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# BV CONTINUOUS SWEEPING PROCESSES

VINCENZO RECUPERO

ABSTRACT. We consider a large class of continuous sweeping processes and we prove that they are well posed with respect to the *BV* strict metric.

## 1. INTRODUCTION

Let  $\mathcal{X}$  be real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and let  $\mathcal{C}(t) \subseteq \mathcal{X}$  be a family of nonempty closed convex sets parametrized by the time variable  $t \in [0, T]$ , where  $T > 0$ . A *sweeping process* is the following evolution differential inclusion in the unknown  $\xi : [0, T] \rightarrow \mathcal{X}$ :

$$-\xi'(t) \in N_{\mathcal{C}(t)}(\xi(t)), \quad \text{for a.e. } t \in [0, T], \quad (1.1)$$

$$\xi(0) = \xi^0, \quad (1.2)$$

where  $\xi^0 \in \mathcal{C}(0)$  is a prescribed initial datum and

$$N_{\mathcal{K}}(x_0) := \{\nu \in \mathcal{X} : \langle \nu, x_0 - w \rangle \geq 0 \ \forall w \in \mathcal{K}\} \quad (1.3)$$

is the exterior normal cone to a closed convex set  $\mathcal{K} \subseteq \mathcal{X}$  at the point  $x_0 \in \mathcal{K}$ . Notice that it is implicitly assumed that

$$\xi(t) \in \mathcal{C}(t) \quad \forall t \in [0, T]. \quad (1.4)$$

Sweeping processes were introduced by J.J. Moreau in the fundamental paper [21] and originated a research which is still active: see, e.g., the monograph [20], the expository papers [17, 29], and the references therein.

In the present paper we continue the analysis of [27], where we studied some continuity properties of the solution operator  $\mathcal{C} \mapsto \xi$  of the sweeping processes by setting it in the wider framework of *rate independent operators*, indeed problem (1.1)–(1.2) has the following property, called *rate independence*: if  $\phi : [0, T] \rightarrow [0, T]$  is an increasing surjective reparametrization of time and  $\xi$  is the solution associated to  $\mathcal{C}(t)$ , then  $\xi(\phi(t))$  is the solution corresponding to  $\mathcal{C}(\phi(t))$ . Rate independent evolution problems are strictly connected to elasto-plasticity and hysteresis and have been deeply studied from the mathematical point of view in the monographs [12, 30, 6, 13, 19]. The study of continuity properties with respect to various topologies has been recently performed also in, e.g., [17, 5, 16, 31] and these properties are important since they ensure robustness of the model.

Here we address the sweeping process in the following formulation provided in [5]: a Banach space  $\mathcal{Y}$ , two functions  $u : [0, T] \rightarrow \mathcal{X}$ ,  $r : [0, T] \rightarrow \mathcal{Y}$ , and a family of closed convex sets  $\mathcal{Z}(r) \subseteq \mathcal{X}$  parametrized by  $r \in \mathcal{Y}$  are given, and we have to find a function  $\xi : [0, T] \rightarrow \mathcal{X}$  such that

$$\langle u(t) - \xi(t) - z, \xi'(t) \rangle \geq 0, \quad \text{for a.e. } t \in [0, T], \quad \forall z \in \mathcal{Z}(r(t)), \quad (1.5)$$

$$u(0) - \xi(0) = x^0. \quad (1.6)$$

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Again it is implicitly assumed that  $u(t) - \xi(t) \in \mathcal{Z}(r(t))$  for all  $t \in [0, T]$  (all the precise definitions, assumptions and formulations will be given in the next Sections 2 and 3).

Note that (1.5)–(1.6) is actually a reformulation of (1.1)–(1.2), indeed, as observed in [5], one can reduce (1.5)–(1.6) to (1.1)–(1.2) by setting  $u(t) = 0$ ,  $r(t) = t$ ,  $x^0 = -\xi^0$ ,  $\mathcal{C} = -\mathcal{Z}$ ; vice versa with the position  $\mathcal{C}(t) = u(t) - \mathcal{Z}(r(t))$ ,  $\xi^0 = u(0) - x^0$  one can reduce the first problem to the second. However formulation (1.5)–(1.6) introduces the new parameters  $u(t), r(t)$  that are relevant in applications, so that it is useful to study the properties of the sweeping process with respect to  $u$  and  $r$ . This analysis is performed in [5] where it is shown that the solution operator  $\mathbb{S} : (u, r) \rightarrow \xi$  of (1.5)–(1.6) is continuous with respect to the  $W^{1,1}$ -topology (or the strong  $BV$  topology, see (2.9)), i.e. if  $u_n \rightarrow u$  in  $W^{1,1}(0, T; \mathcal{X})$  and  $r_n \rightarrow r$  in  $W^{1,1}(0, T; \mathcal{Y})$ , then  $\mathbb{S}(u_n, r_n) \rightarrow \mathbb{S}(u, r)$  in  $W^{1,1}(0, T; \mathcal{X})$ . This property is essentially proved under some geometrical assumptions on  $\mathcal{Z}(r)$  (cf. Assumption 3.1) which however turn out to be not so restrictive for applications.

In [21, 15, 16] the  $BV$ -generalization of (1.5)–(1.6) is considered:  $\mathcal{Z}(r)$  is given as above, but  $u$  and  $r$  are with bounded variation, and one has to find a continuous function  $\xi : [0, T] \rightarrow \mathcal{X}$  of bounded variation such that (1.6) holds together with the condition

$$\int_0^T \langle u(t) - \xi(t) - z(t), dD\xi(t) \rangle \geq 0, \\ \forall z \in BV([0, T]; \mathcal{X}), \quad z(t) \in \mathcal{Z}(r(t)) \quad \forall t \in [0, T], \quad (1.7)$$

where the integral is meant in the sense of the Stieltjes or differential measures (see [21, 16]). In [16] it is proved that also in this case the corresponding solution operator  $\bar{\mathbb{S}} : (u, r) \mapsto \xi$  is continuous with respect to the  $BV$ -norm.

Here instead we prove that the well posedness of (1.7)–(1.6) (and (1.5)–(1.6)) with respect to the  $BV$  strict metric (cf. (2.10)) when  $u$  and  $r$  are continuous in time (for non-continuous data the  $BV$ -strict discontinuity is proved in [26] when  $\mathcal{Z}(r) = \mathcal{Z}$  for every  $r$ ,  $\mathcal{Z}$  belonging to wide class of constant convex sets). The strict metric is very natural, especially when one deals with approximation procedures (see [1]): indeed given a function of bounded variation  $v$ , by means of the classical convolution operation one can find a sequence of regular functions  $v_n$  converging strictly to  $v$ . The geometric meaning is clear, two curves  $u$  and  $v$  are near with respect to the strict metric if they are near in the  $L^1$ -norm and if their lengths are near.

In connection with rate independent problems the strict metric has been studied for instance in [7, 30, 13, 22, 24, 25]. In particular, concerning the specific sweeping process when the data are continuous and  $\mathcal{Z}(r(t)) = \mathcal{Z}$ , a fixed closed convex subset of  $\mathcal{X}$ , in [13] it is proved its continuity with respect to the strict metric provided the boundary  $\mathcal{Z}$  satisfies certain smoothness assumptions. This requirement was completely removed in [26]. Since in the present paper we address the more general case (1.7)–(1.6), where the product  $\mathcal{X} \times \mathcal{Y}$  of a Hilbert and a Banach space is involved, the Hilbert technique used in [26] does not apply due to some uniform convexity issues (see Remark 4.2).

A byproduct of our result is that only the analysis of the sweeping process for Lipschitz data is needed: then the analogous results for the continuous  $BV$  case are a straightforward consequence of standard measure theory arguments.

We conclude this introduction with a brief plan of the paper. In the next section we recall all the necessary rigorous and precise preliminaries. In Section 3 we state the main theorems of the paper and in Section 4 we prove them. Finally in the Appendix we prove some technical properties about the strict convergence of sequences of Banach valued functions of bounded variation.

## 2. PRELIMINARIES

If  $\mathcal{B}$  is a real Banach space with norm  $\|\cdot\|_{\mathcal{B}}$ , then  $\mathcal{B}^*$  will denote its topological dual space and  $\mathcal{B}^*\langle \cdot, \cdot \rangle_{\mathcal{B}}$  the duality between  $\mathcal{B}$  and  $\mathcal{B}^*$ . We use the notation  $B_{\rho}(v_0) := \{v \in \mathcal{B} : \|v - v_0\|_{\mathcal{B}} < \rho\}$  for open balls with center  $v_0 \in \mathcal{B}$  and radius  $\rho > 0$ . The topological interior of a set  $\mathcal{S}$  is indicated by  $\text{int}(\mathcal{S})$ . If  $v, v_n \in \mathcal{B}$  for every  $n \in \mathbb{N}$  and  $v_n$  converges weakly to  $v$  as  $n \rightarrow \infty$ , we will write  $v_n \rightharpoonup v$  in  $\mathcal{B}$  as  $n \rightarrow \infty$ . We also set

$$\mathcal{C}_{\mathcal{B}} := \{\mathcal{K} \subseteq \mathcal{B} : \mathcal{K} \text{ nonempty, closed and convex}\}. \quad (2.1)$$

If

$$\mathcal{K} \in \mathcal{C}_{\mathcal{B}} \text{ is bounded and } 0 \in \text{int}(\mathcal{K}), \quad (2.2)$$

we recall that the *Minkowski functional associated with  $\mathcal{K}$*  is the function  $M_{\mathcal{K}} : \mathcal{B} \rightarrow [0, \infty[$  defined by

$$M_{\mathcal{K}}(v) := \inf \left\{ \lambda > 0 : \frac{v}{\lambda} \in \mathcal{K} \right\}, \quad v \in \mathcal{B}. \quad (2.3)$$

Here are some properties of the Minkowski functional that will be implicitly used in the sequel (cf., e.g., [28, Theorems 1.34–1.36] and recall that (2.2) holds):

$$M(x + y) \leq M(x) + M(y), \quad M(\lambda x) = \lambda M(x) \quad \forall x, y \in \mathcal{B}, \forall \lambda \geq 0, \quad (2.4)$$

$$M \text{ is continuous}, \quad (2.5)$$

$$\mathcal{K} = \{x \in \mathcal{K} : M(x) \leq 1\}, \quad (2.6)$$

$$M(x) = 0 \iff x = 0. \quad (2.7)$$

In the sequel  $T > 0$  will be a fixed positive number denoting the final time of the sweeping process. If  $\mathcal{L}^1$  is the one-dimensional Lebesgue measure and if  $p \in [1, \infty]$ , then the space of  $\mathcal{B}$ -valued Lebesgue functions which are integrable on  $[0, T]$  with respect to  $\mathcal{L}^1$  will be denoted by  $L^p(0, T; \mathcal{B})$  (see [18, Chapter III]).

For a function  $v : [0, T] \rightarrow \mathcal{B}$  we set  $\|v\|_{\infty} := \sup_{t \in [0, T]} \|v(t)\|_{\mathcal{B}}$ . Moreover if  $J \subseteq [0, T]$  is an interval, the *variation of  $v$  on  $J$*  is the real extended number  $V(v, J)$  defined by

$$V(v, J) := \sup \left\{ \sum_{j=1}^m \|v(t_j) - v(t_{j-1})\|_{\mathcal{B}} : m \in \mathbb{N}, t_j \in J, t_1 < \dots < t_m \right\}, \quad (2.8)$$

and we say that  $v$  is of *bounded variation on  $J$*  if  $V(v, J) < \infty$ . We set

$$BV([0, T]; \mathcal{B}) := \{v : [0, T] \rightarrow \mathcal{B} : V(v, [0, T]) < \infty\}.$$

Let us recall two natural topologies in  $BV$ : the *strong topology* induced by the semimetric

$$d_{BV}(u, v) := \|u - v\|_{L^1(0, T; \mathcal{B})} + |V(u - v, [0, T])|, \quad u, v \in BV([0, T]; \mathcal{B}), \quad (2.9)$$

and the *strict topology*, induced by the strict semimetric

$$d_s(u, v) := \|u - v\|_{L^1(0, T; \mathcal{B})} + |V(u, [0, T]) - V(v, [0, T])|, \quad u, v \in BV([0, T]; \mathcal{B}). \quad (2.10)$$

When we restrict to continuous functions, then  $d_{BV}$  and  $d_s$  are actually metrics. If  $v, v_n \in BV([0, T]; \mathcal{B})$ , we say that  $v_n \rightarrow v$  *strictly on  $[0, T]$*  if  $d_s(v_n, v) \rightarrow 0$  as  $n \rightarrow \infty$ . Geometrically this means that  $v_n \rightarrow v$  in  $L^1$  and the lengths of the curves  $v_n$  converge to the length of  $v$ .

If  $p \in [1, \infty]$  we denote by  $W^{1,p}(0, T; \mathcal{B})$  the Sobolev spaces of  $\mathcal{B}$ -valued function: we recall that  $v \in W^{1,p}(0, T; \mathcal{B})$  if and only if there exists  $w \in L^p(0, T; \mathcal{B})$  such that  $v(t) = v(0) + \int_0^t w(s) ds$  for every  $t \in [0, T]$ , in other words  $w$  is the distributional derivative of  $v$ . If  $v \in W^{1,p}(0, T; \mathcal{B})$  then we have that  $v$  is differentiable  $\mathcal{L}^1$ -a.e. and any representative of  $v'$  is the distributional derivative of  $v$ , moreover  $v \in BV([0, T]; \mathcal{B})$  and  $V(v, [0, T]) = \int_0^T \|v'(t)\|_{\mathcal{B}} dt$ . If  $1 \leq p \leq q \leq \infty$  we obviously have that  $W^{1,q}([0, T]; \mathcal{B}) \subseteq W^{1,p}([0, T]; \mathcal{B}) \subseteq C([0, T]; \mathcal{B})$ , the space of  $\mathcal{B}$ -valued continuous functions. For any  $v : [0, T] \rightarrow \mathcal{B}$  we set  $\text{Lip}(v) := \sup_{t \neq s} \|v(t) - v(s)\|_{\mathcal{B}} / |t - s|$ .

$v(s)\|_{\mathcal{B}}/|t-s|$  and  $Lip([0, T]; \mathcal{B}) := \{v : [0, T] \rightarrow \mathcal{B} : Lip(v) < \infty\}$ . Clearly  $W^{1,\infty}(0, T; \mathcal{B}) \subseteq Lip([0, T]; \mathcal{B})$ . If  $\mathcal{B}$  is reflexive then  $W^{1,\infty}(0, T; \mathcal{B}) = Lip([0, T]; \mathcal{B})$  (we refer to [3, Appendix] for vector valued Sobolev spaces).

### 3. MAIN RESULTS

In the sequel of the paper we will assume that

$$\mathcal{X} \text{ is a real Hilbert space with inner product } \langle \cdot, \cdot \rangle \text{ and norm } \|\cdot\|_{\mathcal{X}} = \langle \cdot, \cdot \rangle^{1/2}, \quad (3.1)$$

$$\mathcal{Y} \text{ is a reflexive real Banach space with norm } \|\cdot\|_{\mathcal{Y}}, \quad (3.2)$$

$$\mathcal{R} \in \mathcal{C}_{\mathcal{Y}} \text{ and } \text{int}(\mathcal{R}) \neq \emptyset. \quad (3.3)$$

There will be given a multivalued map

$$\mathcal{Z} : \mathcal{R} \rightarrow \mathcal{C}_{\mathcal{X}} \quad (3.4)$$

and the functional  $M : \mathcal{X} \times \mathcal{Y} \rightarrow [0, \infty]$  defined by

$$M(x, r) := M_{\mathcal{Z}(r)}(x), \quad (x, r) \in \mathcal{X} \times \mathcal{R}. \quad (3.5)$$

the Minkowski functional of  $\mathcal{Z}(r)$ .

Now we can state the problem defining the sweeping process in the absolutely continuous framework.

**Problem 3.1.** Assume that  $\mathcal{Z} : \mathcal{R} \rightarrow \mathcal{C}_{\mathcal{X}}$ ,  $u \in W^{1,1}(0, T; \mathcal{X})$ ,  $r \in W^{1,1}(0, T; \mathcal{Y})$ , and  $x^0 \in \mathcal{Z}(r(0))$  are given such that  $r([0, T]) \subseteq \mathcal{R}$ . Find  $\xi \in W^{1,1}(0, T; \mathcal{X})$  such that

$$u(t) - \xi(t) \in \mathcal{Z}(r(t)) \quad \forall t \in [0, T], \quad (3.6)$$

$$u(0) - \xi(0) = x^0, \quad (3.7)$$

$$\langle u(t) - \xi(t) - z, \xi'(t) \rangle \geq 0, \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0, T], \quad \forall z \in \mathcal{Z}(r(t)). \quad (3.8)$$

We need the following set of assumptions (cf. [5]).

**Assumption 3.1.** *There exists  $C > 0$  such that*

$$0 \in \mathcal{Z}(r) \subseteq B_C(0) \quad \forall r \in \mathcal{R}. \quad (3.9)$$

*There exist the partial (Fréchet) derivatives  $\partial_x M(x, r) \in \mathcal{X}$ ,  $\partial_r M(x, r) \in \mathcal{Y}^*$  for every  $(x, r) \in \mathcal{X} \times \mathcal{R}$ , and there are positive constants  $K_0$ ,  $C_J$ ,  $C_K$  such that the maps  $J : (\mathcal{X} \setminus \{0\}) \times \text{int}(\mathcal{R}) \rightarrow \mathcal{X}$ ,  $K : (\mathcal{X} \setminus \{0\}) \times \text{int}(\mathcal{R}) \rightarrow \mathcal{Y}^*$  defined by*

$$J(x, r) := M(x, r)\partial_x M(x, r), \quad (x, r) \in (\mathcal{X} \setminus \{0\}) \times \text{int}(\mathcal{R}), \quad (3.10)$$

$$K(x, r) := M(x, r)\partial_r M(x, r), \quad (x, r) \in (\mathcal{X} \setminus \{0\}) \times \text{int}(\mathcal{R}), \quad (3.11)$$

*can be continuously extended to  $(0, r) \in \mathcal{X} \times \mathcal{R}$  for any  $r \in \mathcal{R}$ , and*

$$\|J(x_1, r_1) - J(x_2, r_2)\|_{\mathcal{X}} \leq C_J(\|x_1 - x_2\|_{\mathcal{X}} + \|r_1 - r_2\|_{\mathcal{Y}}), \quad (3.12)$$

$$\|K(x_1, r_1) - K(x_2, r_2)\|_{\mathcal{Y}^*} \leq C_K(\|x_1 - x_2\|_{\mathcal{X}} + \|r_1 - r_2\|_{\mathcal{Y}}), \quad (3.13)$$

$$\|K(x, r)\|_{\mathcal{Y}^*} \leq K_0. \quad (3.14)$$

*for every  $x_1, x_2 \in B_C(0)$  and  $r_1, r_2 \in \mathcal{R}$ .*

**Remark 3.1.** The map  $J$  can be seen as the partial derivative with respect to  $x$  of the function  $(x, r) \mapsto (M(x, r))^2/2$ , i.e.  $J$  associates to every  $(x, r)$  the vector  $\partial_x M(x, r)$  multiplied by the scalar  $M(x, r)$ . A similar remark holds for  $K$ .

Let us recall two consequences of Assumption 3.1. In [16, Lemma 2.3] it is proved that there exists  $c \in ]0, C[$  such that

$$B_c(0) \subseteq \mathcal{Z}(r) \quad \forall r \in \mathcal{R}. \quad (3.15)$$

Moreover if  $r \in \mathcal{R}$  then (cf. [5, Lemma 3.1])

$$J(x, r) \neq 0, \quad N_{\mathcal{Z}(r)}(x) = \left\{ \lambda \frac{J(x, r)}{\|J(x, r)\|_{\mathcal{X}}} : \lambda \geq 0 \right\} \quad \forall r \in \mathcal{R}, \forall x \in \partial \mathcal{Z}(r) \quad (3.16)$$

where  $N_{\mathcal{Z}(r)}(x) := \{\nu \in \mathcal{X} : \langle \nu, x_0 - w \rangle \geq 0 \forall w \in \mathcal{K}\}$  is the normal cone of convex analysis. In other words the normal cone to  $\mathcal{Z}(r)$  at  $x$  is a half-line whose direction is  $J(x, r)/\|J(x, r)\|_{\mathcal{X}}$ .

Observe that condition (3.9) assumed here and in [5] is not very restrictive for applications, indeed the function  $u(t)$  allows a translation of the moving convex set  $\mathcal{C}(t)$  of (1.2), whereas (3.9) and (3.15) require that  $\mathcal{C}(t)$  remains uniformly bounded and does not shrink to a point.

In [5, Proposition 4.1, Theorem 7.1] the following theorem is proved.

**Theorem 3.1.** *Let us assume that Assumption 3.1 holds. Then Problem 3.1 admits a unique solution. Let*

$$D := \left\{ (u, r, x^0) \in W^{1,1}(0, T; \mathcal{X}) \times W^{1,1}(0, T; \mathcal{Y}) \times \mathcal{X} : r([0, T]) \subseteq \mathcal{R}, x^0 \in \mathcal{Z}(r(0)) \right\} \quad (3.17)$$

and let  $S : D \rightarrow W^{1,1}(0, T; \mathcal{X})$  be the operator assigning to each  $(r, u, x^0) \in D$  the unique  $\xi \in W^{1,1}(0, T; \mathcal{X})$  satisfying (3.6)–(3.8). Then  $S$  is continuous with respect to the  $W^{1,1}$ -topology, in the following sense: if  $(u, r, x^0), (u_n, r_n, x_n^0) \in D$  for every  $n \in \mathbb{N}$  and

$$u_n \rightarrow u \text{ in } W^{1,1}(0, T; \mathcal{X}), \quad (3.18)$$

$$r_n \rightarrow r \text{ in } W^{1,1}(0, T; \mathcal{Y}), \quad (3.19)$$

$$x_n^0 \rightarrow x^0 \text{ in } \mathcal{X} \quad (3.20)$$

as  $n \rightarrow \infty$ , then  $S(u_n, r_n, x_n^0) \rightarrow S(u, r, x^0)$  in  $W^{1,1}(0, T; \mathcal{X})$ .

A key tool in our arguments will rely on the following proposition whose proof is straightforward. Its content is described by saying that Problem 3.1 (or the operator  $S$ ) is *rate independent*.

**Proposition 3.1.** *Let  $S : D \rightarrow W^{1,1}(0, T; \mathcal{X})$  be the operator defined by Theorem 3.1. If  $\phi : [0, T] \rightarrow [0, T]$  is absolutely continuous and increasing, then*

$$S(u \circ \phi, r \circ \phi, x^0) = S(u, r, x^0) \circ \phi \quad (3.21)$$

for every  $(u, r, x^0) \in D$ .

**Remark 3.2.** In the previous proposition the function  $\phi$  may have some constancy intervals.

In [16] it is considered the following *BV* version of the sweeping processes (analogous to the *BV*-version in [21]):

**Problem 3.2.** Assume that  $\mathcal{Z} : \mathcal{Y} \rightarrow \mathcal{C}_{\mathcal{X}}$ ,  $u \in BV([0, T]; \mathcal{Y}) \cap C([0, T]; \mathcal{Y})$ ,  $r \in BV([0, T]; \mathcal{Y}) \cap C([0, T]; \mathcal{Y})$ , and  $x^0 \in \mathcal{Z}(r(0))$  are given such that  $r([0, T]) \subseteq \mathcal{R}$ . Find  $\xi \in BV([0, T]; \mathcal{X}) \cap C([0, T]; \mathcal{X})$  such that

$$u(t) - \xi(t) \in \mathcal{Z}(r(t)) \quad \forall t \in [0, T], \quad (3.22)$$

$$u(0) - \xi(0) = x^0, \quad (3.23)$$

$$\int_0^T \langle u(t) - \xi(t) - z(t), dD\xi(t) \rangle \geq 0, \quad \forall z \in BV([0, T]; \mathcal{X}), \quad z(t) \in \mathcal{Z}(r(t)) \quad \forall t \in [0, T], \quad (3.24)$$

where the integral in (3.24) is meant in the Riemann-Stieltjes sense (cf., e.g., [18, Chapter 10]) or equivalently in the ordinary Lebesgue sense with respect to the Stieltjes vector measure  $D\xi$ , the function  $\xi$  being continuous (see [8, Section III.17] or [26, Section 2]).

**Remark 3.3.** In the reference [16], the integral of (3.24) is considered in the sense of Kurzweil or Young (cf. [14, 15]). However in [26] it is proved that when  $\xi$  is left continuous and with bounded variation, then these integrals coincide with the ordinary Lebesgue integral with respect to the differential measure  $D\xi$ . Moreover in [16] the test functions of (3.24) are allowed to belong to  $\text{Reg}([0, T]; \mathcal{Y})$ , the space of *regulated functions on*  $[0, T]$ , i.e. those functions  $v : [0, T] \rightarrow \mathcal{Y}$  such that there exist the left and right limits  $u(t-), u(t+)$  in  $\mathcal{Y}$  at any point  $t \in [0, T]$ , with the convention that  $u(0-) = u(0)$  and  $u(T+) = u(T)$ . Actually this more restrictive condition is implied by (3.24), indeed it is enough to approximate any  $z \in \text{Reg}([0, T]; \mathcal{X})$  with a uniformly convergent sequence  $z_n \in \text{BV}([0, T]; \mathcal{X})$  (cf. [2, Section II.1.3]) and pass to the limit in (3.2) where  $z$  is replaced by  $z_n$  (see also [14, Theorem 3.9]).

In [15] it is shown that Problem 3.2 admits a unique solution by means of an *approximation-a priori estimates-limit procedure*. In Theorem 4.1 below we will give a different short proof of this result making use of basic measure theory tools. This proof will provide a sort of representation formula for the solution that will allow to prove our main result, i.e. that Problem 3.2 is well-posed with respect to the strict metric. Here is the precise formulation.

**Theorem 3.2.** *Let us assume that Assumption 3.1 holds. Let*

$$\bar{D} := \left\{ (r, u, x^0) \in [\text{BV}([0, T]; \mathcal{X}) \cap C([0, T]; \mathcal{X})] \times [\text{BV}([0, T]; \mathcal{Y}) \cap C([0, T]; \mathcal{Y})] \times \mathcal{X} : \right. \\ \left. r([0, T]) \subseteq \mathcal{R}, x^0 \in \mathcal{Z}(r(0)) \right\}. \quad (3.25)$$

For every  $(r, u, x^0) \in \bar{D}$  there exists a unique  $\xi =: \bar{S}(r, u, x^0) \in \text{BV}([0, T]; \mathcal{X}) \cap C([0, T]; \mathcal{X})$  satisfying (3.22)–(3.24). The resulting solution operator  $\bar{S} : \bar{D} \rightarrow \text{BV}([0, T]; \mathcal{X}) \cap C([0, T]; \mathcal{X})$  is continuous with respect to the strict metric, in the following sense: if  $(u, r, x^0), (u_n, r_n, x_n^0) \in \bar{D}$  for every  $n \in \mathbb{N}$ , and

$$u_n \rightarrow u \text{ strictly on } [0, T], \quad (3.26)$$

$$r_n \rightarrow r \text{ strictly on } [0, T], \quad (3.27)$$

$$x_n^0 \rightarrow x^0 \text{ in } \mathcal{X} \quad (3.28)$$

as  $n \rightarrow \infty$ , then  $\bar{S}(u_n, r_n, x_n^0) \rightarrow \bar{S}(u, r, x^0)$  strictly on  $[0, T]$ .

#### 4. PROOFS

In general, for a real Banach space  $\mathcal{B}$  and a function  $v \in \text{BV}(0, T; \mathcal{B}) \cap C([0, T]; \mathcal{B})$ , we can define the following increasing (continuous) surjective arc length function  $\ell_v : [0, T] \rightarrow [0, T]$  by setting

$$\ell_v(t) := \begin{cases} \frac{T}{V(v, [0, T])} V(v, [0, t]) & \text{if } V(v, [0, T]) \neq 0 \\ 0 & \text{if } V(v, [0, T]) = 0 \end{cases} \quad (4.1)$$

(the only difference with the usual arc length function is given by a multiplicative factor allowing the range of  $\ell_v$  to be  $[0, T]$ ). Arguing as in [11, Section 2.5.16, p. 109] we infer that there exists a unique  $\tilde{v} \in \text{Lip}([0, T]; \mathcal{B})$  such that

$$v(t) = \tilde{v}(\ell_v(t)) \quad \forall t \in [0, T], \quad (4.2)$$

$$\|\tilde{v}'\|_{L^\infty(0, T; \mathcal{B})} \leq \frac{V(v, [0, T])}{T}. \quad (4.3)$$

The function  $\tilde{v}$  is the *reparametrization of  $v$  by the arc length  $\ell_v$* . Clearly we have

$$V(\tilde{v}, [0, T]) = V(v, [0, T]). \quad (4.4)$$

In the sequel we will set

$$\mathcal{B} := \mathcal{X} \times \mathcal{Y} \quad (4.5)$$

endowed with the norm

$$\|(x, y)\|_{\mathcal{B}} := \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}, \quad (x, y) \in \mathcal{B}. \quad (4.6)$$

Note that with this norm the space  $\mathcal{B}$  is not uniformly convex because  $\mathbb{R}^2$  is not uniformly convex with the 1-norm. This fact prevents from applying the Hilbert techniques used in [26] (cf. Remark 4.2 below). Nevertheless  $\mathcal{B}$  is reflexive, due to the reflexivity of  $\mathcal{X}$  and  $\mathcal{Y}$  and to Kakutani's theorem (cf., e.g., [4, Theorem 3.17]). In this case if

$$v = (v_x, v_y) : [0, T] \longrightarrow \mathcal{B}, \quad (4.7)$$

from (2.8), (4.5) and (4.6) we immediately infer that

$$V(v, [0, T]) = V(v_x, [0, T]) + V(v_y, [0, T]). \quad (4.8)$$

Therefore if  $v_x \in BV([0, T]; \mathcal{X}) \cap C([0, T]; \mathcal{X})$  and  $v_y \in BV([0, T]; \mathcal{Y}) \cap C([0, T]; \mathcal{Y})$  then  $v = (v_x, v_y) \in BV([0, T]; \mathcal{B}) \cap C([0, T]; \mathcal{B})$  and there exist  $\bar{v}_x \in Lip([0, T]; \mathcal{X})$ ,  $\bar{v}_y \in Lip([0, T]; \mathcal{Y})$  such that

$$\tilde{v} = (\bar{v}_x, \bar{v}_y) : [0, T] \longrightarrow \mathcal{B} \quad (4.9)$$

and

$$(v_x(t), v_y(t)) = v(t) = \tilde{v}(\ell_v(t)) = (\bar{v}_x(\ell_v(t)), \bar{v}_y(\ell_v(t))) \quad \forall t \in [0, T]. \quad (4.10)$$

By Proposition 3.1 we immediately have that

$$S(u, r, x^0) = S(\bar{u}, \bar{r}, x^0) \circ \ell_v \quad \forall (u, r, x^0) \in D. \quad (4.11)$$

We start by showing that such formula also holds for  $BV$ -solutions. The following theorem also provides an alternative proof for the existence of Problem 3.2.

**Theorem 4.1.** *If  $(u, v, x^0) \in \bar{D}$  then*

$$\bar{S}(u, r, x^0) = S(\bar{u}, \bar{r}, x^0) \circ \ell_v \quad (4.12)$$

*is the unique solution of Problem 3.2.*

*Proof.* The uniqueness of a solution for Problem 3.2 is standard and we refer to [15]. Now we prove formula (4.12). We set  $v := (u, r) \in BV([0, T]; \mathcal{B}) \cap C([0, T]; \mathcal{B})$  and we prove that

$$\xi := S(\bar{u}, \bar{r}, x^0) \circ \ell_v$$

solves Problem 3.2. Formulas (3.22), (3.23) are obvious. In order to check (3.24) let  $z \in Reg([0, T]; \mathcal{Y})$  be such that  $z(t) \in \mathcal{Z}(r(t))$  for every  $t \in [0, T]$ . Then by a change of variable in the Stieltjes integral (cf. [23, Lemma 5.1]) we have

$$\begin{aligned} & \int_0^T \langle u(t) - \xi(t) - z(t), dD\xi(t) \rangle \\ &= \int_0^T \left\langle \bar{u}(\ell_v(t)) - (S(\bar{u}, \bar{r}, x^0) \circ \ell_v)(t) - z(t), d(S(\bar{u}, \bar{r}, x^0) \circ \ell_v)(t) \right\rangle \\ &= \int_0^T \left\langle \bar{u}(\ell_v(t)) - S(\bar{u}, \bar{r}, x^0)((\ell_v)(t)) - z(t), (S(\bar{u}, \bar{r}, x^0))'(\ell_v(t)) \right\rangle dD\ell_v(t) \end{aligned} \quad (4.13)$$

Now let

$$A = \left\{ \sigma \in [0, T] : \left\langle \bar{u}(\sigma) - S(\bar{u}, \bar{r}, x^0)(\sigma) - z, (S(\bar{u}, \bar{r}, x^0))'(\sigma) \right\rangle < 0 \forall z \in \mathcal{Z}(\bar{r}(\sigma)) \right\}.$$

From (3.8) it follows that  $A$  has Lebesgue measure zero, hence  $Dl_v(\ell_v^{-1}(A)) = 0$  (cf. [23, Proposition 2.2]) and, since  $z(t) \in \mathcal{Z}(r(t)) = \mathcal{Z}(\bar{r}(\ell_v(t)))$ , we find that

$$\begin{aligned} & Dl_v \left( \left\{ t \in [0, T] : \left\langle \bar{u}(\ell_v(t)) - \mathcal{S}(\bar{u}, \bar{r}, x^0)(\ell_v(t)) - z(t), (\mathcal{S}(\bar{u}, \bar{r}, x^0))'(\ell_v(t)) \right\rangle < 0 \right\} \right) \\ & \leq Dl_v(\{s \in [0, t] : \ell_v(s) \in A\}) = 0 \end{aligned}$$

Consequently from (4.14) we infer that  $\int_0^T \langle u(t) - \xi(t) - z(t), dD\xi(t) \rangle \geq 0$  and we are done.  $\square$

**Remark 4.1.** Let us observe that Theorem 4.1 provides a proof for the existence/uniqueness of Problem 3.2 which allows to reduce to the Lipschitz case by means of basic measure theoretical facts. The same argument shows that the operator  $\bar{\mathcal{S}}$  is rate independent.

**Proposition 4.1.** *Assume that  $u, u_n \in BV([0, T]; \mathcal{X}) \cap C([0, T]; \mathcal{X})$  and  $r, r_n \in BV([0, T]; \mathcal{Y}) \cap C([0, T]; \mathcal{Y})$  for every  $n \in \mathbb{N}$  and set*

$$v := (u, r) : [0, T] \longrightarrow \mathcal{B}, \quad (4.15)$$

$$v_n := (u_n, r_n) : [0, T] \longrightarrow \mathcal{B}, \quad n \in \mathbb{N}. \quad (4.16)$$

If  $u_n \rightarrow u$  and  $r_n \rightarrow r$  strictly as  $n \rightarrow \infty$ , then

$$\tilde{v}_n \rightarrow \tilde{v} \text{ strictly on } [0, T], \quad (4.17)$$

where  $\tilde{v}_n$  and  $\tilde{v}$  are the arc length reparametrizations defined above in (4.2)–(4.3). Moreover if  $\tilde{v} := (\bar{u}, \bar{r})$  and  $\tilde{v}_n := (\bar{u}_n, \bar{r}_n)$ , then

$$\bar{u}_n \rightarrow \bar{u} \text{ uniformly on } [0, T], \quad (4.18)$$

$$\bar{r}_n \rightarrow \bar{r} \text{ uniformly on } [0, T]. \quad (4.19)$$

*Proof.* From the continuity of the functions involved and from (4.8), it follows that  $v, v_n \in BV([0, T]; \mathcal{B}) \cap C([0, T]; \mathcal{B})$  for every  $n \in \mathbb{N}$  and

$$v_n \rightarrow v \quad \text{strictly in } BV([0, T]; \mathcal{B}) \quad (4.20)$$

as  $n \rightarrow \infty$ . Moreover  $\bar{u}, \bar{u}_n \in Lip([0, T]; \mathcal{X})$ ,  $\bar{r}, \bar{r}_n \in Lip([0, T]; \mathcal{Y})$  and

$$u(t) = \bar{u}(\ell_v(t)), \quad r(t) = \bar{r}(\ell_v(t)), \quad (4.21)$$

$$u_n(t) = \bar{u}_n(\ell_{v_n}(t)), \quad r_n(t) = \bar{r}_n(\ell_{v_n}(t)) \quad (4.22)$$

for every  $t \in [0, T]$  and every  $n \in \mathbb{N}$ .

If  $s \in [0, T]$  and  $n \in \mathbb{N}$  we have that

$$\|\tilde{v}_n(s)\|_{\mathcal{B}} \leq \|\tilde{v}_n(0)\|_{\mathcal{B}} + V(\tilde{v}_n, [0, T]) = \|v_n(0)\|_{\mathcal{B}} + V(v_n, [0, T]),$$

therefore from (4.2), (4.20) and Lemma 5.4 of the Appendix we infer that

$$\|\tilde{v}_n\|_{L^\infty(0, T; \mathcal{B})} \leq C_1 \quad (4.23)$$

for some  $C_1 > 0$  independent of  $n \in \mathbb{N}$ . Moreover by (4.3) we have  $\|\tilde{v}_n'\|_\infty \leq V(v_n, [0, T])/T$  for every  $n \in \mathbb{N}$ , hence there exists  $C_2 > 0$  such that

$$\|\tilde{v}_n'\|_\infty \leq C_2 \quad (4.24)$$

for all  $n \in \mathbb{N}$ . It follows that  $\tilde{v}_n$  is bounded in  $W^{1,p}(0, T; \mathcal{B})$  for every  $p \in [1, \infty]$ . The reflexivity of  $L^p(0, T; \mathcal{B})$  for  $p \in ]1, \infty[$  (cf. [10, Theorem 8.20.5, p. 607]) and a standard Sobolev spaces argument imply that there exists  $\hat{v} \in W^{1,1}(0, T; \mathcal{B})$  such that, at least for a subsequence that we do not relabel,

$$\tilde{v}_n \rightharpoonup \hat{v} \quad \text{in } W^{1,p}(0, T; \mathcal{B}) \quad \forall p \in ]1, \infty[. \quad (4.25)$$

Now let us fix  $\sigma \in [0, T]$  and for every  $x^* \in \mathcal{B}^*$  let us consider the linear functional  $\phi_{x^*}^\sigma : W^{1,p}(0, T; \mathcal{B}) \longrightarrow \mathbb{R} : v \longmapsto \mathcal{B}^* \langle x^*, v(\sigma) \rangle_{\mathcal{B}}$ . Since  $W^{1,p}(0, T; \mathcal{B})$  is continuously embedded in

$C([0, T]; \mathcal{B})$  (cf. Corollary 5.2), we have that  $\phi_{x^*}^\sigma$  is also continuous, thus from (4.25) we infer that

$$\lim_{n \rightarrow \infty} \mathcal{B}^* \langle x^*, \tilde{v}_n(\sigma) \rangle_{\mathcal{B}} = \lim_{n \rightarrow \infty} \phi_{x^*}^\sigma(\tilde{v}_n) = \phi_{x^*}^\sigma(\hat{v}) = \mathcal{B}^* \langle x^*, \hat{v}(\sigma) \rangle_{\mathcal{B}},$$

i.e.

$$\tilde{v}_n(\sigma) \rightarrow \hat{v}(\sigma) \quad \text{in } \mathcal{B} \quad \forall \sigma \in [0, T] \quad (4.26)$$

as  $n \rightarrow \infty$ . Now for every  $x^* \in \mathcal{B}^*$  and every  $n \in \mathbb{N}$  let us define the functions  $f_n^{x^*} : [0, T] \rightarrow \mathbb{R}$  and  $f^{x^*} : [0, T] \rightarrow \mathbb{R}$  by

$$f_n^{x^*}(\sigma) := \mathcal{B}^* \langle x^*, \tilde{v}_n(\sigma) \rangle_{\mathcal{B}}, \quad f^{x^*}(\sigma) := \mathcal{B}^* \langle x^*, \hat{v}(\sigma) \rangle_{\mathcal{B}}, \quad \sigma \in [0, T]. \quad (4.27)$$

From the continuity of  $\tilde{v}_n$  and  $\hat{v}$  we infer that  $f_n^{x^*}$  and  $f^{x^*}$  are continuous, moreover from (4.29) it follows that  $f_n^{x^*} \rightarrow f^{x^*}$  pointwise in  $[0, T]$ . Moreover if  $\sigma, \tau \in [0, T]$  we have, thanks to (4.24), that

$$|f_n^{x^*}(\sigma) - f_n^{x^*}(\tau)| \leq \|x^*\| \|\tilde{v}_n(\sigma) - \tilde{v}_n(\tau)\| \leq \|x^*\| \|\tilde{v}_n'\|_{L^\infty(0, T; \mathcal{B})} |\sigma - \tau| \leq C \|x^*\| |\sigma - \tau|,$$

thus  $(f_n^{x^*})_n$  is equicontinuous and  $f_n^{x^*} \rightarrow f^{x^*}$  uniformly on  $[0, T]$  for every  $x^* \in \mathcal{B}^*$ . But  $\ell_{v_n}(t) \rightarrow \ell_v(t)$  pointwise on  $[0, T]$  by Lemma 5.2, hence  $f_n^{x^*}(\ell_{v_n}(t)) \rightarrow f^{x^*}(\ell_v(t))$  for every  $t \in [0, T]$ , i.e.

$$\tilde{v}_n(\ell_{v_n}(t)) \rightarrow \hat{v}(\ell_v(t)) \quad \text{in } \mathcal{B} \quad \forall t \in [0, T]. \quad (4.28)$$

On the other hand by the strict convergence of  $v_n$  and by Lemma 5.4 we have that

$$\lim_{n \rightarrow \infty} \tilde{v}_n(\ell_{v_n}(t)) = \lim_{n \rightarrow \infty} v_n(t) = v(t) = \tilde{v}(\ell_v(t)) \quad \forall t \in [0, T],$$

hence, as  $\ell_v$  is surjective, we get that  $\hat{v} = \tilde{v}$ . Hence from (4.25)–(4.26) we infer that

$$\tilde{v}_n(\sigma) \rightarrow \tilde{v}(\sigma) \quad \text{in } \mathcal{B} \quad \forall \sigma \in [0, T] \quad (4.29)$$

and

$$\tilde{v}_n \rightarrow \tilde{v} \quad \text{in } W^{1,p}(0, T; \mathcal{B}) \quad \forall p \in ]1, \infty[. \quad (4.30)$$

If  $\sigma \in [0, T]$  is fixed, then for every  $n \in \mathbb{N}$  there exists  $t_n \in [0, T]$  such that

$$\tilde{v}_n(\sigma) = \tilde{v}_n(\ell_{v_n}(t_n)) = v_n(t_n). \quad (4.31)$$

Passing to a subsequence, not relabeled, we have that  $t_n \rightarrow t_*$  for some  $t_* \in [0, T]$ . Hence, thanks to the uniform convergence of  $v_n$ ,  $v_n(t_n) \rightarrow v(t_*)$  as  $n \rightarrow \infty$ . It follows, as  $v(t_*) = \tilde{v}(\ell_v(t_*))$ , that

$$\tilde{v}_n(\sigma) \rightarrow \tilde{v}(\ell_v(t_*)) \quad (4.32)$$

as  $n \rightarrow \infty$ . From (4.29) we get that

$$\tilde{v}_n(\sigma) \rightarrow \tilde{v}(\sigma) \quad \text{in } \mathcal{B} \quad \forall \sigma \in [0, T] \quad (4.33)$$

and the whole sequence is converging by the uniqueness of the limit. Hence, taking into account (4.23), we can apply the dominated convergence theorem and infer that  $\tilde{v}_n \rightarrow \tilde{v}$  in  $L^1(0, T; \mathcal{B})$ . Since it is clear that  $V(\tilde{v}_n, [0, T]) \rightarrow V(\tilde{v}, [0, T])$ , we have that  $\tilde{v}_n \rightarrow \tilde{v}$  strictly on  $[0, T]$ . Therefore, by Proposition 5.1, we get that  $\tilde{v}_n \rightarrow \tilde{v}$  uniformly on  $[0, T]$  and (4.18)–(4.19) follow.  $\square$

**Lemma 4.1.** *Assume that  $(u, r, x^0), (u_n, r_n, x_n^0) \in \overline{D}$  for every  $n \in \mathbb{N}$ ,  $u_n \rightarrow u$ ,  $r_n \rightarrow r$  strictly on  $[0, T]$ , and  $x_n^0 \rightarrow x^0$  in  $\mathcal{X}$ , as  $n \rightarrow \infty$ . With the same notations of Proposition 4.1, we have that  $\mathcal{S}(\bar{u}_n, \bar{r}_n, x_n^0) \rightarrow \mathcal{S}(\bar{u}, \bar{r}, x^0)$  strictly on  $[0, T]$ .*

*Proof.* Let us set

$$\bar{\xi} := \mathcal{S}(\bar{u}, \bar{r}, x^0), \quad \bar{\xi}_n := \mathcal{S}(\bar{u}_n, \bar{r}_n, x_n^0)$$

and

$$\bar{x} := \bar{u} - \bar{\xi}, \quad \bar{x}_n := \bar{u}_n - \bar{\xi}_n$$

for every  $n \in \mathbb{N}$ . Observe that from (4.24) we get

$$\max\{\|\bar{u}'_n\|_\infty, \|\bar{r}'_n\|_\infty\} \leq \|\bar{v}'_n\|_\infty \leq C_2. \quad (4.34)$$

Since  $\bar{u}, \bar{r}, \bar{u}_n, \bar{r}_n$  are Lipschitz continuous, the following basic estimate holds (cf. [17, Theorem 4] or [5, Formulas (36)–(38), (46)]):

$$\begin{aligned} \|\bar{\xi}(t) - \bar{\xi}_n(t)\|_{\mathcal{X}} &\leq (\|x^0 - x_n^0\|_{\mathcal{X}} + \|\bar{u}(0) - \bar{u}_n(0)\|_{\mathcal{X}})^2 \\ &\quad + L_n \int_0^t (\|\bar{u}(s) - \bar{u}_n(s)\|_{\mathcal{X}} + C^3 K_0 \|\bar{r}(s) - \bar{r}_n(s)\|_{\mathcal{Y}}) \, ds \end{aligned} \quad (4.35)$$

where

$$L_n := 2(\|\bar{u}'\|_\infty + \|\bar{u}'_n\|_\infty + C^3 K_0 (\|\bar{r}'\|_\infty + \|\bar{r}'_n\|_\infty)). \quad (4.36)$$

The sequence  $L_n$  is bounded by virtue of (4.34), therefore from (4.18)–(4.19) and from (4.35)–(4.36) we infer that

$$\bar{\xi}_n \rightarrow \bar{\xi} \text{ uniformly on } [0, T], \quad (4.37)$$

which together with (4.18) yields

$$\bar{x}_n \rightarrow \bar{x} \text{ uniformly on } [0, T]. \quad (4.38)$$

Therefore from (3.12), (4.38), and (4.19) we infer that

$$J(\bar{x}_n(t), \bar{r}_n(t)) \rightarrow J(\bar{x}(t), \bar{r}(t)) \quad \forall t \in [0, T]. \quad (4.39)$$

as  $n \rightarrow \infty$ . If  $(v, \rho, z^0) \in D$ ,  $\eta := \mathcal{S}(v, \rho, z^0)$ , and  $y := v - \eta$ , then [5, Lemma 5.2] yields the following implication:

$$\eta'(t) \neq 0 \implies \begin{cases} y(t) \in \partial \mathcal{Z}(\rho(t)) \\ \|\eta'(t)\|_{\mathcal{X}} = \left\langle \eta'(t), \frac{J(y(t), \rho(t))}{\|J(y(t), \rho(t))\|_{\mathcal{X}}} \right\rangle \end{cases} \text{ for } \mathcal{L}^1\text{-a.e. } t \in [0, T]. \quad (4.40)$$

Let us define  $H : \mathcal{X} \times \mathcal{R} \rightarrow \mathcal{X}$  by

$$H(y, \rho) := \begin{cases} M(y, \rho) \frac{J\left(\frac{y}{M(y, \rho)}, \rho(t)\right)}{\left\|J\left(\frac{y}{M(y, \rho)}, \rho(t)\right)\right\|_{\mathcal{X}}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad (4.41)$$

The map  $H$  is well-defined thanks to (3.16) and to the fact that  $y/M(y, \rho) \in \partial \mathcal{Z}(\rho)$ , therefore we have that

$$\|\eta'(t)\|_{\mathcal{X}} = \langle \eta'(t), H(y(t), \rho(t)) \rangle \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0, T]. \quad (4.42)$$

Moreover, since  $M(x) \rightarrow 0$  as  $x \rightarrow 0$ , from (3.5) and Assumption 3.1 we infer that  $H$  is continuous, hence  $H(\bar{x}_n(t), \bar{r}_n(t)) \rightarrow H(\bar{x}(t), \bar{r}(t))$  for every  $t \in [0, T]$ . On the other hand the sequence  $H(\bar{x}_n(\cdot), \bar{r}_n(\cdot))$  is uniformly bounded, thus by the dominated convergence theorem

$$H(\bar{x}_n(\cdot), \bar{r}_n(\cdot)) \rightarrow H(\bar{x}(\cdot), \bar{r}(\cdot)) \quad \text{in } L^q(0, T; \mathcal{X}) \quad \forall q \in ]1, \infty[. \quad (4.43)$$

Observe that (cf. [5, Formula 50])

$$\|\bar{\xi}'_n(t)\|_{\mathcal{X}} \leq \|\bar{u}'_n(t)\|_{\mathcal{X}} + C K_0 \|\bar{r}'_n(t)\|_{\mathcal{Y}} \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0, T], \quad (4.44)$$

hence, thanks to (4.34),  $\bar{\xi}_n$  is bounded in  $W^{1,p}(0, T; \mathcal{X})$  for every  $p \in ]1, \infty[$ , and (4.37) implies that

$$\bar{\xi}_n \rightharpoonup \bar{\xi} \quad \text{in } W^{1,p}(0, T; \mathcal{X}) \text{ for every } p \in ]1, \infty[. \quad (4.45)$$

Thus from (4.42), (4.43), and (4.45) we get that

$$\begin{aligned} \lim_{n \rightarrow \infty} V(\bar{\xi}_n, [0, T]) &= \lim_{n \rightarrow \infty} \int_0^T \|\bar{\xi}'_n(t)\|_{\mathcal{X}} dt = \lim_{n \rightarrow \infty} \int_0^T \langle \bar{\xi}'_n(t), H(\bar{x}_n(t), \bar{r}_n(t)) \rangle dt \\ &= \int_0^T \langle \bar{\xi}'(t), H(\bar{x}(t), \bar{r}(t)) \rangle dt = \int_0^T \|\bar{\xi}'(t)\|_{\mathcal{X}} dt = V(\bar{\xi}, [0, T]), \end{aligned} \quad (4.46)$$

which together with (4.37) proves the lemma.  $\square$

*Proof of Theorem 3.2.* We are left to prove the continuity property. to this aim let  $(u, r, x^0)$ ,  $(u_n, r_n, x_n^0) \in \bar{D}$  be such that  $u_n \rightarrow u$ ,  $r_n \rightarrow r$  strictly on  $[0, T]$  and  $x_n^0 \rightarrow x^0$  in  $\mathcal{X}$ . If  $v = (u, r)$  and  $v_n = (u_n, r_n)$  then by Lemma 5.2 we have that

$$\ell_{v_n}(t) \rightarrow \ell_v(t) \quad \forall t \in [0, T]. \quad (4.47)$$

Observe that by Theorem 4.1 we have

$$S(u, r, x^0)(t) = S(\bar{u} \circ \ell_v, \bar{r} \circ \ell_v, x^0)(t) = S(\bar{u}, \bar{r}, x^0)(\ell_v(t)), \quad (4.48)$$

$$S(u_n, r_n, x_n^0)(t) = S(\bar{u}_n \circ \ell_{v_n}, \bar{r}_n \circ \ell_{v_n}, x_n^0)(t) = S(\bar{u}_n, \bar{r}_n, x_n^0)(\ell_{v_n}(t)). \quad (4.49)$$

Moreover from Lemma 4.1 we get that

$$S(\bar{u}_n, \bar{r}_n, x_n^0) \rightarrow S(\bar{u}, \bar{r}, x^0) \quad \text{strictly on } [0, T], \quad (4.50)$$

in particular  $S(\bar{u}_n, \bar{r}_n, x_n^0) \rightarrow S(\bar{u}, \bar{r}, x^0)$  uniformly on  $[0, T]$  thanks to Proposition 5.1, therefore from (4.47) we get

$$\begin{aligned} \lim_{n \rightarrow \infty} S(u_n, r_n, x_n^0)(t) &= \lim_{n \rightarrow \infty} S(\bar{u}_n \circ \ell_{v_n}, \bar{r}_n \circ \ell_{v_n}, x_n^0)(t) \\ &= \lim_{n \rightarrow \infty} S(\bar{u}_n, \bar{r}_n, x_n^0)(\ell_{v_n}(t)) \\ &= S(\bar{u}, \bar{r}, x^0)(\ell_v(t)) \\ &= S(\bar{u} \circ \ell_v, \bar{r} \circ \ell_v, x^0)(t) \\ &= S(u, r, x^0)(t) \end{aligned} \quad (4.51)$$

Now  $\|S(u_n, r_n, x_n^0)\|_{\infty} = \|S(\bar{u}_n, \bar{r}_n, x_n^0) \circ \ell_{v_n}\|_{\infty} = \|S(\bar{u}_n, \bar{r}_n, x_n^0)\|_{\infty}$ , thus  $S(u_n, r_n, x_n^0)$  is uniformly bounded because of the strict convergence of  $S(\bar{u}_n, \bar{r}_n, x_n^0)$ , and by the dominated convergence theorem we infer that

$$S(u_n, r_n, x_n^0) \rightarrow S(u, r, x^0) \quad \text{in } L^1(0, T; \mathcal{X}). \quad (4.52)$$

Finally we have to prove the convergence of the variations. From (4.48)–(4.49) and from the continuity of  $\ell_v$  we have that

$$V(S(u_n, r_n, x_n^0), [0, T]) = V(S(\bar{u}_n, \bar{r}_n, x_n^0), [0, T]), \quad (4.53)$$

$$V(S(u, r, x^0), [0, T]) = V(S(\bar{u}, \bar{r}, x^0), [0, T]), \quad (4.54)$$

moreover (4.50) yields

$$\lim_{n \rightarrow \infty} V(S(\bar{u}_n, \bar{r}_n, x_n^0), [0, T]) = V(S(\bar{u}, \bar{r}, x_0), [0, T])$$

and the theorem is completely proved.  $\square$

**Remark 4.2.** As we mentioned in the Introduction, when  $\mathcal{Z}(r(t)) = \mathcal{Z}$ , a fixed closed convex subset of the Hilbert space  $\mathcal{X}$ , the solution operator  $\mathbf{S}$  is actually acting on  $BV([0, T]; \mathcal{X}) \cap C([0, T]; \mathcal{X})$  only, and its strict continuity was deduced in [26] by applying the general implication

$$\mathbf{R} \text{ } d_{BV}\text{-continuous} \implies \mathbf{R} \text{ } d_s\text{-continuous}, \quad (4.55)$$

holding for a rate independent operator  $\mathbf{R} : BV([0, T]; \mathcal{X}) \cap C([0, T]; \mathcal{X}) \longrightarrow BV([0, T]; \mathcal{X}) \cap C([0, T]; \mathcal{X})$ . Property (4.55) (proved in [26, Theorem 3.4]) cannot be applied in our new framework where  $\mathcal{B} = \mathcal{X} \times \mathcal{Y}$  replaces  $\mathcal{X}$  in the domain of  $\mathbf{R}$ , because the norm (4.6) is not uniformly convex, and property (4.55) does not hold in the non-uniformly convex case, even if  $\mathcal{B}$  is reflexive. Let us show this fact with a counterexample by considering the space  $\mathcal{B}_1 = \mathbb{R}^2$  endowed with the 1-norm  $\|(x, y)\|_{\mathcal{B}_1} := |x| + |y|$ ,  $(x, y) \in \mathbb{R}^2$ . Notice that  $\mathcal{B}_1$  is reflexive but is not uniformly convex. By  $\mathcal{B}_2$  we denote the space  $\mathbb{R}^2$  endowed with the euclidean norm  $\|(x, y)\|_{\mathcal{B}_2} := (|x|^2 + |y|^2)^{1/2}$ ,  $(x, y) \in \mathbb{R}^2$ . If an interval  $J \subseteq [0, T]$  and  $v : [0, T] \longrightarrow \mathbb{R}^2$  are given, for  $k = 1, 2$  we denote by  $V_k(u, J)$  the variation of  $u$  on  $J$  with respect to the norm  $\|\cdot\|_{\mathcal{B}_k}$ , and we also set  $V_k(u)(t) := V_k(u, [0, t])$ ,  $t \in [0, T]$ . Accordingly we denote by  $d_{BV}^k$  and by  $d_s^k$  the distances defined in (2.9) and in (2.10) with  $\mathcal{B} = \mathcal{B}_k$ ,  $k = 1, 2$ , while  $d_{BV}$  will be used for the case  $\mathcal{B} = \mathbb{R}$ . Observe that the metrics  $d_{BV}^1$  and  $d_{BV}^2$  are equivalent, hence they generate the same topology. Let us define  $\mathbf{R} : BV([0, T]; \mathcal{B}_1) \cap C([0, T]; \mathcal{B}_1) \longrightarrow BV([0, T]; \mathcal{B}_1) \cap C([0, T]; \mathcal{B}_1)$  by

$$\mathbf{R}(u)(t) := (V_1(u)(t), V_2(u)(t)), \quad u \in BV([0, T]; \mathcal{B}_1) \cap C([0, T]; \mathcal{B}_1)$$

(we could take  $V_2(u)$  in both components, but we prefer to keep them distinct). Clearly  $\mathbf{R}$  is rate independent. In order to prove that it is  $d_{BV}^1$ -continuous, assume that  $d_{BV}^1(u_n, u) \rightarrow 0$ , thus  $d_{BV}^2(u_n, u) \rightarrow 0$  as well. Since  $V_k(u_n)$  and  $V_k(u)$  are increasing functions, a straightforward computation shows that

$$V(V_k(v) - V_k(w), J) = V_k(u - w, J) \quad (4.56)$$

for every  $v, w$ , and  $J$ , therefore, using also the inequality  $|V_k(v, J) - V_k(w, J)| \leq V_k(v - w, J)$ , we have

$$\begin{aligned} d_{BV}(V_k(u_n), V_k(u)) &= \|V_k(u_n) - V_k(u)\|_{L^1(0, T; \mathbb{R})} + |V(V_k(u_n) - V_k(u), [0, T])| \\ &= \int_0^T |V_k(u_n)(t) - V_k(u)(t)| dt + V_k(u_n - u, [0, T]) \\ &\leq \int_0^T V_k(u_n - u, [0, t]) dt + V_k(u_n - u, [0, T]) \\ &\leq (T + 1) V_k(u_n - u, [0, T]), \end{aligned}$$

hence  $d_{BV}(V_k(u_n), V_k(u)) \rightarrow 0$  as  $n \rightarrow \infty$  for  $k = 1, 2$ , and this implies that  $d_{BV}^1(\mathbf{R}(u_n), \mathbf{R}(u)) \rightarrow 0$  as  $n \rightarrow \infty$ , and  $\mathbf{R}$  is  $d_{BV}^1$ -continuous. Now we show that  $\mathbf{R}$  is *not*  $d_s^1$ -continuous. To this aim we consider a sequence of Lipschitz curves  $u_n$  whose trace is a kind of “staircase with  $n$  steps” laid upon the line  $y = x$ , going from the origin to the point  $(1, 1)$ . More precisely, for every  $n \in \mathbb{N}$  we split  $[0, 1]$  into  $n$  subintervals  $[(j-1)/2^{n-1}, j/2^{n-1}]$ ,  $j = 1, \dots, 2^{n-1}$ , and let  $u_n : [0, 1] \longrightarrow \mathbb{R}^2$  be the unique Lipschitz curve such that

$$u_n(t) = \begin{cases} ((j-1)/2^{n-1}, g_n(t)) & \text{if } t \in [(j-1)/2^{n-1}, (2j-1)/2^n] \\ (h_n(t), j/2^{n-1}) & \text{if } t \in [(2j-1)/2^n, j/2^{n-1}] \end{cases}, \quad j = 1, \dots, 2^{n-1},$$

where  $g_n : [(j-1)/2^{n-1}, (2j-1)/2^n] \longrightarrow [(j-1)/2^{n-1}, j/2^{n-1}]$  and  $h_n : [(2j-1)/2^n, j/2^{n-1}] \longrightarrow [(j-1)/2^{n-1}, j/2^{n-1}]$  are affine increasing surjective functions. If  $u : [0, 1] \longrightarrow \mathbb{R}^2$  is defined by  $u(t) := (t, t)$ , then we have  $\|u_n - u\|_{L^1(0, 1; \mathcal{B}_k)} \rightarrow 0$  for  $k = 1, 2$ . Since  $V_1(u_n, [0, 1]) = V_2(u_n, [0, 1]) = 2$  for every  $n \in \mathbb{N}$ , by (4.56) we have that  $V_1(\mathbf{R}(u_n), [0, 1]) = V(V_1(u_n), [0, 1]) +$

$V(V_2(u_n), [0, 1]) = 2 + 2 = 4$ . On the other hand  $V_1(u, [0, 1]) = 2$  and  $V_2(u, [0, 1]) = \sqrt{2}$ , therefore  $V_1(R(u), [0, 1]) = V(V_1(u), [0, 1]) + V(V_2(u), [0, 1]) = 2 + \sqrt{2}$ , hence  $R(u_n)$  is not  $d_s^1$ -convergent and  $R$  is not  $d_s^1$ -continuous.

**Remark 4.3.** In the simpler case  $\mathcal{Z}(r(t)) = \mathcal{Z}$  the strict continuity of the solution operator  $S$  was deduced in [26] without any smoothness assumption on  $\mathcal{Z}$ . Therefore it seems natural to wonder if Assumption 3.1 can be relaxed in the present framework. This question is still open.

## 5. APPENDIX

In this section we show some properties about the strict convergence in  $BV([0, T]; \mathcal{B})$ .

**Lemma 5.1.** *Assume that  $v_n, v \in BV([0, T]; \mathcal{B}) \cap C([0, T]; \mathcal{B})$  and let  $J \subseteq [0, T]$  be an interval. If  $v_n(t) \rightarrow v(t)$  for a.e.  $t \in J$ , then  $V(v, J) \leq \liminf_{n \rightarrow \infty} V(v_n, J)$ .*

*Proof.* Let  $0 = s_0 < \dots < s_m = T$  be such that

$$V(v, J) < \varepsilon/2 + \sum_{j=0}^m \|v(s_j) - v(s_{j-1})\|_{\mathcal{B}}.$$

The set  $E := \{t \in [0, T] : v_n(t) \rightarrow v(t) \text{ as } n \rightarrow \infty\}$  has full measure in  $[0, T]$ , therefore we can find points  $t_j \in E$ ,  $j = 1, \dots, m$  such that  $0 < t_1 < \dots < t_m = T$  and  $\|v(t_j) - v(s_j)\|_{\mathcal{B}} < m\varepsilon/4$  for  $j = 1, \dots, m$ , and we have

$$\begin{aligned} V(u, [0, T]) &< \varepsilon/2 + \sum_{j=0}^m \|v(s_j) - v(s_{j-1})\|_{\mathcal{B}} \\ &\leq \varepsilon/2 + \sum_{j=0}^m (\|v(s_j) - v(t_j)\|_{\mathcal{B}} + \|v(t_j) - v(t_{j-1})\|_{\mathcal{B}} + \|v(t_{j-1}) - v(s_{j-1})\|_{\mathcal{B}}) \\ &< \varepsilon + \sum_{j=0}^m \|v(t_j) - v(t_{j-1})\|_{\mathcal{B}}. \end{aligned}$$

For every  $n \in \mathbb{N}$  we have

$$V(v_n, [0, T]) \geq \sum_{j=0}^m \|v_n(t_j) - v_n(t_{j-1})\|_{\mathcal{B}}, \quad (5.1)$$

therefore taking the lower limit we get

$$\begin{aligned} \liminf_{n \rightarrow \infty} V(v_n, [0, T]) &\geq \liminf_{n \rightarrow \infty} \sum_{j=0}^m \|v_n(t_j) - v_n(t_{j-1})\|_{\mathcal{B}} \\ &\geq \sum_{j=0}^m \liminf_{n \rightarrow \infty} \|v_n(t_j) - v_n(t_{j-1})\|_{\mathcal{B}} \\ &= \sum_{j=0}^m \|v(t_j) - v(t_{j-1})\|_{\mathcal{B}} > V(v, [0, T]) - \varepsilon. \end{aligned}$$

and the statement follows from the arbitrariness of  $\varepsilon$ .  $\square$

**Corollary 5.1.** *Let  $v, v_n \in BV([0, T]; \mathcal{B}) \cap C([0, T]; \mathcal{B})$  be such that  $v_n \rightarrow v$  strictly on  $[0, T]$  as  $n \rightarrow \infty$ . Let  $J \subseteq [0, T]$  be an interval. Then*

$$V(v, J) \leq \liminf_{n \rightarrow \infty} V(v_n, J).$$

*Proof.* Let  $(n_k)_k$  be a sequence of positive integers such that  $n_k \rightarrow \infty$  and  $V(v_{n_k}, I) \rightarrow \ell$  as  $k \rightarrow \infty$  for some  $\ell \geq 0$ . By the strict convergence it follows that there is a further subsequence  $n_{k_h}$  such that  $v_{n_{k_h}} \rightarrow u$  almost everywhere. Hence by Lemma 5.1  $V(v, J) \leq \ell$  and we are done.  $\square$

**Lemma 5.2.** *Assume that  $v_n, v \in BV([0, T]; \mathcal{B}) \cap C([0, T]; \mathcal{B})$  for every  $n \in \mathbb{N}$ . If  $v_n \rightarrow v$  strictly on  $[0, T]$  as  $n \rightarrow \infty$ , then  $V(v_n, [s, t]) \rightarrow V(v, [s, t])$  for every  $s, t \in [0, T]$ ,  $s < t$ .*

*Proof.* Thanks to Corollary 5.1 we have that

$$V(v, [s, t]) = \liminf_{n \rightarrow \infty} V(v_n, [s, t]).$$

On the other hand, using again Corollary 5.1 and the strict convergence, we infer that

$$\begin{aligned} \limsup_{n \rightarrow \infty} V(v_n, [s, t]) &= \limsup_{n \rightarrow \infty} (V(v_n, [0, T]) - V(v_n, [0, s]) - V(v_n, [t, T])) \\ &\leq V(v, [0, T]) - V(v, [0, s]) - V(v, [t, T]) = V(v, [s, t]). \end{aligned}$$

$\square$

**Lemma 5.3.** *Assume that  $v_n, v \in BV([0, T]; \mathcal{B}) \cap C([0, T]; \mathcal{B})$  for every  $n \in \mathbb{N}$ . If  $v_n \rightarrow v$  strictly as  $n \rightarrow \infty$ , then for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $c, d \in [0, T]$  we have*

$$0 < d - c < \delta \implies \sup_{n \in \mathbb{N}} V(v_n, [c, d]) < \varepsilon. \quad (5.2)$$

*Proof.* Thanks to Lemma 5.2, the sequence of real functions  $V_n : [0, T] \rightarrow \mathbb{R} : t \mapsto V(v_n, [0, t])$  is pointwise converging to the continuous function  $V : [0, T] \rightarrow \mathbb{R} : t \mapsto V(v, [0, t])$ . Moreover  $V_n$  is a monotone function for every  $n \in \mathbb{N}$ , therefore from the Polya Lemma (cf. [9, Theorem 10, p. 166]) we deduce that  $V_n \rightarrow V$  uniformly on  $[0, T]$ , hence for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $\sup_{n \in \mathbb{N}} |V(d) - V(c)| < \varepsilon$  whenever  $0 < d - c < \delta$ ,  $c, d \in [0, T]$ . This is what we wanted to prove.  $\square$

**Lemma 5.4.** *Assume that  $v_n, v \in BV([0, T]; \mathcal{B}) \cap C([0, T]; \mathcal{B})$  for every  $n \in \mathbb{N}$ . If  $v_n \rightarrow v$  strictly as  $n \rightarrow \infty$ , then  $v_n(t) \rightarrow v(t)$  as  $n \rightarrow \infty$  for every  $t \in [0, T]$ .*

*Proof.* If  $t \in [0, T]$  is fixed and a subsequence  $v_{n'_k}(t)$  of  $v_{n_k}(t)$  is given, we can extract a further subsequence  $(n''_k)_k$  such that  $v_{n''_k} \rightarrow v$  a.e. in  $[0, T]$ . If  $\varepsilon > 0$  there exists  $\delta > 0$  such that (5.2) holds. We can find a point  $t_0$  such that  $0 \leq t - t_0 < \delta$  and  $v_{n''_k}(t_0) \rightarrow v(t_0)$ . Hence we get

$$\begin{aligned} \|v_{n''_k}(t), v(t)\|_{\mathcal{B}} &\leq \|v_{n''_k}(t_0) - v(t_0)\|_{\mathcal{B}} + \|v_{n''_k}(t) - v_{n''_k}(t_0)\|_{\mathcal{B}} + \|v_{n''_k}(t) - v(t)\|_{\mathcal{B}} \\ &\leq \|v_{n''_k}(t_0) - v(t_0)\|_{\mathcal{B}} + V(v_{n''_k}, [t_0, t]) + V(v, [t_0, t]) \leq 3\varepsilon, \end{aligned}$$

provided  $k$  is large enough. The thesis follows.  $\square$

**Proposition 5.1.** *Assume  $v, v_n \in BV([0, T]; \mathcal{B}) \cap C([0, T]; \mathcal{B})$  and  $v_n \rightarrow v$  strictly as  $n \rightarrow \infty$ . Then  $v_n \rightarrow v$  uniformly on  $[0, T]$ .*

*Proof.* It is enough to apply the Ascoli theorem for  $\mathcal{B}$  valued functions (cf. [18, Theorem 3.1, p. 57]). The pointwise convergence of  $v_n$  is proved in Lemma 5.4, the equicontinuity follows immediately from Lemma 5.3.  $\square$

Notice that as a consequence of Proposition 5.1 we can also obtain the following

**Corollary 5.2.**  *$W^{1,1}([0, T]; \mathcal{B})$  is continuously embedded in  $C([0, T]; \mathcal{B})$ .*

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