|\!|\!|\cdot|\!|\!|_\omega$ the norm in $\mathrm{H}^1_0(\omega)$. Further $(\cdot,\cdot)_{\alpha,\omega}$ and $\|\cdot\|_{\alpha,\omega}$ indicate the scalar product and the norm on $\mathrm{H}^\alpha(\omega)$, respectively. For any given segment $\sigma \subset F_i$, $i \in \mathcal{I}$, $\gamma^i_\sigma : \mathrm{H}^1_0(F_i) \to \mathrm{H}^{\frac{1}{2}}(\sigma)$ is the trace operator and

$$\langle \mu, \beta \rangle_\sigma := {}_{\mathrm{H}^{-\frac{1}{2}}(\sigma)}\langle \mu, \beta \rangle_{\mathrm{H}^{\frac{1}{2}}(\sigma)} \quad \forall \mu \in \mathrm{H}^{-\frac{1}{2}}(\sigma), \forall \beta \in \mathrm{H}^{\frac{1}{2}}(\sigma)$$

is the duality between $\mathrm{H}^{\frac{1}{2}}(\sigma)$ and $\mathrm{H}^{-\frac{1}{2}}(\sigma)$. For any given function $v \in \mathrm{H}^1_0(F_i)$, $\gamma_{\mathcal{M}_i}(v) \in \prod_{m \in \mathcal{M}_i} \mathrm{H}^{\frac{1}{2}}(\Gamma_m)$ is the tuple of functions $\gamma^i_{\Gamma_m}(v)$, $m \in \mathcal{M}_i$ ordered by increasing trace index m, and we denote the duality between product spaces on the set of the traces of a fracture as

$$\forall \boldsymbol{\mu} \in \prod_{m \in \mathcal{M}_i} \mathrm{H}^{-\frac{1}{2}}(\Gamma_m), \forall \boldsymbol{\beta} \in \prod_{m \in \mathcal{M}_i} \mathrm{H}^{\frac{1}{2}}(\Gamma_m), \;\; \langle \boldsymbol{\mu}, \boldsymbol{\beta} \rangle_{\mathcal{M}_i} := \sum_{m \in \mathcal{M}_i} \langle \mu_m, \beta_m \rangle_{\Gamma_m}.$$

Let us introduce the functional space $V := V_1 \times \cdots \times V_N$, where $V_i := \mathrm{H}^1_0(F_i)$ $\forall i \in \mathcal{I}$. For any function $\boldsymbol{g} \in V$, we define the jump operator across a trace Γ_m as $[\![\boldsymbol{g}]\!]_{\Gamma_m} := \gamma^i_{\Gamma_m}(g_i) - \gamma^j_{\Gamma_m}(g_j)$ $\forall m \in \mathcal{M}$ and $(i,j) = \mathcal{I}_m$. Then $[\![\boldsymbol{g}]\!]_{\mathcal{M}_i}$ is the vector of jumps of $\boldsymbol{g}$ across the traces in $\mathcal{M}_i$, ordered according to trace index: $[\![\boldsymbol{g}]\!]_{\mathcal{M}_i} := \left([\![\boldsymbol{g}]\!]_{\Gamma_{\mathcal{M}_i(1)}}, \ldots, [\![\boldsymbol{g}]\!]_{\Gamma_{\mathcal{M}_i(M_i)}} \right)$. Similarly, given a function $g_i \in V_i$, $\left[\!\left[\frac{\partial g_i}{\partial \boldsymbol{n}_{m,i}}\right]\!\right]_{\Gamma_m}$ is the

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jump of the conormal derivative across Γ_m on F_i, and we define the tuple

$$\left[\!\left[\frac{\partial g_i}{\partial \boldsymbol{n}}\right]\!\right]_{\mathcal{M}_i} := \left(\left[\!\left[\frac{\partial g_i}{\partial \boldsymbol{n}_{\mathcal{M}_i(1),i}}\right]\!\right]_{\Gamma_{\mathcal{M}_i(1)}}, \dots, \left[\!\left[\frac{\partial g_i}{\partial \boldsymbol{n}_{\mathcal{M}_i(M_i),i}}\right]\!\right]_{\Gamma_{\mathcal{M}_i(M_i)}}\right).$$

3. Problem formulation. Let us denote the unknown hydraulic head in Ω as $\boldsymbol{h} = (h_1, \dots, h_N) \in V$, where $h_i \in V_i$ for $i = 1, \dots, N$ is the hydraulic head on F_i. Then, in a simplified setting, using homogeneous Dirichlet boundary conditions, the DFN problem can be stated as follows: *find* $\boldsymbol{h} \in V$ *such that* $\forall i \in \mathcal{I}$

$$(\nabla h_i, \nabla v)_{F_i} = (f_i, v)_{F_i} + \left\langle \left[\!\left[\frac{\partial h_i}{\partial \boldsymbol{n}}\right]\!\right]_{\mathcal{M}_i}, \gamma_{\mathcal{M}_i}(v) \right\rangle_{\mathcal{M}_i} \quad \forall v \in V_i, \tag{3.1}$$

where $f_i \in \mathrm{L}^2(F_i)$ $\forall i \in \mathcal{I}$ is a function representing source terms on the fracture. At fracture intersections, additional matching conditions are added, enforcing continuity of the hydraulic head and conservation of fluxes: $\forall m \in \mathcal{M}$, $\mathcal{I}_m = (i, j)$

$$\gamma^i_{\Gamma_m}(h_i) - \gamma^j_{\Gamma_m}(h_j) = 0, \tag{3.2}$$

$$\left[\!\left[\frac{\partial h_i}{\partial \boldsymbol{n}_{m,i}}\right]\!\right]_{\Gamma_m} + \left[\!\left[\frac{\partial h_j}{\partial \boldsymbol{n}_{m,j}}\right]\!\right]_{\Gamma_m} = 0. \tag{3.3}$$

3.1. Formulation as an optimization problem. We now aim at a different formulation of the above problem as an optimization problem of a suitable functional. First of all, we define the fluxes (see Figure 1b)

$$\forall i \in \mathcal{I}, \forall m \in \mathcal{M}_i,\ u_i^m := \left[\!\left[\frac{\partial h_i}{\partial \boldsymbol{n}_{m,i}}\right]\!\right]_{\Gamma_m} \in \mathrm{H}^{-1/2}(\Gamma_m)$$
$$\forall i \in \mathcal{I},\ \boldsymbol{u}_i := \left(u_i^{\mathcal{M}_i(1)}, \dots, u_i^{\mathcal{M}_i(M_i)}\right) \in \prod_{m \in \mathcal{M}_i} \mathrm{H}^{-1/2}(\Gamma_m) := U_i,$$
$$\boldsymbol{u} := (\boldsymbol{u}_1, \dots, \boldsymbol{u}_N) \in \prod_{i \in \mathcal{I}} U_i := U.$$

In general, an element $\boldsymbol{w}$ of U is a $2\,(\#\mathcal{M})$-tuple of functions each belonging to $\mathrm{H}^{-\frac{1}{2}}(\Gamma_m)$ for some $m \in \mathcal{M}$. For all $\boldsymbol{w} \in U$, we indicate by $\boldsymbol{w}_i$ the M_i-tuple of functions in $\boldsymbol{w}$ which are defined on the traces lying on fracture F_i. The component of $\boldsymbol{w}$ related to the trace Γ_m and the fracture F_i is denoted by $w_i^m \in \mathrm{H}^{-\frac{1}{2}}(\Gamma_m)$. Moreover, for any $\boldsymbol{w} \in U$ we set $\{\{\boldsymbol{w}\}\}_{\Gamma_m} = w_i^m + w_j^m$ $\forall m \in \mathcal{M}$ with $\mathcal{I}_m = (i, j)$ and indicate by $\{\{\boldsymbol{w}\}\}_{\mathcal{M}_i}$ the vector whose kth component is $\{\{\boldsymbol{w}\}\}_{\Gamma_{\mathcal{M}_i(k)}}$. Let us define by U_i^* and U^* the dual spaces of U_i and U, respectively.

Let us define the operator $\mathcal{H} : U \to V$, which associates to each vector $\boldsymbol{w} \in U$ a vector $\mathcal{H}(\boldsymbol{w}) = (h_1^w, \dots, h_N^w)$ of solutions to a Darcy's problem on each fracture independently, that is,

$$(\nabla h_i^w, \nabla v_i)_{F_i} = (f_i, v_i)_{F_i} + \left\langle \boldsymbol{w}_i, \gamma_{\mathcal{M}_i}(v_i) \right\rangle_{\mathcal{M}_i} \quad \forall i \in \mathcal{I}, \forall v_i \in V_i. \tag{3.4}$$

Moreover, for each $m \in \mathcal{M}$, we define the constrained functional $J_m : U \to \mathbb{R}$ such that

$$J_m(\boldsymbol{w}) = \left\| [\![\boldsymbol{h}^w]\!]_{\Gamma_m} \right\|^2_{\mathrm{H}^{\frac{1}{2}}(\Gamma_m)} + \left\| \{\{\boldsymbol{w}\}\}_{\Gamma_m} \right\|^2_{\mathrm{H}^{-\frac{1}{2}}(\Gamma_m)}, \quad \text{where } \boldsymbol{h}^w = \mathcal{H}(\boldsymbol{w}). \tag{3.5}$$

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The first term of the functional $J_m(\boldsymbol{w})$ represents the jump of the hydraulic head on the two fractures sharing the trace Γ_m; we call this functional "constrained" because we assume that these hydraulic heads satisfy equations (3.4) on the two fractures sharing Γ_m. The second term of the functional represents the flux conservation at the trace Γ_m.

We can define the global (constrained) functional $J : U \to \mathbb{R}$ such that $J(\boldsymbol{w}) = \sum_{m\in\mathcal{M}} J_m(\boldsymbol{w})$. We formulate the problem (3.1)–(3.3) as a constrained optimization problem:

$$\text{find } \boldsymbol{u} \in U \text{ such that } \boldsymbol{u} = \underset{\boldsymbol{w}\in U}{\arg\min}\, J(\boldsymbol{w}). \tag{3.6}$$

The functional $J(\boldsymbol{w})$ is positive $\forall \boldsymbol{w} \in U \setminus \{\boldsymbol{u}\}$ and $J(\boldsymbol{u}) = 0$.

3.2. Equivalence with an elliptic differential problem. We recall here the equivalence between problem (3.6) and a system of PDEs involving $\boldsymbol{h}$, $\boldsymbol{u}$, and an auxiliary pressure $\boldsymbol{p} \in V$ (see [7, Proposition 2.4]).

PROPOSITION 3.1. *The unique minimum of the functional $J(\boldsymbol{w})$ corresponds to the first order stationary conditions $\forall i \in \mathcal{I}$:*

$$\left\langle \{\{\boldsymbol{u}\}\}_{\mathcal{M}_i}, \mu_i \right\rangle_{\mathcal{M}_i} = -\left\langle \gamma_{\mathcal{M}_i}(p_i), \mu_i \right\rangle_{\mathcal{M}_i} \quad \forall \mu_i \in U_i^*, \tag{3.7}$$

$$(\nabla p_i, \nabla q_i)_{F_i} = \left\langle [\![\boldsymbol{h}]\!]_{\mathcal{M}_i}, \gamma_{\mathcal{M}_i}(q_i) \right\rangle_{\mathcal{M}_i} \quad \forall q_i \in V_i, \tag{3.8}$$

$$(\nabla h_i, \nabla v_i)_{F_i} = (f_i, v_i)_{F_i} + \left\langle \boldsymbol{u}_i, \gamma_{\mathcal{M}_i}(v_i) \right\rangle_{\mathcal{M}_i} \quad \forall v_i \in V_i. \tag{3.9}$$

Remark 1. From (3.2)–(3.3), we see that $[\![\boldsymbol{h}]\!]_{\mathcal{M}}$ is the null vector as well as $\{\{\boldsymbol{u}\}\}_{\mathcal{M}}$. Therefore, the exact solution of (3.8) corresponds to $p_i \equiv 0 \ \forall i \in \mathcal{I}$.

We now want to find a suitable elliptic operator that describes our problem. We define the functional spaces $L := V \times V \times U$ and $L^* := V \times V \times U^*$, whose norms are

$$\|(\boldsymbol{h}, \boldsymbol{p}, \boldsymbol{u})\|_L := \left[\sum_{i\in\mathcal{I}} \left(\|h_i\|_{F_i}^2 + \|p_i\|_{F_i}^2 + \sum_{m\in\mathcal{M}_i} \|u_i^m\|^2_{\mathrm{H}^{-\frac{1}{2}}(\Gamma_m)} \right)\right]^{\frac{1}{2}}, \tag{3.10}$$

$$\|(\boldsymbol{v}, \boldsymbol{q}, \boldsymbol{\mu})\|_{L^*} := \left[\sum_{i\in\mathcal{I}} \left(\|v_i\|_{F_i}^2 + \|q_i\|_{F_i}^2 + \sum_{m\in\mathcal{M}_i} \|\mu_i^m\|^2_{\mathrm{H}^{\frac{1}{2}}(\Gamma_m)} \right)\right]^{\frac{1}{2}}. \tag{3.11}$$

We can define the bilinear continuous operator $\mathscr{L} : L \times L^* \to \mathbb{R}$ such that

$$\begin{aligned} \mathscr{L}\left((\boldsymbol{h}, \boldsymbol{p}, \boldsymbol{u}), (\boldsymbol{v}, \boldsymbol{q}, \boldsymbol{\mu})\right) := \sum_{i\in\mathcal{I}} \Big\{ & (\nabla h_i, \nabla v_i)_{F_i} - \left\langle \boldsymbol{u}_i, \gamma_{\mathcal{M}_i}(v_i) \right\rangle_{\mathcal{M}_i} + (\nabla p_i, \nabla q_i)_{F_i} \\ & - \left\langle [\![\boldsymbol{h}]\!]_{\mathcal{M}_i}, \gamma_{\mathcal{M}_i}(q_i) \right\rangle_{\mathcal{M}_i} + \left\langle \gamma_{\mathcal{M}_i}(p_i) + \{\{\boldsymbol{u}\}\}_{\mathcal{M}_i}, \boldsymbol{\mu}_i \right\rangle_{\mathcal{M}_i} \Big\}. \end{aligned} \tag{3.12}$$

Using this definition, the system of equations of Proposition 3.1 can be written in compact form as

$$\forall (\boldsymbol{v}, \boldsymbol{q}, \boldsymbol{\mu}) \in L^*, \quad \mathscr{L}\left((\boldsymbol{h}, \boldsymbol{p}, \boldsymbol{u}), (\boldsymbol{v}, \boldsymbol{q}, \boldsymbol{\mu})\right) = \sum_{i\in\mathcal{I}} (f_i, v_i)_{F_i}. \tag{3.13}$$

This problem has a unique solution since it is equivalent to (3.1)–(3.3); thus applying the Nečas theorem (see, for example, [20, Theorem 3.3]), we can say that $\mathscr{L}$ satisfies an inf-sup condition:

$$\exists \beta > 0 : \|(\boldsymbol{h}, \boldsymbol{p}, \boldsymbol{u})\|_L \leq \beta \sup_{(\boldsymbol{v}, \boldsymbol{q}, \boldsymbol{\mu})\in L^*} \frac{\mathscr{L}\left((\boldsymbol{h}, \boldsymbol{p}, \boldsymbol{u}), (\boldsymbol{v}, \boldsymbol{q}, \boldsymbol{\mu})\right)}{\|(\boldsymbol{v}, \boldsymbol{q}, \boldsymbol{\mu})\|_{L^*}}. \tag{3.14}$$

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3.3. Problem discretization. An important advantage of the formulation introduced in the previous sections is that the discretization of each fracture does not have to conform to the traces; i.e., triangles can freely cross traces. In the following, we assume that each fracture is meshed by a good quality triangulation $\mathcal{T}_{\delta i}$ [11]. Let $\mathcal{T}_\delta = \bigcup_{i\in\mathcal{I}} \mathcal{T}_{\delta i}$ be the set of all the triangles on the DFN. Let $\mathcal{V}_\delta$ be the set of the vertices of the triangles in $\mathcal{T}_\delta$, and let $\mathcal{E}_\delta$ be the set of all the edges of the triangles in $\mathcal{T}_\delta$. $\mathcal{V}_{\delta i}$ and $\mathcal{E}_{\delta i}$ coherently are the subsets of $\mathcal{V}_\delta$ and $\mathcal{E}_\delta$ containing the objects defined on fracture F_i. For all $T \in \mathcal{T}_\delta$, we indicate by $\mathring{T}$ the interior of T, by $\mathcal{M}_T$ the set of indices of those traces having nonempty intersection with $\mathring{T}$, and by ℓ_T^m the segment $\Gamma_m \cap \mathring{T}$ $\forall m \in \mathcal{M}_T$. Coherently, for any given $\sigma \in \mathcal{E}_\delta$ we indicate by $\mathcal{M}_\sigma$ the set of those $m \in \mathcal{M}$ such that $|\,\Gamma_m \cap \sigma\,| \neq \emptyset$. Moreover, on each trace Γ_m shared by two fractures F_i and F_j, we fix two discretizations $\Lambda_{m,i}$ and $\Lambda_{m,j}$ defined on the two fractures, respectively. In the following, the symbol $h_\sharp$ denotes the diameter of an arbitrary geometrical object $\sharp$.

To solve the minimization problem in (3.6) we start by discretizing (3.13). Let us define the following finite-dimensional subspaces: $V_{\delta,i} \subset V_i$ $\forall i \in \mathcal{I}$, $U_{\delta i}^m \subset \mathrm{L}^2(\Gamma_m) \subset \mathrm{H}^{-\frac{1}{2}}(\Gamma_m) = U_i^m$ $\forall i \in \mathcal{I}, m \in \mathcal{M}_i$, and let us set $U_{\delta i} := \prod_{m\in\mathcal{M}_i} U_{\delta i}^m$ $\forall i \in \mathcal{I}$, $U_\delta := \prod_{i\in\mathcal{I}} U_{\delta i}$, $V_\delta := \prod_{i\in\mathcal{I}} V_{\delta i}$. Our discrete problem is to find $\boldsymbol{h}_\delta, \boldsymbol{p}_\delta \in V_\delta$ and $\boldsymbol{u}_\delta \in U_\delta$ such that

$$\mathscr{L}\left((\boldsymbol{h}_\delta,\, \boldsymbol{p}_\delta,\, \boldsymbol{u}_\delta),\, (\boldsymbol{v}_\delta,\, \boldsymbol{q}_\delta,\, \boldsymbol{\mu}_\delta)\right) = \sum_{i\in\mathcal{I}} (f_i, v_{\delta i})_{F_i} \quad \forall \boldsymbol{v}_\delta,\, \boldsymbol{q}_\delta \in V_\delta,\, \boldsymbol{\mu}_\delta \in U_\delta\,, \tag{3.15}$$

that is, to solve the following system of equations $\forall i \in \mathcal{I}$:

$$\left(\{\{\boldsymbol{u}_\delta\}\}_{\mathcal{M}_i}, \mu_{\delta i}\right)_{\mathcal{M}_i} = -\left(\gamma_{\mathcal{M}_i}(p_{\delta i}), \mu_{\delta i}\right)_{\mathcal{M}_i} \quad \forall \mu_{\delta i} \in U_{\delta i}\,, \tag{3.16}$$

$$(\nabla p_{\delta i}, \nabla v_{\delta i})_{F_i} = \left([\![\boldsymbol{h}_\delta]\!]_{\mathcal{M}_i}, \gamma_{\mathcal{M}_i}(v_{\delta i})\right)_{\mathcal{M}_i} \quad \forall v_{\delta i} \in V_{\delta i}\,, \tag{3.17}$$

$$(\nabla h_{\delta i}, \nabla v_{\delta i})_{F_i} = (f_i, v_{\delta i})_{F_i} + \left(\boldsymbol{u}_{\delta i}, \gamma_{\mathcal{M}_i}(v_{\delta i})\right)_{\mathcal{M}_i} \quad \forall v_{\delta i} \in V_{\delta i}\,. \tag{3.18}$$

This is equivalent (see [7]) to minimizing a functional with the same structure of J but involving $\mathrm{L}^2(\Gamma_m)$ norms of the discrete functions $\boldsymbol{h}_\delta$ and $\boldsymbol{u}_\delta$. Indeed, if we define $\mathcal{H}_\delta : U_\delta \to V_\delta$ such that $(h_{\delta 1}, \ldots, h_{\delta N}) = \mathcal{H}_\delta(\boldsymbol{u}_\delta)$ is the solution vector of

$$(\nabla h_{\delta i}, \nabla v_{\delta i})_{F_i} = (f_i, v_{\delta i})_{F_i} + \left(\boldsymbol{u}_{\delta i}, \gamma_{\mathcal{M}_i}(v_{\delta i})\right)_{\mathcal{M}_i} \quad \forall v_{\delta i} \in V_{\delta i},\ \forall i \in \mathcal{I}\,, \tag{3.19}$$

then we can define, for any given $\boldsymbol{w}_\delta \in U_\delta$ and any $m \in \mathcal{M}$, the functional $J_{m\delta}$ such that

$$J_{m\delta}(\boldsymbol{w}) = \left\|[\![\boldsymbol{h}_\delta^w]\!]_{\Gamma_m}\right\|_{\Gamma_m}^2 + \left\|\{\{\boldsymbol{w}_\delta\}\}_{\Gamma_m}\right\|_{\Gamma_m}^2 \quad \text{with } \boldsymbol{h}_\delta^w = \mathcal{H}_\delta(\boldsymbol{w}_\delta)\,. \tag{3.20}$$

The system (3.16)–(3.18) is equivalent to the following minimum problem:

$$\boldsymbol{u}_\delta = \operatorname*{arg\,min}_{\boldsymbol{w}_\delta \in U_\delta} J_\delta(\boldsymbol{w}_\delta) = \operatorname*{arg\,min}_{\boldsymbol{w}_\delta \in U_\delta} \sum_{m\in\mathcal{M}} J_{m\delta}(\boldsymbol{w}_\delta)\,. \tag{3.21}$$

4. Error and error estimators. The following quantities define the error performed approximating (3.13) by (3.15):

$$e_{\boldsymbol{h}} = \boldsymbol{h} - \boldsymbol{h}_\delta, \quad e_{\boldsymbol{p}} = \boldsymbol{p} - \boldsymbol{p}_\delta, \quad e_{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_\delta\,. \tag{4.1}$$

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Since $(e_{\boldsymbol{h}},\, e_{\boldsymbol{p}},\, e_{\boldsymbol{u}}) \in L$, we have the following lemma.

LEMMA 4.1. *Let* $\mathscr{L}$ *be defined by* (3.12) *and* $e_{\boldsymbol{h}},\, e_{\boldsymbol{p}},\, e_{\boldsymbol{u}}$ *by* (4.1). *Then, for any* $\boldsymbol{v}_\delta,\, \boldsymbol{q}_\delta \in V_\delta$ *and* $\boldsymbol{\mu}_\delta \in U_\delta$, $\mathscr{L}\left((e_{\boldsymbol{h}},\, e_{\boldsymbol{p}},\, e_{\boldsymbol{u}}), (\boldsymbol{v}_\delta,\, \boldsymbol{q}_\delta,\, \boldsymbol{\mu}_\delta)\right) = 0$.

We define the *error measure* as

$$err := \left\|(e_{\boldsymbol{h}}, e_{\boldsymbol{p}}, e_{\boldsymbol{u}})\right\|_L . \tag{4.2}$$

The main result of section 5 is that the error measure (4.2) can be controlled by the following quantities $\forall i \in \mathcal{I}$.

Residual estimator:

$$\eta_{R,T} := h_T \left\| f_i + \Delta h_{\delta i} \right\|_T \quad \forall T \in \mathcal{T}_{\delta i} . \tag{4.3}$$

Estimator for the approximation of the flux through edges: $\forall \sigma \in \mathcal{E}_{\delta i}$,

$$\xi_{F,\sigma} := (h_\sigma)^{\frac{1}{2}} \left\| \left[\!\left[\frac{\partial h_{\delta i}}{\partial \boldsymbol{n}_\sigma} \right]\!\right]_\sigma - \tilde{u}_{\delta i,\sigma} \right\|_\sigma , \text{ where } \quad \tilde{u}_{\delta i,\sigma} := \begin{cases} u^m_{\delta i} & \forall m \in \mathcal{M}_\sigma, \\ 0 & \text{elsewhere} . \end{cases} \tag{4.4}$$

Estimator for the nonconformity of the discretization:

$$\xi^m_{NC,T} := \left(h_{\ell^m_T}\right)^{\frac{1}{2}} \left\| u^m_{\delta i} \right\|_{\ell^m_T} \quad \forall T \in \mathcal{T}_{\delta i},\, m \in \mathcal{M}_T . \tag{4.5}$$

Local estimator for the pressure induced by discontinuity:

$$\eta_{P,T} := \left\| p_{\delta i} \right\|_T \quad \forall T \in \mathcal{T}_{\delta i}. \tag{4.6}$$

Local estimator for the pressure induced by the unbalancing of fluxes:

$$\xi^i_{P,\lambda} := h_\lambda^{\frac{1}{2}} \left\| \gamma^i_{\Gamma_m}\left(p_{\delta i}\right) \right\|_\lambda \quad \forall m \in \mathcal{M}_i,\, \lambda \in \Lambda_{m,i}. \tag{4.7}$$

Local estimator of the minimization error:

$$J_{\delta,\lambda}(\boldsymbol{u}_\delta) := h_\lambda^{\frac{1}{2}} \left(\left\| \{\!\{\boldsymbol{u}_\delta\}\!\}_{\Gamma_m} \right\|_\lambda + \left\| [\![\boldsymbol{h}_\delta]\!]_{\Gamma_m} \right\|_\lambda \right) \quad \forall m \in \mathcal{M}_i,\, \lambda \in \Lambda_{m,i}. \tag{4.8}$$

The symbols $\lesssim$ and $\simeq$ are used to compare functions with the following meaning:

$$\begin{aligned} f \lesssim g &\iff \exists C > 0 \colon f \leq C\, g\,, \\ f \simeq g &\iff \exists c, C > 0 \colon c\, g \leq f \leq C\, g\,, \end{aligned}$$

where all the constants are independent of the meshsize.

5. Reliability. This section is devoted to obtaining an a posteriori upper bound for the error norm (4.2) based on condition (3.14). After stating some auxiliary results (subsection 5.1), we obtain (Theorem 5.2) our estimate.

5.1. Auxiliary results. In the following, we apply the well-known properties of Clement's pseudointerpolation operator on the fracture F_i, $i \in \mathcal{I}$, denoted by $\Pi_{\delta i}$. Given a fracture F_i, $i \in \mathcal{I}$, let us consider a triangle $T \in \mathcal{T}_{\delta i}$ and an edge $\sigma \in \mathcal{E}_{\delta i}$. Then, for any $v \in \mathrm{H}^1_0(F_i)$,

$$\left\| v - \Pi_{\delta i}(v) \right\|_T \lesssim h_T \left\| v \right\|_{\omega_T} , \tag{5.1}$$

$$\left\|\!\left| v - \Pi_{\delta i}(v) \right|\!\right\|_T \lesssim \left\|\!\left| v \right|\!\right\|_{\omega_T} , \tag{5.2}$$

$$\left\| \gamma^i_\sigma \left(v - \Pi_{\delta i}(v)\right) \right\|_\sigma \lesssim (h_\sigma)^{\frac{1}{2}} \left\|\!\left| v \right|\!\right\|_{\omega_\sigma} , \tag{5.3}$$

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where ω_T is the union of all triangles having a side or a vertex in common with T and ω_σ is the union of the two triangles having σ in common.

Concerning trace spaces, given a bounded open set $\Omega \subset \mathbb{R}^2$, a segment $\lambda \subseteq \partial\Omega$, and a function $g \in \mathrm{H}^{\frac{1}{2}}(\lambda)$, one can define the set $\mathrm{H}^1_{g,\lambda}(\Omega) := \left\{v \in \mathrm{H}^1(\Omega) :\ \gamma_\lambda(v) = g\right\}$ $\subseteq \mathrm{H}^1(\Omega)$ and the seminorm

$$|g|_{\frac{1}{2},\lambda} := \inf_{v \in \mathrm{H}^1_{g,\lambda}} \|\nabla v\|_\Omega .$$

The following result holds.

THEOREM 5.1. *Let λ be a segment of length h_λ and $P :\ \mathrm{H}^{\frac{1}{2}}(\lambda) \to \mathrm{L}^2(\lambda)$ a continuous linear operator preserving a.e. constant functions. Then*

$$\exists C > 0 :\ \forall g \in \mathrm{H}^{\frac{1}{2}}(\lambda),\ \|g - Pg\|_{0,\lambda} \le C h_\lambda^{\frac{1}{2}} |g|_{\frac{1}{2},\lambda} . \tag{5.4}$$

5.2. Upper bound. In this section we derive an upper bound for the error.

THEOREM 5.2. *Let $e_{\boldsymbol{h}}$, $e_{\boldsymbol{p}}$, $e_{\boldsymbol{u}}$ be defined by* (4.1), *and let all the quantities defined in* (4.3)–(4.8) *be given. Then*

$$\begin{aligned} err \lesssim \sum_{i\in\mathcal{I}} \Bigg[\sum_{T\in\mathcal{T}_{\delta i}} \Bigg(\eta_{R,T} + \eta_{P,T} + \sum_{m\in\mathcal{M}_T} \xi^m_{NC,T} \Bigg) + \sum_{\sigma\in\mathcal{E}_{\delta i}} \xi^m_{F,\sigma} \\ + \sum_{m\in\mathcal{M}_i} \sum_{\lambda\in\Lambda_{m,i}} \left(\xi^i_{P,\lambda} + J_{\delta,\lambda}(\boldsymbol{u}_\delta) \right) \Bigg] . \end{aligned}$$

Proof. From (3.14) we have

$$err = \|(e_{\boldsymbol{h}}, e_{\boldsymbol{p}}, e_{\boldsymbol{u}})\|_L \lesssim \sup_{(\boldsymbol{v},\boldsymbol{q},\boldsymbol{\mu})\in L^*} \frac{\mathscr{L}\left((e_{\boldsymbol{h}}, e_{\boldsymbol{p}}, e_{\boldsymbol{u}}),\, (\boldsymbol{v}, \boldsymbol{q}, \boldsymbol{\mu})\right)}{\|(\boldsymbol{v}, \boldsymbol{q}, \boldsymbol{\mu})\|_{L^*}} .$$

From Lemma 4.1 we know that, for any given $\boldsymbol{v}_\delta$, $\boldsymbol{q}_\delta \in V_\delta$ and $\boldsymbol{\mu}_\delta \in U_\delta$,

$$\begin{aligned} \mathscr{L}\left((e_{\boldsymbol{h}},\, e_{\boldsymbol{p}},\, e_{\boldsymbol{u}}),\, (\boldsymbol{v},\, \boldsymbol{q},\, \boldsymbol{\mu})\right) &= \mathscr{L}\left((e_{\boldsymbol{h}},\, e_{\boldsymbol{p}},\, e_{\boldsymbol{u}}),\, (\boldsymbol{v} - \boldsymbol{v}_\delta,\, \boldsymbol{q} - \boldsymbol{q}_\delta,\, \boldsymbol{\mu} - \boldsymbol{\mu}_\delta)\right) \\ &= \sum_{i\in\mathcal{I}} \Big\{ \left(\nabla(h_i - h_{\delta i}), \nabla(v_i - v_{\delta i})\right)_{F_i} - \left\langle \boldsymbol{u} - \boldsymbol{u}_\delta, \gamma_{\mathcal{M}_i}(v_i - v_{\delta i}) \right\rangle_{\mathcal{M}_i} \\ &\quad + \left(\nabla(p_i - p_{\delta i}), \nabla(q_i - q_{\delta i})\right)_{F_i} - \left\langle [\![\boldsymbol{h} - \boldsymbol{h}_\delta]\!]_{\mathcal{M}_i}, \gamma_{\mathcal{M}_i}(q_i - q_{\delta i}) \right\rangle_{\mathcal{M}_i} \\ &\qquad + \left\langle \gamma_{\mathcal{M}_i}(p_i - p_{\delta i}) + \{\{\boldsymbol{u} - \boldsymbol{u}_\delta\}\}_{\mathcal{M}_i}, \boldsymbol{\mu}_i - \boldsymbol{\mu}_{\delta i} \right\rangle_{\mathcal{M}_i} \Big\} . \end{aligned}$$

We now proceed by estimating separately the terms involving different test functions. Let $i \in \mathcal{I}$ be fixed:

- *Terms with test function $v_i - v_{\delta i}$.* Since $\boldsymbol{h} = \mathcal{H}(\boldsymbol{u})$ (thus h_i and $\boldsymbol{u}_i$ are linked by (3.9)), applying Green's formula on $\mathcal{T}_{\delta i}$, we obtain

$$\begin{aligned} &\left(\nabla(h_i - h_{\delta i}), \nabla(v_i - v_{\delta i})\right)_i - \left\langle (\boldsymbol{u}_i - \boldsymbol{u}_{\delta i}), \gamma_{\mathcal{M}_i}(v_i - v_{\delta i}) \right\rangle_{\mathcal{M}_i} \\ &\quad = \sum_{T\in\mathcal{T}_{\delta i}} \left(f_i + \Delta h_{\delta i}, v_i - v_{\delta i}\right)_T - \sum_{\sigma\in\mathcal{E}_{\delta i}} \left\langle \left[\!\!\left[\frac{\partial h_{\delta i}}{\partial \boldsymbol{n}_\sigma} \right]\!\!\right]_\sigma, \gamma^i_\sigma(v_i - v_{\delta i}) \right\rangle_\sigma \\ &\qquad + \sum_{m\in\mathcal{M}_i} \left\langle \boldsymbol{u}^m_{\delta i}, \gamma^i_{\Gamma_m}(v_i - v_{\delta i}) \right\rangle_{\Gamma_m} . \end{aligned}$$

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Then, since $\left[\!\left[\frac{\partial h_{\delta i}}{\partial \boldsymbol{n}_\sigma}\right]\!\right]_\sigma \in \mathrm{L}^2(\sigma)$, $u^m_{\delta i} \in \mathrm{L}^2(\Gamma_m) \subset \mathrm{H}^{-\frac{1}{2}}(\Gamma_m)$, $\gamma^i_\sigma(v_i - v_{\delta i}) \in \mathrm{H}^{\frac{1}{2}}(\sigma) \subset \mathrm{L}^2(\sigma)$, $\gamma^i_{\Gamma_m}(v_i - v_{\delta i}) \in \mathrm{H}^{\frac{1}{2}}(\Gamma_m) \subset \mathrm{L}^2(\Gamma_m)$, it is possible to write the duality product on each trace as a scalar product in L^2:

$$
\begin{aligned}
&-\sum_{\sigma\in\mathcal{E}_{\delta i}} \left\langle \left[\!\left[\frac{\partial h_{\delta i}}{\partial \boldsymbol{n}_\sigma}\right]\!\right]_\sigma, \gamma^i_\sigma(v_i - v_{\delta i}) \right\rangle_\sigma + \sum_{m\in\mathcal{M}_i} \left\langle u^m_{\delta i}, \gamma^i_{\Gamma_m}(v_i - v_{\delta i}) \right\rangle_{\Gamma_m} \\
&= \sum_{\sigma\in\mathcal{E}_\delta} \left(\tilde{u}_{\delta i,\sigma} - \left[\!\left[\frac{\partial h_{\delta i}}{\partial \boldsymbol{n}_\sigma}\right]\!\right]_\sigma, \gamma^i_\sigma(v_i - v_{\delta i}) \right)_\sigma + \sum_{\substack{T\in\mathcal{T}_{\delta i}\\ m\in\mathcal{M}_T}} \left(u^m_{\delta i}, \gamma^i_{\Gamma_m}(v_i - v_{\delta i}) \right)_{\ell^m_T}.
\end{aligned}
$$

Then, taking $v_{\delta i} = \Pi_{\delta i}(v_i)$ and using inequalities (5.1) and (5.3),

$$
\begin{aligned}
&(\nabla(h_i - h_{\delta i}), \nabla(v_i - v_{\delta i}))_{F_i} - \left\langle \boldsymbol{u}_i - \boldsymbol{u}_{\delta i}, \gamma_{\mathcal{M}_i}(v_i - v_{\delta i}) \right\rangle_{\mathcal{M}_i} \\
&\qquad \le \left\{ \sum_{T\in\mathcal{T}_{\delta i}} \left(h_T \left\| f_i + \Delta h_{\delta i} \right\|_T + \sum_{m\in\mathcal{M}_T} h^{\frac{1}{2}}_{\ell^m_T} \left\| u^m_{\delta i} \right\|_{\ell^m_T} \right) \right. \\
&\qquad\qquad \left. + \sum_{\sigma\in\mathcal{E}_\delta} h^{\frac{1}{2}}_\sigma \left\| \left[\!\left[\frac{\partial h_{\delta i}}{\partial \boldsymbol{n}_\sigma}\right]\!\right]_\sigma - \tilde{u}_{\delta i,\sigma} \right\|_\sigma \right\} \left\| v_i \right\|_{F_i}.
\end{aligned}
$$

- *Terms with test function $q_i - q_{\delta i}$.* Using (3.8), Green's formula applied on $\mathcal{T}_{\delta i}$, and the fact that $[\![\boldsymbol{h}]\!]_{\mathcal{M}_i} = \boldsymbol{0}$ and $p_i = 0$,

$$
\begin{aligned}
&(\nabla(p_i - p_{\delta i}), \nabla(q_i - q_{\delta i}))_{F_i} - \left\langle [\![\boldsymbol{h} - \boldsymbol{h}_\delta]\!]_{\mathcal{M}_i}, \gamma_{\mathcal{M}_i}(q_i - q_{\delta i}) \right\rangle_{\mathcal{M}_i} \\
&\quad = \sum_{T\in\mathcal{T}_{\delta i}} (-\nabla p_{\delta i}, \nabla(q_i - q_{\delta i}))_T + \sum_{m\in\mathcal{M}_i} \left(-[\![\boldsymbol{h}_\delta]\!]_{\Gamma_m}, \gamma^i_{\Gamma_m}(q_i - q_{\delta i}) \right)_{\Gamma_m}.
\end{aligned}
$$

For any given $m \in \mathcal{M}_i$, we introduce a discretization $\Lambda_{m,i}$ writing

$$
\left(-[\![\boldsymbol{h}_\delta]\!]_{\Gamma_m}, \gamma^i_{\Gamma_m}(q_i - q_{\delta i}) \right)_{\Gamma_m} \le \sum_{\lambda\in\Lambda_{m,i}} \left\| [\![\boldsymbol{h}_\delta]\!]_{\Gamma_m} \right\|_\lambda \left\| \gamma^i_{\Gamma_m}(q_i - q_{\delta i}) \right\|_\lambda;
$$

then, choosing $q_{\delta i} = \Pi_{\delta i}(q_i)$ and using inequalities (5.1) and (5.3),

$$
\begin{aligned}
&(\nabla(p_i - p_{\delta i}), \nabla(q_i - q_{\delta i}))_{F_i} - \left\langle [\![\boldsymbol{h} - \boldsymbol{h}_\delta]\!]_{\mathcal{M}_i}, \gamma_{\mathcal{M}_i}(q_i - q_{\delta i}) \right\rangle_{\mathcal{M}_i} \\
&\qquad \le \left(\sum_{T\in\mathcal{T}_{\delta i}} \left\| p_{\delta i} \right\|_T + \sum_{m\in\mathcal{M}_i} \sum_{\lambda\in\Lambda_{m,i}} h^{\frac{1}{2}}_\lambda \left\| [\![\boldsymbol{h}_\delta]\!]_{\Gamma_m} \right\|_\lambda \right) \left\| q_i \right\|_{F_i}.
\end{aligned}
$$

- *Term with test function $\boldsymbol{\mu}_i - \boldsymbol{\mu}_{\delta i}$.* Using (3.7), and since $\gamma^i_{\Gamma_m}(p_{\delta i})$, $\mu^m_i \in \mathrm{H}^{\frac{1}{2}}(\Gamma_m) \subset \mathrm{L}^2(\Gamma_m)$, $u^m_{\delta i} \in \mathrm{L}^2(\Gamma_m) \subset \mathrm{H}^{-\frac{1}{2}}(\Gamma_m)$, we obtain the following by rewriting the duality product as a scalar product in $\mathrm{L}^2(\Gamma_m)$ and using discretization $\Lambda_{m,i}$:

$$
\begin{aligned}
&\left\langle \gamma_{\mathcal{M}_i}(p_i - p_{\delta i}) + \{\!\{\boldsymbol{u}_i - \boldsymbol{u}_{\delta i}\}\!\}_{\mathcal{M}_i}, \boldsymbol{\mu}_i - \boldsymbol{\mu}_{\delta i} \right\rangle_{\mathcal{M}_i} \\
&\quad = \sum_{m\in\mathcal{M}_i} \sum_{\lambda\in\Lambda_{m,i}} \left(-\gamma^i_{\Gamma_m}(p_{\delta i}), \mu^m_i - \mu^m_{\delta i} \right)_\lambda + \left(\{\!\{\boldsymbol{u}_\delta\}\!\}_{\Gamma_m}, \mu^m_i - \mu^m_{\delta i} \right)_\lambda \\
&\qquad \lesssim \sum_{m\in\mathcal{M}_i} \sum_{\lambda\in\Lambda_{m,i}} h^{\frac{1}{2}}_\lambda \left(\left\| \gamma^i_{\Gamma_m}(p_{\delta i}) \right\|_\lambda + \left\| \{\!\{\boldsymbol{u}_\delta\}\!\}_{\Gamma_m} \right\|_\lambda \right) \left\| \mu^m_i \right\|_{\frac{1}{2},\Gamma_m},
\end{aligned}
$$

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where the last estimate is obtained by supposing $\mu_{\delta i}^m$ is the image of μ_i^m through a linear continuous operator that preserves constants by applying Theorem 5.1 and since

$$|\mu_i^m|_{\mathrm{H}^{\frac{1}{2}}(\Gamma_m)} \leq \|\mu_i^m\|_{\mathrm{H}^{\frac{1}{2}}(\Gamma_m)} = \inf_{\substack{v \in \mathrm{H}^1(\omega_{\Gamma_m,i}) \\ \gamma_m^i(v) = \mu_i^m}} \|v\|_{1,\omega_{\Gamma_m,i}} ,$$

where $\omega_{\Gamma_m,i}$ is a subregion of F_i having Γ_m on its boundary.

The proof is concluded since $\forall i \in \mathcal{I}$ and $\forall m \in \mathcal{M}_i$

$$\|v_i\|_{F_i} \leq \|(\boldsymbol{v}, \boldsymbol{q}, \boldsymbol{\mu})\|_{L^*} , \; \|q_i\|_{F_i} \leq \|(\boldsymbol{v}, \boldsymbol{q}, \boldsymbol{\mu})\|_{L^*} , \; \|\mu_i^m\|_{\frac{1}{2},\Gamma_m} \leq \|(\boldsymbol{v}, \boldsymbol{q}, \boldsymbol{\mu})\|_{L^*} . \qquad \square$$

For the sake of notational simplicity, we define the global estimator

$$est_\delta := \sum_{i \in \mathcal{I}} \left[\sum_{T \in \mathcal{T}_{\delta i}} \left(\eta_{R,T} + \eta_{P,T} + \sum_{m \in \mathcal{M}_T} \xi_{NC,T}^m \right) + \sum_{\sigma \in \mathcal{E}_\delta} \xi_{F,\sigma}^m \right.$$
$$\left. + \sum_{m \in \mathcal{M}_i} \sum_{\lambda \in \Lambda_{m,i}} \left(\xi_{P,\lambda}^i + J_{\delta,\lambda}(\boldsymbol{u}_\delta) \right) \right] . \tag{5.5}$$

6. Efficiency of the a posteriori error estimate. In this section we prove the efficiency of the estimators presented in Theorem 5.2; i.e., we show that for the a posteriori error estimator of Theorem 5.2 we can write a lower bound in terms of a multiple of the error norm defined by (4.2).

From now on, $\forall i \in \mathcal{I}$ we assume that the discretization $\Lambda_{m,i}$ which was fixed in subsection 3.3 is the one induced on Γ_m by the triangulation $\mathcal{T}_{\delta i}$, that is, $\Lambda_{m,i} = \bigcup_{T \in \mathcal{T}_{\delta i}} \overline{\ell_T^m}$. For any triangle $T \in \mathcal{T}_\delta$, the following nonconformity measure can be defined:

$$h_{NC,T} := \sum_{m \in \mathcal{M}_T} h_{\ell_T^m} . \tag{6.1}$$

Such a quantity is zero for all triangles having an empty intersection with all traces and is less than or equal to $\#\mathcal{M}_T \, h_T$ for those intersecting some of them. It is not too restrictive to suppose $h_{NC,T} < 1$, assuming that the problem is written in a nondimensional way. The results in subsections 6.1, 6.2, and 6.3 together prove the following theorem.

THEOREM 6.1. *Let* est_δ *be defined by* (5.5), *let* $e_{\boldsymbol{h}}$, $e_{\boldsymbol{u}}$, *and* $e_{\boldsymbol{p}}$ *be defined by* (4.1), *and let* $h_{NC,T}$ *be defined by* (6.1) $\forall T \in \mathcal{T}_\delta$. *Then, if* $h_{NC,T} < 1 \; \forall T \in \mathcal{T}_\delta$,

$$est_\delta \lesssim \|e_{\boldsymbol{p}}\| + C_{NC} \left[\max_{\sigma \in \mathcal{E}_\delta} \{1, h_\sigma\} \left[\|e_{\boldsymbol{h}}\| + \sum_{m \in \mathcal{M}} \|e_{\boldsymbol{u}}\|_U \right] + \max_{\substack{i \in \mathcal{I} \\ T \in \mathcal{T}_{\delta i}}} h_T \, \|f_i - f_T\| \right] ,$$

where f_T *is the mean of* f_i *on triangle* $T \in \mathcal{T}_{\delta i}$ *and*

$$C_{NC} := \max_{T \in \mathcal{T}_\delta} \left(\frac{1 + h_{NC,T}}{1 - h_{NC,T}} \right) .$$

Remark 2. We remark that the efficiency of the a posteriori estimate depends on the nonconformity of the triangulation through C_{NC}, which tends to 1 by refining the mesh. In a nondimensional formulation of the problem, it is, however, always possible to ask that the coarsest considered triangulation satisfy $h_{NC,T} \leq \frac{1}{2}$ (thus having $C_{NC} \leq 3$).

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6.1. Auxiliary results. We apply classical results about suitable cut-off functions [25], which exploit the properties of special polynomial functions with compact support, called *bubble functions*.

Given $i \in \mathcal{I}$ and any triangle $T \subset \mathcal{T}_{\delta i}$, let us denote by $\mathfrak{b}_T$ the triangle bubble function as defined in [25]. It has the following properties:

$$\operatorname{supp} \mathfrak{b}_T = T, \quad 0 \leq \mathfrak{b}_T \leq 1, \quad \max_{\boldsymbol{x} \in T} \mathfrak{b}_T(\boldsymbol{x}) = 1, \quad (\mathfrak{b}_T, 1)_T = \frac{9}{20} \left| T \right| ,$$

from which, since $\mathfrak{b}_T \in \mathrm{H}^1_0(T)$, the following estimates can be obtained (see [25, Lemma 1.3]):

$$\|\mathfrak{b}_T\|_T = \left[(\mathfrak{b}_T, \mathfrak{b}_T)_T\right]^{\frac{1}{2}} \leq \left[(\mathfrak{b}_T, 1)_T\right]^{\frac{1}{2}} \Rightarrow \|\mathfrak{b}_T\|_T \lesssim h_T \,, \tag{6.2}$$

$$\|\nabla \mathfrak{b}_T\|_T \lesssim h_T^{-1} \|\mathfrak{b}_T\|_T \Rightarrow \|\nabla \mathfrak{b}_T\|_T \lesssim 1 \,. \tag{6.3}$$

Let $l \subset T$ be a segment not necessarily intersecting ∂T, and let L be its prolongation up to ∂T. Then, since $\gamma_L(\mathfrak{b}_T) \in \mathrm{H}^{\frac{1}{2}}_{00}(L)$ and $h_L \leq h_T$, applying the continuity of the trace operator on L, we have

$$\|\gamma_l(\mathfrak{b}_T)\|_{\mathrm{H}^{\frac{1}{2}}(l)} \leq \|\gamma_L(\mathfrak{b}_T)\|_{\mathrm{H}^{\frac{1}{2}}(L)} \lesssim \|\gamma_L(\mathfrak{b}_T)\|_{\mathrm{H}^{\frac{1}{2}}_{00}(L)} \lesssim \|\nabla \mathfrak{b}_T\|_T \lesssim 1 \,. \tag{6.4}$$

All the constants depend on the quality of the considered triangle, namely, on the minimum of its angles. It is possible to prove the following useful result [24, Lemma 4.1].

LEMMA 6.2. *Let $i \in \mathcal{I}$, let $T \in \mathcal{T}_{\delta i}$, and let $\mathfrak{b}_T$ be the bubble function on T. Let $\mathcal{P}(T) \subset \mathrm{H}^1(T)$ be a finite-dimensional space. Then, for any given $v \in \mathcal{P}(T)$,*

$$\|v\|_T^2 \lesssim \left(\mathfrak{b}_T, v^2\right)_T, \qquad \|v \mathfrak{b}_T\|_T \leq \|v\|_T \,, \tag{6.5}$$

$$|\!|\!| \mathfrak{b}_T \, v |\!|\!|_T \lesssim h_T^{-1} \|v\|_T \,. \tag{6.6}$$

We now consider a segment $\sigma \subset F_i$ shared by two triangles R and L belonging to a regular triangulation, that is, such that $h_R \simeq h_L \simeq h_\sigma$. We denote by $\mathfrak{b}_\sigma$ the side bubble function of σ as defined in [25]. It has the following properties:

$$\operatorname{supp} \mathfrak{b}_\sigma = \omega_\sigma, \quad 0 \leq \mathfrak{b}_\sigma \leq 1, \quad \max_{\boldsymbol{x} \in \omega_\sigma} \mathfrak{b}_\sigma = 1, \quad \left(\gamma_\sigma(\mathfrak{b}_\sigma), 1\right)_\sigma = \frac{2}{3} h_\sigma \simeq h_\sigma \,,$$
$$\forall T \in \{R, L\}, \quad (\mathfrak{b}_\sigma, 1)_T = \frac{1}{3} \left| T \right| ,$$

from which, since $\mathfrak{b}_\sigma \in \mathrm{H}^1_0(\omega_\sigma)$,

$$\|\gamma_\sigma(\mathfrak{b}_\sigma)\|_\sigma^2 = \left(\gamma_\sigma(\mathfrak{b}_\sigma), \gamma_\sigma(\mathfrak{b}_\sigma)\right)_\sigma \leq \left(\gamma_\sigma(\sigma), 1\right)_\sigma \Rightarrow \|\gamma_\sigma(\mathfrak{b}_\sigma)\|_\sigma \lesssim h_\sigma^{\frac{1}{2}} \,,$$
$$\|\mathfrak{b}_\sigma\|_{\omega_\sigma}^2 = (\mathfrak{b}_\sigma, \mathfrak{b}_\sigma)_{\omega_\sigma} \leq (\mathfrak{b}_\sigma, 1)_{\omega_\sigma} \Rightarrow \|\mathfrak{b}_\sigma\|_{\omega_\sigma} \lesssim h_\sigma \,,$$
$$\|\nabla \mathfrak{b}_\sigma\|_{\omega_\sigma} \lesssim h_\sigma^{-1} \|\mathfrak{b}_\sigma\|_{\omega_\sigma} \Rightarrow \|\nabla \mathfrak{b}_\sigma\|_{\omega_\sigma} \lesssim 1 \,.$$

Let $l \subset \omega_\sigma$ be a segment that does not necessarily intersect $\partial\omega_\sigma$, and let L be its straight prolongation whose extrema intersect $\partial\omega_\sigma$. Then, since $\mathfrak{b}_\sigma \in \mathrm{H}^{\frac{1}{2}}_{00}(L)$ and $h_L \leq h_\sigma$, by applying the continuity of the trace operator on L,

$$\|\gamma_l(\mathfrak{b}_\sigma)\|_{\mathrm{H}^{\frac{1}{2}}(l)} \leq \|\gamma_L(\mathfrak{b}_\sigma)\|_{\mathrm{H}^{\frac{1}{2}}(L)} \lesssim \|\gamma_L(\mathfrak{b}_\sigma)\|_{\mathrm{H}^{\frac{1}{2}}_{00}(L)} \lesssim \|\nabla \mathfrak{b}_\sigma\|_{\omega_\sigma} \lesssim 1 \,. \tag{6.7}$$

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Again, constants depend on the minimum angle in ω_σ. The following lemma, analogous to Lemma 6.2, also involves the *continuation operator* defined in [25], which extends a function from a side σ of a triangle T to the whole triangle. We denote this operator by $\mathcal{C}_T : L^\infty(E) \to L^\infty(T)$. The following lemmas can be proved using [24, Lemma 4.1] and the techniques in [25].

Lemma 6.3. *Let $\sigma \subset F_i$ be a segment shared by two triangles R and L, and let $\mathfrak{b}_\sigma$ be its bubble function defined on the union of the two triangles. Let $\mathcal{P}(\sigma) \subset \mathrm{H}^{\frac{1}{2}}(\sigma)$ be a finite-dimensional space. Then, for any given $v \in \mathcal{P}(\sigma)$ and any triangle $T \in \{R, L\}$,*

$$\|v\|_\sigma^2 \lesssim (v, v\,\gamma_\sigma(\mathfrak{b}_\sigma))_\sigma, \qquad \|v\mathfrak{b}_\sigma\|_\sigma \leq \|v\|_\sigma, \tag{6.8}$$

$$|\!|\!|\mathcal{C}_T(v)\ \mathfrak{b}_\sigma|\!|\!|_T \lesssim h_T^{-\frac{1}{2}}\|v\|_\sigma, \qquad \|\mathcal{C}_T(v)\ \mathfrak{b}_\sigma\|_T \lesssim h_T^{\frac{3}{2}}\|v\|_\sigma. \tag{6.9}$$

6.2. Efficiency of estimators. In this subsection we report the results about the efficiency of the estimators. We show only the proof of Proposition 6.8, which is responsible for the introduction of the constant C_{NC}. We refer the reader to the supplementary material for the other proofs. These results, together with the ones in the following subsection, prove Theorem 6.1. Lemmas 6.4 and 6.5 are used in the proofs of the subsequent propositions.

Lemma 6.4. *Let $\boldsymbol{u} \in U$, $\boldsymbol{u}_\delta \in U_\delta$ be the solutions of* (3.6) *and* (3.21). *Let $h = \mathcal{H}(\boldsymbol{u})$ and $h_\delta = \mathcal{H}_\delta(\boldsymbol{u}_\delta)$. Then, for any given $i \in \mathcal{I}$, $v \in V_i$,*

$$\sum_{\sigma\in\mathcal{E}_{\delta i}} \left(\left[\!\!\left[\frac{\partial h_{\delta i}}{\partial \boldsymbol{n}_\sigma} \right]\!\!\right]_\sigma, \gamma_\sigma(v) \right)_\sigma = \sum_{T\in\mathcal{T}_\delta} (f_i + \Delta h_{\delta i}, v)_T + \left\langle \boldsymbol{u}_i, \gamma_{\mathcal{M}_i}(v) \right\rangle_{\mathcal{M}_i} + (\nabla(h_{\delta i} - h_i), \nabla v)_{F_i}.$$

Lemma 6.5. *Let $\boldsymbol{f}$ be the vector of forcing terms in problem* (3.1) *and $f_T = \frac{1}{|T|}(f_i, 1)_T$ its mean value on each $T \in \mathcal{T}_{\delta i}$ ($i \in \mathcal{I}$). Then*

$$\|f_T + \Delta h_{\delta i}\|_T \lesssim \|f_i - f_T\|_T + \frac{1}{h_T}\left[|\!|\!|h_i - h_{\delta i}|\!|\!|_T + \sum_{m\in\mathcal{M}_i} \|u_i^m - u_{\delta i}^m\|_{-\frac{1}{2},\ell_T^m} + \xi_{NC,T}^m \right].$$

Proposition 6.6 (efficiency of $\eta_{R,T}$). *Let $i \in \mathcal{I}$, $T \in \mathcal{T}_{\delta i}$. Let $\eta_{R,T}$ be the estimator defined by* (4.3). *Then*

$$\eta_{R,T} \lesssim |\!|\!|h_i - h_{\delta i}|\!|\!|_T + \sum_{m\in\mathcal{M}_i} \left(\|u_i^m - u_{\delta i}^m\|_{-\frac{1}{2},\ell_T^m} + \xi_{NC,T}^m \right) + h_T\|f_i - f_T\|_T.$$

Proposition 6.7 (efficiency of $\xi_{F,\sigma}$). *Let $i \in \mathcal{I}$, let $\sigma \in \mathcal{E}_{\delta i}$, let $\xi_{F,\sigma}$ be defined by* (4.4), *and let $\xi_{NC,T}^m$ be defined by* (4.5) *and $\eta_{R,T}$ by* (4.3). *Then*

$$\xi_{F,\sigma} \lesssim \sum_{T\in\omega_\sigma} \left(|\!|\!|h_i - h_{\delta i}|\!|\!|_T + h_\sigma\eta_{R,T} + \sum_{m\in\mathcal{M}_i} \xi_{NC,T}^m \right) + \sum_{m\in\mathcal{M}_i} \|u_i^m - u_{\delta i}^m\|_{-\frac{1}{2},\Gamma_m\cap\mathring{\omega}_\sigma}.$$

Proposition 6.8 (efficiency of $\xi_{NC,T}^m$). *Let $i \in \mathcal{I}$, $T \in \mathcal{T}_{\delta i}$, and $m \in \mathcal{M}_T$, and let $\xi_{NC,T}^m$ be the estimator defined by* (4.5). *Then, assuming that $u_{\delta i}^m$ has a finite number of jumps in ℓ_T^m,*

$$\xi_{NC,T}^m \lesssim \|u_i^m - u_{\delta i}^m\|_{-\frac{1}{2},\ell_T^m} + |\!|\!|h_i - h_{\delta i}|\!|\!|_T + h_{\ell_T^m}\eta_{R,T}.$$

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Proof. First, suppose $u_{\delta i}^m$ is continuous on ℓ_T^m. Let $R,\, L \subset T$ be two triangles lying in the interior of T and sharing ℓ_T^m as a side. Let $\mathfrak{b}_{\ell_T^m}$ be the bubble function of ℓ_T^m having support on $\omega_{\ell_T^m} = R \cup L \subset T$, and let $\mathcal{C}_R$ and $\mathcal{C}_L$ be the continuation operators of R and L, respectively. Let $w_{\ell_T^m}$ be the function such that $w_{\ell_T^m}\big|_E := \mathcal{C}_E\left(u_{\delta i}^m\right)\mathfrak{b}_{\ell_T^m}$ $\forall E \in \{R, L\}$. Since $\mathcal{C}_R\left(u_{\delta i}^m\right)$ and $\mathcal{C}_L\left(u_{\delta i}^m\right)$ belong to a finite-dimensional subspace, we can apply (6.8) on the two triangles R and L, obtaining

$$\|u_{\delta i}^m\|_{\ell_T^m}^2 \lesssim \left(u_{\delta i}^m, \gamma_{\ell_T^m}^i\left(w_{\ell_T^m}\right)\right)_{\ell_T^m}.$$

Since $u_{\delta i}^m \in \mathrm{L}^2\left(\ell_T^m\right) \subset \mathrm{H}^{-\frac{1}{2}}(\ell_T^m)$ and $\gamma_{\ell_T^m}^i\left(w_{\ell_T^m}\right) \in \mathrm{H}^{\frac{1}{2}}(\ell_T^m)$, we can rewrite the scalar product above as a duality product. Then, adding and subtracting u_i^m,

$$\begin{aligned}
\|u_{\delta i}^m\|_{\ell_T^m}^2 &\lesssim \left\langle u_{\delta i}^m, \gamma_{\ell_T^m}^i\left(w_{\ell_T^m}\right)\right\rangle_{\ell_T^m} = \left\langle u_{\delta i}^m - u_i^m, \gamma_{\ell_T^m}^i\left(w_{\ell_T^m}\right)\right\rangle_{\ell_T^m} \\
&\quad + \left\langle u_i^m, \gamma_{\ell_T^m}^i\left(w_{\ell_T^m}\right)\right\rangle_{\ell_T^m} = \left\langle u_{\delta i}^m - u_i^m, \gamma_{\ell_T^m}^i\left(w_{\ell_T^m}\right)\right\rangle_{\ell_T^m} + \left(\nabla h_i, \nabla w_{\ell_T^m}\right)_{\omega_{\ell_T^m}} \\
&\quad - \left(f_i, w_{\ell_T^m}\right)_{\omega_{\ell_T^m}} = \left\langle u_{\delta i}^m - u_i^m, \gamma_{\ell_T^m}^i\left(w_{\ell_T^m}\right)\right\rangle_{\ell_T^m} + \left(\nabla(h_i - h_{\delta i}), \nabla w_{\ell_T^m}\right)_{\omega_{\ell_T^m}} \\
&\quad + \left(\nabla h_{\delta i}, \nabla w_{\ell_T^m}\right)_{\omega_{\ell_T^m}} - \left(f_i, w_{\ell_T^m}\right)_{\omega_{\ell_T^m}} = \left\langle u_{\delta i}^m - u_i^m, \gamma_{\ell_T^m}^i\left(w_{\ell_T^m}\right)\right\rangle_{\ell_T^m} \\
&\quad + \left(\nabla(h_i - h_{\delta i}), \nabla w_{\ell_T^m}\right)_{\omega_{\ell_T^m}} + \left(-f_i - \Delta h_{\delta i}, w_{\ell_T^m}\right)_{\omega_{\ell_T^m}} \leq \|u_i^m - u_{\delta i}^m\|_{-\frac{1}{2},\ell_T^m} \\
&\quad \times \left\|\gamma_{\ell_T^m}^i\left(w_{\ell_T^m}\right)\right\|_{\frac{1}{2},\ell_T^m} + \|h_i - h_{\delta i}\|_{\omega_{\ell_T^m}} \left\|w_{\ell_T^m}\right\|_{\omega_{\ell_T^m}} + \|f_i + \Delta h_{\delta i}\|_{\omega_{\ell_T^m}} \left\|w_{\ell_T^m}\right\|_{\omega_{\ell_T^m}},
\end{aligned}$$

where Green's formula has been applied, using the fact that there are no jumps of $\nabla h_{\delta i}$ inside $\omega_{\ell_T^m}$. Using the continuity of the trace operator and (6.9), we obtain, $\forall E \in \{R, L\}$,

$$\left\|\gamma_{\ell_T^m}^i\left(w_{\ell_T^m}\right)\right\|_{\frac{1}{2},\ell_T^m} \lesssim \left\|w_{\ell_T^m}\right\|_E = \left\|\mathcal{C}_E\left(u_{\delta i}^m\right)\mathfrak{b}_{\ell_T^m}\right\|_E \lesssim h_{\ell_T^m}^{-\frac{1}{2}} \|u_{\delta i}^m\|_{\ell_T^m},$$

and, since $h_{\ell_T^m} \leq h_T$,

$$\left\|w_{\ell_T^m}\right\|_E = \left\|\mathcal{C}_E\left(u_{\delta i}^m\right)\mathfrak{b}_{\ell_T^m}\right\|_E \lesssim h_{\ell_T^m}^{\frac{3}{2}} \|u_{\delta i}^m\|_{\ell_T^m} \leq h_T\, h_{\ell_T^m}^{\frac{1}{2}} \|u_{\delta i}^m\|_{\ell_T^m}.$$

The thesis comes from the definitions of $\eta_{R,T}$, $\xi_{NC,T}^m$ and from $\|\cdot\|_{\omega_{\ell_T^m}} \leq \|.\|_T$. If $u_{\delta i}^m$ has some jumps, it is sufficient to apply this procedure on each of the subsegments of ℓ_T^m upon which it is continuous. ∎

Proposition 6.9 (efficiency of $\eta_{P,T}$ and ξ_{P,Γ_m}^i). *Let $i \in \mathcal{I}$, $T \in \mathcal{T}_{\delta i}$. Then*

$$\eta_{P,T} = \|p_i - p_{\delta i}\|_T.$$

Moreover, let $m \in \mathcal{M}_i$, $\lambda \in \Lambda_{m,i}$. Then

$$\xi_{P,\lambda}^i \lesssim \|p_i - p_{\delta i}\|_{\omega_\lambda},$$

where ω_λ is the union of two triangles having λ as one of their sides.

Proposition 6.10 (efficiency of $J_{\delta,\lambda}^i$). *Let $\boldsymbol{u}_\delta$ be the solution of* (3.21), $\boldsymbol{h}_\delta = \mathcal{H}_\delta(\boldsymbol{u}_\delta)$. *Let $i \in \mathcal{I}$, $m \in \mathcal{M}_i$, and $\lambda \in \Lambda_{m,i}$, and let $J_{\delta,\lambda}^i(\boldsymbol{u}_\delta)$ be the quantity defined by* (4.8). *Then, if $\Gamma_m = F_i \cap F_j$,*

$$J_{\delta,\lambda}^i \lesssim \|h_i - h_{\delta i}\|_{F_i} + \|h_j - h_{\delta j}\|_{F_j} + \|u_i^m - u_{\delta i}^m\|_{-\frac{1}{2},\lambda} + \left\|u_j^m - u_{\delta j}^m\right\|_{-\frac{1}{2},\lambda}.$$

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6.3. Final lower bounds. In this subsection we collect the previous efficiency results to complete the proof of Theorem 6.1.

Assuming $h_{NC,T} < 1 \ \forall T \in \mathcal{T}_\delta$, we first look at the result of Proposition 6.6, together with the result of Proposition 6.8. From these we can obtain an efficiency estimate for $\eta_{R,T}$ involving only the exact errors and higher order terms (this is standard; see, for example, [25]). For any given $i \in \mathcal{I}$ and a triangle $T \in \mathcal{T}_{\delta i}$,

$$\eta_{R,T} \lesssim \|h_i - h_{\delta i}\|_T + \sum_{m\in\mathcal{M}_T} \|u_i - u_{\delta i}\|_{-\frac{1}{2},\ell_T^m} + h_T \|f_i - f_T\|_T + \sum_{m\in\mathcal{M}_T} \left(\|u_i - u_{\delta i}\|_{-\frac{1}{2},\ell_T^m} + \|h_i - h_{\delta i}\|_T + h_{\ell_T^m}\eta_{R,T} \right).$$

Then, since $\#\mathcal{M}_T \le \#\mathcal{M}_i$ and $\#\mathcal{M}_i$ is fixed, thanks to the assumption $h_{NC,T} < 1$ we have

$$\eta_{R,T} \lesssim \frac{1}{1-h_{NC,T}} \left[\|h_i - h_{\delta i}\|_T + \sum_{m\in\mathcal{M}_T} \|u_i^m - u_{\delta i}^m\|_{-\frac{1}{2},\ell_T^m} \right] + \frac{h_T}{1-h_{NC,T}} \|f_i - f_T\|_T . \tag{6.10}$$

Now we consider Proposition 6.8. Since $\#\mathcal{M}_T$ is bounded independently on the discretization, summing on all $m \in \mathcal{M}_T$ both members we obtain

$$\begin{aligned} \sum_{m\in\mathcal{M}_T} \xi_{NC,T} &\lesssim \#\mathcal{M}_T \|h_i - h_{\delta i}\|_T + \sum_{m\in\mathcal{M}_T} \|u_i^m - u_{\delta i}^m\|_{-\frac{1}{2},\ell_T^m} + \underbrace{\left(\sum_{m\in\mathcal{M}_T} h_{\ell_T^m} \right)}_{h_{NC,T}} \eta_{R,T} \\ &\lesssim \|h_i - h_{\delta i}\|_T + \sum_{m\in\mathcal{M}_T} \|u_i^m - u_{\delta i}^m\|_{-\frac{1}{2},\ell_T^m} + h_{NC,T}\eta_{R,T}. \end{aligned}$$

We remark that the constants may depend on $\#\mathcal{M}_T \le \#\mathcal{M}_i$. We now make use of Proposition 6.6 for bounding $\eta_{R,T}$ with the exact error and $\xi_{NC,T}$, obtaining

$$\sum_{m\in\mathcal{M}_T} \xi_{NC,T} \lesssim \|h_i - h_{\delta i}\|_T + \sum_{m\in\mathcal{M}_T} \|u_i^m - u_{\delta i}^m\|_{-\frac{1}{2},\ell_T^m} + h_{NC,T} \left[\|h_i - h_{\delta i}\|_T + \sum_{m\in\mathcal{M}_T} \|u_i - u_{\delta i}\|_{-\frac{1}{2},\ell_T^m} + h_T \|f_i - f_T\|_T \right] + h_{NC,T} \sum_{m\in\mathcal{M}_T} \xi_{NC,T}.$$

Then

$$\sum_{m\in\mathcal{M}_T} \xi_{NC,T}^m \lesssim \frac{1+h_{NC,T}}{1-h_{NC,T}} \left[\|h_i - h_{\delta i}\|_T \sum_{m\in\mathcal{M}_T} \|u_i^m - u_{\delta i}^m\|_{-\frac{1}{2},\ell_T^m} \right] + \frac{h_{NC,T}h_T}{1-h_{NC,T}} \|f_i - f_T\|_T . \tag{6.11}$$

The influence of nonconformity on the efficiency of our estimate is clear.

Finally, let's turn to the result of Proposition 6.7. To have an explicit estimate for $\xi_{F,\sigma}$ we use equations (6.10), (6.11) and the fact that $\|u_i^m - u_{\delta i}^m\|_{-\frac{1}{2},\Gamma_m\cap\mathring{\omega}_\sigma} \le$

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TABLE 1
Tables of effectivity indices.

$\max_{T\in\mathcal{T}_\delta} h_T$	ε
0.0047	0.3130
0.0063	0.3251
0.0097	0.3160
0.0112	0.3146
0.0146	0.2907
0.0221	0.3011
0.0305	0.2810
0.0331	0.2595
0.0339	0.2643
0.0773	0.2526
0.0899	0.2892
0.1012	0.2614
0.1250	0.2888
0.2435	0.3507
0.3860	0.3297
0.4373	0.3454
0.4375	0.3518

(a) `test1`

$\max_{T\in\mathcal{T}_\delta} h_T$	ε
0.0049	0.2943
0.0054	0.2672
0.0101	0.2394
0.0103	0.2478
0.0157	0.2324
0.0234	0.2183
0.0317	0.2079
0.0413	0.2075
0.0488	0.2007
0.0815	0.1978
0.0938	0.1933
0.1047	0.1949
0.1415	0.2050
0.2582	0.1980
0.2795	0.2329
0.3953	0.2347
0.5014	0.2146

(b) `test2`

$\max_{T\in\mathcal{T}_\delta} h_T$	ε
0.0180	1.2598
0.0261	1.2769
0.0327	1.2377
0.0450	1.2809
0.0508	1.2676
0.0822	1.2210
0.1115	1.2359
0.1250	1.2780
0.1766	1.2505
0.2111	1.2763
0.4004	1.2003
0.4337	1.2604
0.4719	1.3189
1.1180	1.1613
1.4142	1.8722
1.7321	1.7375
2.2361	1.6907

(c) `7_fract`

$\sum_{T\in\omega_\sigma} \|u_i^m - u_{\delta i}^m\|_{-\frac{1}{2},\ell_T^m}$. For any given $i \in \mathcal{I}$, $\sigma \in \mathcal{E}_{\delta i}$ and indicating by ω_σ the set of triangles in $\mathcal{T}_{\delta i}$ having σ as one of their sides, algebraic calculations yield

(6.12)
$$\xi_{F,\sigma} \lesssim \max\{1, h_\sigma\} \sum_{T\in\omega_\sigma} \frac{1}{1-h_{NC,T}} \left[\|h_i - h_{\delta i}\|_T + + \sum_{m\in\mathcal{M}_T} \|u_i^m - u_{\delta i}^m\|_{-\frac{1}{2},\ell_T^m} \right] + \frac{h_T(1+h_{NC,T})}{1-h_{NC,T}} \|f_i - f_T\|_T \,,$$

where we see the same kind of dependence. Using the results from Propositions 6.9 and 6.10 and (6.10), (6.11), and (6.12), we can prove Theorem 6.1.

7. Numerical results. We show here the results of numerical experiments mainly performed in order to evaluate the *effectivity index*, defined as the ratio between the *true error err* and the *estimated error est*$_\delta$ (see Table 1):

$$\varepsilon := \frac{err}{est_\delta} \,.$$

In order to approximate the norm of the error $\boldsymbol{u} - \boldsymbol{u}_\delta$ in the dual space $\mathrm{H}^{-\frac{1}{2}}(\Gamma_m)$ $\forall m \in \mathcal{M}$, we have used the following weighted $\mathrm{L}^2(\Gamma_m)$ norm:

$$\forall i \in \mathcal{I},\, \forall m \in \mathcal{M}_i,\; \|u_i^m - u_{\delta i}^m\|_{-\frac{1}{2},\Gamma_m} \approx \left(\sum_{\lambda\in\Lambda_{m,i}} h_\lambda \|u_i^m - u_{\delta i}^m\|^2_{\Gamma_m} \right)^{\frac{1}{2}} ,$$

where $\Lambda_{m,i}$ is defined as the discretization of the trace Γ_m induced by the mesh $\mathcal{T}_{\delta i}$.

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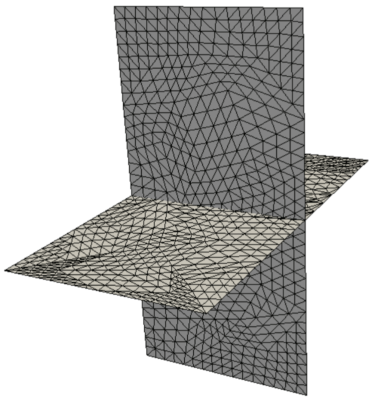
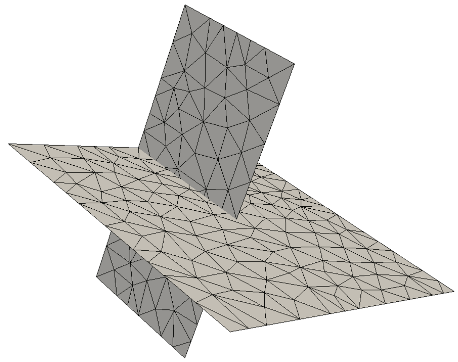
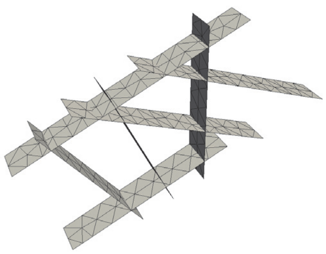


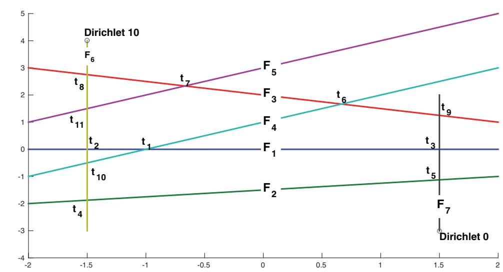
(a) `test1` 3D representation, with mesh



(b) `test2` 3D representation, with mesh



(c) `7_fract` 3D representation, with mesh



(d) `7_fract` 2D representation, with boundary conditions

Fig. 2. *Views of the considered DFNs.*

We consider three DFNs, as shown in Figure 2. All simulations are performed using linear finite elements on a sequence of refined grids, and using, for each trace Γ_m, a continuous piecewise linear approximation for u_i^m and u_j^m on the induced meshes $\Lambda_{m,i}$ and $\Lambda_{m,j}$, respectively. In all the considered meshes, traces are arbitrarily placed with respect to the mesh-edges (full nonconformity between the meshes).

All the results are collected in Table 1, where the effectivity index ε for the different cases is reported. Further, Figure 3 shows the behavior of the error estimator est_δ and of the error err with respect to the meshsize for the three DFNs.

7.1. Problem `test1`. The first test problem deals with two identical fractures intersecting each other orthogonally (see Figure 2a):

$$F_1 = \{(x, y, z) \in \mathbb{R}^3 :\ z \in (-1, 1),\, y \in (0, 1), x = 0\},$$
$$F_2 = \{(x, y, z) \in \mathbb{R}^3 :\ x \in (-1, 1),\, y \in (0, 1), z = 0\}.$$

We have $\mathcal{M} = \{1\}$ and $\Gamma_1 = \{(x, y, z) \in \mathbb{R}^3 :\ x = 0, z = 0, y \in (0, 1)\}$, and we set homogeneous Dirichlet boundary conditions on both fractures. For further details regarding this problem, we refer the reader to [7].

Results for this first problem are reported in Table 1a, where the values of the effectivity indices for different meshsizes are shown. We can see that the effectivity

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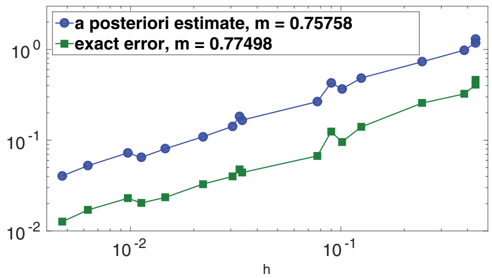
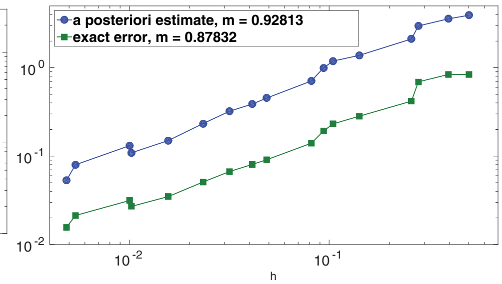
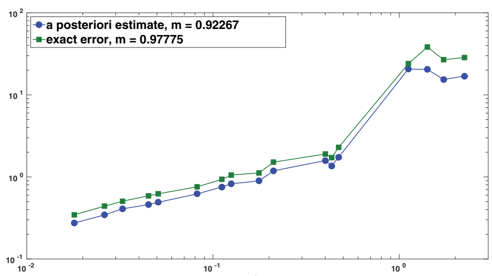


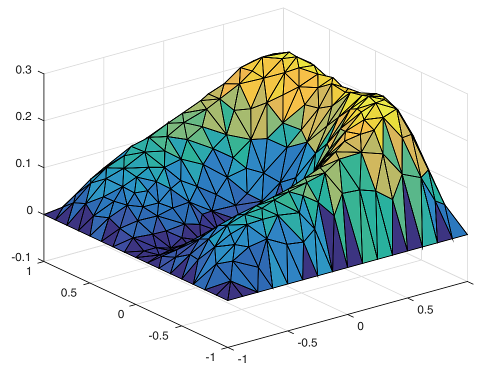
(a) test1 Error estimate (blue ●) and error (green ■) vs. h



(b) test2 Error estimate (blue ●) and error (green ■) vs. h



(c) 7_fract Error estimate (blue ●) and error (green ■) vs. h



(d) test2 Discrete solution in F_1

Fig. 3.

index is almost independent of the meshsize, with values oscillating in a range between 0.2526 and 0.3518 for h spanning two orders of magnitude. In Figure 3a we plot the true errors and the estimated errors. In the legend of this figure we report the exponent m of the fitting of these errors with respect to h ($err \simeq h^m$ and $est_\delta \simeq h^m$). The plots show a good agreement between the error and the estimator.

7.2. Problem `test2`**.** In the second test problem we consider the two-fracture DFN displayed in Figure 2b. In particular, F_1 is not intersected completely by F_2:

$$F_1 = \left\{(x,y,z) \in \mathbb{R}^3 :\ -1 < x < 1,\ -1 < y < 1,\ z = 0\right\},$$
$$F_2 = \left\{(x,y,z) \in \mathbb{R}^3 :\ -1 < x < 0,\ y = 0,\ -1 < z < 1\right\}.$$

Again, we have $\mathcal{M} = \{1\}$ and set $\Gamma_1 = \left\{(x,y,z) \in \mathbb{R}^3 :\ y = z = 0,\ -1 < x < 0\right\}$, and Dirichlet boundary conditions are set on all the boundaries. In this case we have a less regular solution around the trace tip (see [8, 4]). In Figure 3d we report a computed solution on F_1. In Table 1b we report the values of the effectivity indices for different meshsizes. We can see that, again, these values are quite stable with respect to the meshsize. In Figure 3b we plot the true errors and the estimated errors and report the slopes m of the fitting. The plots show a good agreement between the error and the estimator.

7.3. Problem `7_fract`**.** The last test problem considers the DFN of 7 fractures intersecting in 11 traces shown in Figure 2c. We set a constant Dirichlet boundary

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condition $h_D = 10$ on one side of F_6 and an homogeneous Dirichlet boundary condition on one side of F_7 (see Figure 2d). The stated problem has a piecewise linear solution on each fracture (see [7]), which could be exactly computed by the FEM method if one had meshes totally conforming to traces. This is not the case for our meshes; thus this is a good test for the behavior of the nonconformity estimators. In Figure 3c we plot the true errors and the estimated errors, and in Table 1c we report the values of the effectivity indices. We note that for the three coarsest meshes the effectivity indices are larger than the values observed for the other meshes. This shows the influence of nonconformity on the efficiency of the estimate: indeed, these meshes feature a nonconformity indicator $\max_{T\in\mathcal{T}_\delta} h_{NC,T} \approx 0.8571$, which yields $C_{NC} \approx 13$ (Theorem 6.1). With mesh refinement, starting from the fourth coarsest mesh, we have $\max_{T\in\mathcal{T}_\delta} h_{NC,T} \leq 0.5$, and the value of C_{NC} critically drops to values lower than or equal to 3 (see Remark 2) and the effectivity index becomes almost constant.

7.4. Estimators characterization. It is interesting to characterize the estimators with respect to the information they can provide about the distribution of the errors on the domain. With this target we define, $\forall T \in \mathcal{T}_\delta$, two different indicators:

$$\eta_{res,T} := \begin{cases} \eta_{R,T} + \frac{1}{2}\sum_{\sigma\in\mathcal{E}_{\delta T}} \xi_{F,\sigma} & \text{if } \forall m \in \mathcal{M},\ |\,\Gamma_m \cap T\,| = 0, \\ \eta_{R,T} & \text{otherwise,} \end{cases}$$

$$\eta_{tr,T} := \begin{cases} \eta_{P,T} + \xi_{P,T} + \xi_{NC,T} + \frac{1}{2}\sum_{\sigma\in\mathcal{E}_{\delta T}} \xi_{F,\sigma} & \text{if } \exists m \in \mathcal{M}\text{:}\ \ |\,\Gamma_m \cap T\,| \neq 0, \\ \eta_{P,T} + \xi_{P,T} + \xi_{NC,T} & \text{otherwise,} \end{cases}$$

where $\mathcal{E}_{\delta T}$ indicates the set of the edges of T. In Figure 4 we see the behavior of these two quantities for problem `test2` on F_1, whose solution is depicted in Figure 3d. The quantity $\eta_{res,T}$ provides information about the error on each fracture that is related to the finite element approximation of the solution of (3.9) in the interior of the fractures far from the traces. On the other hand, the quantity $\eta_{tr,T}$ provides information about the nonconformity errors and the violation of matching conditions on the traces. In Figure 4 we plot these two quantities on the elements of two different meshes for F_1, the coarsest of which is used for the solution shown in Figure 3d. In the top two panels we report $\eta_{res,T}$ (left) and $\eta_{tr,T}$ (right) on the coarse mesh; we see that $\eta_{res,T}$ is larger where the solution displays strong curvatures, i.e., far from the trace and around the trace tip. Instead, as expected, the conformity indicator $\eta_{tr,T}$ is concentrated around the trace. A similar behavior with different order of magnitude of the estimators is obtained on the finer grid, shown in the bottom panels.

8. Conclusions. In this paper we have derived residual-based "a posteriori" error estimates for the constrained optimization formulation of a simple model for the flow in DFNs. Numerical results have confirmed very weak dependence of the effectivity index on the meshsize, as well as a very good agreement between the estimator and the error distribution, and we have identified a parameter correlating the estimate with the nonconformity of the meshes. The terms of the estimator can be collected in two indicators with a clear meaning: an indicator related to the error inside each fracture, essentially related to the attitude of the finite element space to describe the hydraulic head in each fracture, and a second indicator essentially concerning the lack of continuity of the hydraulic head, the flux mismatch at the traces, and the nonconformity of the meshes of intersecting fractures. The different nature of the estimators makes them useful tools in an adaptive algorithm, and this will be the subject of future investigations.

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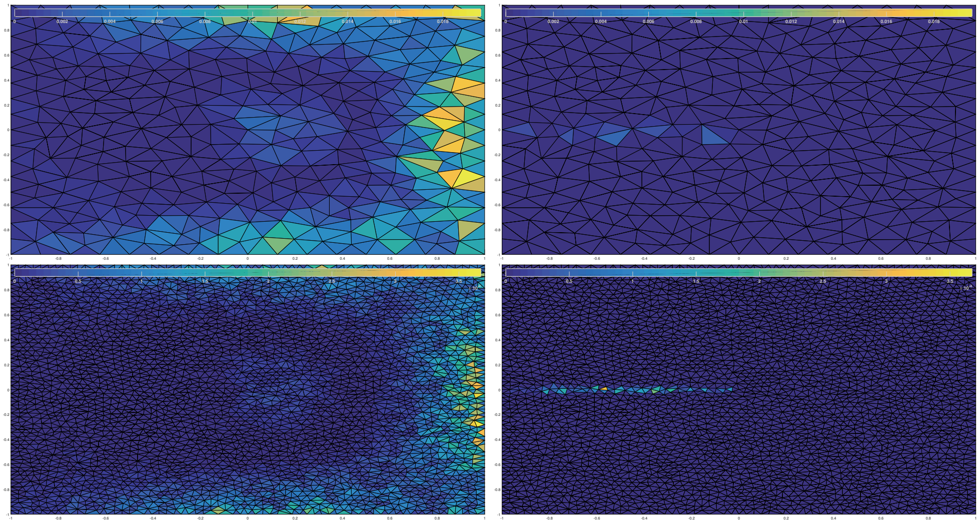

Fig. 4. $\boxed{\texttt{test 2}}$ *Behavior of* $\eta_{res,T}$ *(left) and* $\eta_{tr,T}$ *with two different meshsizes.*

REFERENCES

[1] P. M. Adler, *Fractures and Fracture Networks*, Kluwer Academic, Dordrecht, 1999.

[2] M. Ainsworth and J. T. Oden, *A Posteriori Error Estimation in Finite Element Analysis*, John Wiley & Sons, New York, 2011.

[3] M. Benedetto, S. Berrone, A. Borio, S. Pieraccini, and S. Scialò, *A hybrid mortar virtual element method for discrete fracture network simulations*, J. Comput. Phys., 306 (2016), pp. 148–166.

[4] M. F. Benedetto, S. Berrone, S. Pieraccini, and S. Scialò, *The virtual element method for discrete fracture network simulations*, Comput. Methods Appl. Mech. Engrg., 280 (2014), pp. 135–156.

[5] M. F. Benedetto, S. Berrone, and S. Scialò, *A globally conforming method for solving flow in discrete fracture networks using the virtual element method*, Finite Elem. Anal. Des., 109 (2016), pp. 23–36.

[6] S. Berrone, C. Canuto, S. Pieraccini, and S. Scialò, *Uncertainty quantification in discrete fracture network models: Stochastic fracture transmissivity*, Comput. Math. Appl., 70 (2015), pp. 603–623.

[7] S. Berrone, S. Pieraccini, and S. Scialò, *A PDE-constrained optimization formulation for discrete fracture network flows*, SIAM J. Sci. Comput., 35 (2013), pp. B487–B510.

[8] S. Berrone, S. Pieraccini, and S. Scialò, *On simulations of discrete fracture network flows with an optimization-based extended finite element method*, SIAM J. Sci. Comput., 35 (2013), pp. A908–A935.

[9] S. Berrone, S. Pieraccini, and S. Scialò, *An optimization approach for large scale simulations of discrete fracture network flows*, J. Comput. Phys., 256 (2014), pp. 838–853.

[10] S. Berrone, S. Pieraccini, S. Scialò, and F. Vicini, *A parallel solver for large scale DFN simulations*, SIAM J. Sci. Comput., 37 (2015), pp. C285–C306.

[11] P. G. Ciarlet, *Numerical Analysis of the Finite Element Method*, Sémin. Math. Supér. 59, Presses de l'Université de Montréal, Montreal, 1976.

[12] W. S. Dershowitz and C. Fidelibus, *Derivation of equivalent pipe networks analogues for three-dimensional discrete fracture networks by the boundary element method*, Water Resource Res., 35 (1999), pp. 2685–2691.

[13] J. Erhel, J.-R. de Dreuzy, and B. Poirriez, *Flow simulation in three-dimensional discrete fracture networks*, SIAM J. Sci. Comput., 31 (2009), pp. 2688–2705.

[14] A. Ern and M. Vohralík, *Flux reconstruction and a posteriori error estimation for discontinuous Galerkin methods on general nonmatching grids*, C. R. Math. Acad. Sci. Paris, 347 (2009), pp. 441–444.

[15] C. Fidelibus, G. Cammarata, and M. Cravero, *Hydraulic characterization of fractured rocks*,

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in Rock Mechanics: New Research, Nova Science Publishers, New York, 2009, pp. 233–262.

[16] J. D. Hyman, C. W. Gable, S. L. Painter, and N. Makedonska, *Conforming Delaunay triangulation of stochastically generated three dimensional discrete fracture networks: A feature rejection algorithm for meshing strategy*, SIAM J. Sci. Comput., 36 (2014), pp. A1871–A1894.

[17] P. Jiránek, Z. Strakoš, and M. Vohralík, *A posteriori error estimates including algebraic error and stopping criteria for iterative solvers*, SIAM J. Sci. Comput., 32 (2010), pp. 1567–1590.

[18] H. Mustapha and R. Dimitrakopoulos, *High-order stochastic simulation of complex spatially distributed natural phenomena*, Math. Geosci., 42 (2010), pp. 457–485.

[19] H. Mustapha and K. Mustapha, *A new approach to simulating flow in discrete fracture networks with an optimized mesh*, SIAM J. Sci. Comput., 29 (2007), pp. 1439–1459.

[20] J. Nečas, *Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnelle*, Ann. Scuola Norm. Sup. Pisa (3), 16 (1962), pp. 305–326.

[21] G. Pichot, J. Erhel, and J. de Dreuzy, *A mixed hybrid mortar method for solving flow in discrete fracture networks*, Appl. Anal., 89 (2010), pp. 1629–1643.

[22] G. Pichot, J. Erhel, and J.-R. de Dreuzy, *A generalized mixed hybrid mortar method for solving flow in stochastic discrete fracture networks*, SIAM J. Sci. Comput., 34 (2012), pp. B86–B105.

[23] G. Pichot, B. Poirriez, J. Erhel, and J.-R. de Dreuzy, *A mortar BDD method for solving flow in stochastic discrete fracture networks*, in Domain Decomposition Methods in Science and Engineering XXI, Springer, New York, 2014, pp. 99–112.

[24] R. Verfürth, *A posteriori error estimation and adaptive mesh-refinement techniques*, J. Comput. Appl. Math., 50 (1994), pp. 67–83.

[25] R. Verfürth, *A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*, Advances in Numerical Mathematics Series, Teubner B.G. GmbH, 1996.

[26] R. Verfürth, *A Posteriori Error Estimation Techniques for Finite Element Methods*, Oxford University Press, Oxford, 2013.

[27] M. Vohralík, *Guaranteed and fully robust a posteriori error estimates for conforming discretizations of diffusion problems with discontinuous coefficients*, J. Sci. Comput., 46 (2011), pp. 397–438.

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