A Theoretical Approach to Memristor Devices

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***Abstract*—Recently, scientific communities in electrical engineering, material science, biophysics and nano–technologies have paid special attention to memristor devices. Several physical and mathematical memristor models have been proposed to describe devices developed for non–volatile memory applications and neuromorphic systems.**

The aim of the paper is to provide a theoretical approach to the various classes of memristive devices as nonlinear dynamical systems whose voltage–current curves (i.e. dynamic characteristics) are pinched at the origin when driven by bipolar excitations. Off-origin dynamic characteristics are discussed and mathematical criteria to model such devices are provided as well. Finally, passivity and losslessness properties of memristor are briefly analized.

***Index Terms*—memristor, nonlinear circuit theory, nano–technology, nano–devices**

I. Introduction

THE *MEMRISTOR* (contraction of *memory resistor*) was originally introduced in 1971 [1] as the fourth ideal circuit element, in addition to the resistor, the capacitor and the inductor.

Its original definition was given by a relation of the form $F(\varphi, q) = 0$ where φ and q are the time integrals in $(-\infty, t]$ of the port voltage and current respectively. If this relation is expressible as a map $q \to \varphi$ ($\varphi \to q$), the memristor is said to be *charge (flux) controlled*. By assuming that either map is differentiable, we can express the voltage in terms of the current (or vice versa) in the form $v = R(q)i$ ($i = G(\varphi)v$), where $R(q) = \mathrm{d}\varphi/\mathrm{d}q$ and $G(\varphi) = \mathrm{d}q/\mathrm{d}\varphi$. The quantity $R(q)$ ($G(\varphi)$), that depends on the entire time history of the input variable, is the *memristance* (*memductance*) of the memristor. The memristor is passive if $R(q) \geq 0$, $\forall q$ ($G(\varphi) \geq 0$, $\forall \varphi$), i.e. if the map $q \to \varphi$ ($\varphi \to q$) is monotone increasing.

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The concept of memristor was successively extended to that of *memristor system* in 1976 [2]. The extension preserves all the above properties, but the state of the device is defined not only by the time integral of the input quantity (current or voltage), but also by a set of internal not necessarily electrical variables.

The fingerprint, that distinguishes a passive memristor within the set of all nonlinear dynamical systems, is that:

- any zero–mean periodic input (voltage or current) yields a current–voltage loop confined to the first and the third quadrants of the i–v plane and above all the loop is *pinched at the origin* (additional intersections may occur as well). This property is preserved even in presence of more general inputs (i.e. bipolar periodic input)
- the loop has a shape varying with both the amplitude and frequency.

Remarkably, the memristance may depend on a set of internal state variables. No further conditions are imposed, in particular the property of being a memristor is not bound to the choice of particular materials or to specific physical mechanisms. As clearly reported in [4], "*all 2-terminal non-volatile memory devices based on resistance switching are memristors, regardless of the device material and physical operating mechanisms*" (e.g. examples of voltage vs. current pinched loops are observed in many unrelated fields, such as biology, chemistry, physics, etc., and from many unrelated phenomena, such as gas discharge arcs, mercury lamps, power conversion devices, solid–state and/or nano devices).

An impressively huge number of works followed the breakthrough discovery of memristive behavior in a two-terminal nanoscale device based on a thin oxide film ending at each side with a metallic contact made up of platinum (Pt)[3]. The oxide film is composed of two layers, the upper more conductive with oxygen-deficient titanium dioxide (TiO_{2-x}, $x = 0.05$) and the lower more insulating with stoichiometric titanium dioxide (TiO_2).

Recently, pinched hysteresis loops recorded from the experiments in redox–based resistive switches have revealed a small offset of the pinched point from the origin due to inherent emf voltages [11]. These experimental observations can be easily modeled including constant voltage E and/or current A sources in the original memristor definition with a pinched loop at the origin (e.g. just consider $v - E = R(q)(i - A)$). As reported in [5], "*Pinched hysteresis loops are the hallmarks of all memristors, ideal or otherwise*".

It appears suitable by now to present the entire subject in a systematic form endowed with sufficient flexibility to incorporate the particular aspects of the physical realizations that exhibit memristor properties. This paper is devoted to such an aim. Without losing any generality we mainly present the theory of memristor exhibiting the pinched hysteresis loop at the origin. The present theoretical approach can be extended, *mutatis mutandis*, to memcapacitor and meminductor. This systematic presentation can help memristor researchers to identify the proper memristor model describing the device subjected to different input excitations.

II. The class of memristor devices

The main contribution of this section is the systematic description of memristor in terms of current and voltage momenta. As a consequence, the classification in terms of voltage and current provided in [5] can be easily derived.

Let us consider a two-terminal device described by a current $i(t)$ and a voltage $v(t)$ at its terminals. Let us introduce the following electrical port variables:

- the *current momentum* $q(t)$, such that

$$q(t) = \int_{-\infty}^{t} i(\tau) d\tau$$

- the *voltage momentum* $\varphi(t)$, such that

$$\varphi(t) = \int_{-\infty}^{t} v(\tau) d\tau$$

It must be understood that the current momentum does coincide with a stored electric charge only in the case of a capacitor, the voltage momentum with a magnetic flux only in the case of an inductor. Since the memristor has nothing to do with such circuit elements, the use of the above nomenclature is suitable to avoid misunderstandings, although in the following the terms *flux* and *charge* may be occasionally used for the sake of brevity.

Moreover we introduce an *internal vector with n state variables* $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbf{R}^n$. The internal vector contains state variables different from the port variables (i.e. input voltage or current momentum or voltage and current) and occasionally non–state variables depending on them, which are introduced for the sake of convenience; the dynamics of all these variables, whose nature can be in general nonelectrical, is governed by a nonlinear Ordinary Differential Equation (ODE)

$$\dot{\mathbf{x}} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{g}(u, \dot{u}, \mathbf{x})$$

where the input u can be $u = q$ in current–controlled memristors or $u = \varphi$ in voltage–controlled memristors and $\mathbf{g}(\cdot, \cdot, \cdot) : \mathbf{R}^{n+2} \Rightarrow \mathbf{R}^n$.

A. Description in terms of voltage and current momenta

Without losing any generality, let us assume that the memristor is *current controlled*, i.e. that the input is the current, the output is the voltage; in the opposite case duality applies. Explicit time dependency of variables is reported only if necessary.

By adopting the terminology introduced in [5], the following definition comprises all those given in [1], [2], [5]

Definition 1: A *current–controlled extended memristor* is a two-terminal circuit element defined in terms of the current momentum $q(t)$ and the voltage momentum $\varphi(t)$ by the following equations

$$\begin{aligned} &F(\varphi, q, i, \mathbf{x}) = 0 \\ &\dot{\mathbf{x}} = \mathbf{g}(q, i, \mathbf{x}) \\ &\dot{q} = i \end{aligned} \qquad (1)$$

with the following v–i constraint:

$$i = 0 \Rightarrow v = 0, \ \forall \mathbf{x} \qquad (2)$$

Function F is assumed to possess first derivatives with respect to all its arguments.

The mathematical relationship $F(\varphi, q, i, \mathbf{x}) = 0$ is named the *static characteristic equation* and the collection of all points $(\varphi, q, i, \mathbf{x}) \in \mathbf{R}^{(n+3)}$ satisfying the static characteristic equation is the *static characteristic* $\bar{\mathcal{F}}$, i.e.

$$\bar{\mathcal{F}} = \left\{ (\varphi, q, i, \mathbf{x}) \in \mathbf{R}^{(n+3)} : F(\varphi, q, i, \mathbf{x}) = 0 \right\}$$

It turns out from Definition 1 that, a *current–controlled extended memristor* is explicitly defined in terms of the current and voltage momenta as follows (under the assumption that there exists a function $\varphi(\cdot, \cdot, \cdot)$).

Definition 2: A *current–controlled extended memristor*[1] is defined in terms of the voltage momentum φ and the current momentum q

[1]By swapping the current momentum and the current with the voltage momentum and voltage, a similar definition explicitly specifies the voltage–controlled extended memristor.

- the static characteristic equation

$$\varphi = f(q, i, \mathbf{x}) \tag{3}$$

- the ODEs

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{g}(q, i, \mathbf{x}) \\ \dot{q} &= i \end{aligned} \tag{4}$$

- the constraint on the pinched loop at the origin

$$i = 0 \Rightarrow v = 0, \ \forall \mathbf{x}.$$

The static characteristic turns out to be

$$\mathcal{F} = \left\{ (\varphi, q, i, \mathbf{x}) \in \mathbf{R}^{(n+3)} : \varphi = f(q, i, \mathbf{x}) \right\}.$$

In the following the current–controlled extended memristor is just named as *extended memristor* for the sake of brevity.

B. Description in terms of voltage and current

This section provides conditions on the static characteristic equation $f : \mathbf{R}^{n+2} \to \mathbf{R}$ (see Definition 2) and the nonlinear dynamics of the internal variables $\mathbf{x}$, such that the memristor (described in terms of the port voltage v and current i) exhibits a pinched i–v curve at the origin.

Theorem 1: An *extended memristor* is described in terms of the voltage v and the current i by the following set of differential and algebraic or transcendental nonlinear equations:

$$\begin{aligned} v &= R(q, i, \mathbf{x}) i \\ \dot{\mathbf{x}} &= \mathbf{g}(q, i, \mathbf{x}) \\ \dot{q} &= i \end{aligned} \tag{5}$$

if and only if $\forall \mathbf{x}$

$$\frac{\partial f}{\partial i} \dot{i} + \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \dot{x}_k = L(q, i, \mathbf{x}) \dot{i} + \mathbf{J}(q, i, \mathbf{x}) \cdot \dot{\mathbf{x}} = 0 \tag{6}$$

where

$$\begin{aligned} R(q, i, \mathbf{x}) &= \frac{\partial f(q, i, \mathbf{x})}{\partial q} \\ L(q, i, \mathbf{x}) &= \frac{\partial f(q, i, \mathbf{x})}{\partial i} \\ \mathbf{J}(q, i, \mathbf{x}) &= \left(\frac{\partial f(q, i, \mathbf{x})}{\partial x_1}, \ldots, \frac{\partial f(q, i, \mathbf{x})}{\partial x_n} \right) \\ \mathbf{J}_f(q, i, \mathbf{x}) &= (R(q, i, \mathbf{x}),\ L(q, i, \mathbf{x}),\ \mathbf{J}(q, i, \mathbf{x})) \end{aligned}$$

In equations (7) $\mathbf{J}_f$ is the Jacobian of the static characteristic equation $\varphi = f(q, i, \mathbf{x})$, $\mathbf{J}$ the part of $\mathbf{J}_f$ with only the partial derivatives respect to $x_1, \ldots, x_n$, $R(q, i, \mathbf{x})$ is the *memristance*[2] and $\mathbf{g}(q, i, \mathbf{x})$s the evolution of the internal variables vector $\mathbf{x}$ driven by the current i.[3] All the partial derivatives are assumed to be *bounded* over their domains of definition.

Proof: From Definition 2, it follows that, the time derivative of both sides of the static characteristic equation (3) gives:

$$v = R(q, i, \mathbf{x}) i + L(q, i, \mathbf{x}) \dot{i} + \mathbf{J}(q, i, \mathbf{x}) \cdot \dot{\mathbf{x}}$$

Due to the boundedness of $R(q, i, \mathbf{x})$, $L(q, i, \mathbf{x})$ and $\mathbf{J}(q, i, \mathbf{x})$, the v–i constraint in Definition 2 is fulfilled if and only if (6) holds. ■

Remark 1: The extended memristor defined in Definition 2 includes that given in Table 1 of [5] and the original definition of *memristive systems* given in [2] under the assumptions that the condition (6) holds and

$$\begin{aligned} \varphi &= q \hat{f}(i, \mathbf{x}) && (7) \\ \dot{\mathbf{x}} &= \mathbf{g}(i, \mathbf{x}) && (8) \\ \dot{q} &= i && (9) \end{aligned}$$

In such a case the extended memristor in terms of the voltage v and the current i results to be a memristive systems, i.e.

$$\begin{aligned} v &= \hat{f}(i, \mathbf{x}) i = R(i, \mathbf{x}) i && (10) \\ \dot{\mathbf{x}} &= \mathbf{g}(i, \mathbf{x}) && (11) \\ \dot{q} &= i && (12) \end{aligned}$$

This special class of extended memristors can be easily implemented by means of circuits made of passive components (see for instance the circuit proposed by F. Corinto and A. Ascoli in [6]).

In the following, we assume that (6) *is always satisfied*. In particular, $L(q, i, \mathbf{x}) \dot{i} + \mathbf{J}(q, i, \mathbf{x}) \cdot \dot{\mathbf{x}} = L(q, i, \mathbf{x}) \dot{i} + \mathbf{J}(q, i, \mathbf{x}) \cdot \mathbf{g}(q, i, \mathbf{x}) = 0$, is satisfied if:

- $f(q, i, \mathbf{x})$ is flat along the "coordinate" i, i.e. $L(q, i, \mathbf{x}) = \frac{\partial f(q, i, \mathbf{x})}{\partial i} = 0$ and the function $\mathbf{g}(q, i, \mathbf{x})$ is such that, *for all* q, i *and* $\mathbf{x}$, the vector tangent to the trajectory of $\mathbf{x}(t)$ is orthogonal to the Jacobian of the function $f(q, i, \mathbf{x})$.
- $L(q, i, \mathbf{x}) \neq 0$ and $\mathbf{g}(q, i, \mathbf{x})$ is such that, *for all* q, i *and* $\mathbf{x}$, the scalar product $\mathbf{J}(q, i, \mathbf{x}) \cdot \mathbf{g}(q, i, \mathbf{x}) = -L(q, i, \mathbf{x}) \dot{i}$

In other words, the condition (6) provides the form of the state equation (i.e. the form of $\mathbf{g}(\cdot, \cdot, \cdot)$) governing

[2] The memristance is defined under the conditions providing the existence of the Jacobian $\mathbf{J}_f$.

[3] Additional constraints on $\mathbf{g}(q, i, \mathbf{x})$ depend on the properties of memristor (e.g. shrinking v–i curves, non–volatile/volatile properties, etc.).

the evolution of the vector $\mathbf{x}$. For example, if $\mathbf{x} = x \in \mathbf{R}$ and $L(q, i, x) \neq 0$, the function

$$g(q, i, x) = -\frac{L(q, i, x)}{\frac{\partial f(q,i,x)}{\partial x}} \dot{i}$$

is such that (6) is fulfilled for all q, i and x.

Hence, the following corollary specifies extended memristor in terms of the voltage v and the current i:

Corollary 1: An *extended memristor*[4] is defined in terms of the voltage v and the current i by

$$v = R(q, i, \mathbf{x}) i \quad (13)$$
$$\dot{\mathbf{x}} = \mathbf{g}(q, i, \mathbf{x}) \quad (14)$$
$$\dot{q} = i \quad (15)$$

where

$$R(q, i, \mathbf{x}) = \frac{\partial f(q, i, \mathbf{x})}{\partial q}$$

The equations (13) and (14) define the *dynamic characteristic equation*. The collection of all points (v, i) satisfying the dynamic characteristic equation is called the *dynamic characteristic* $\mathcal{C}$, i.e.

$$\mathcal{C} = \big\{(v(t), i(t)) \in \mathbf{R}^2 : v = R(q, i, \mathbf{x}) i,\ \dot{\mathbf{x}} = g(q, i, \mathbf{x}),\ \dot{q} = i\}$$

It turns out that the dynamical characteristic $\mathcal{C}$ has the following properties:

- $\forall t$ such that $i = 0$ then $v = 0$ (because (6) holds.)
- Any dynamic characteristic is in a one–to–one correspondence with a curve lying on $\mathcal{F}$.

Theorem 1 shows that the constraint $i = 0 \Rightarrow v = 0$, $\forall \mathbf{x}$ due to the pinched hysteresis loop at the origin implies the condition (6) in which two contributions can be identified:

- The term $\mathbf{J} \cdot \dot{\mathbf{x}}$ corresponds to an extra voltage source E in series with the extended memristor having a pinched hysteresis loop at the origin. This term expresses the orthogonality condition between $\mathbf{J}$ (i.e. the shape of the static characteristic) and $\dot{\mathbf{x}}$ (i.e. the nonlinear dynamics of the internal variables vector).
- a parasitic inductive term $L(q, i, \mathbf{x})\dot{i}$ in series with the extended memristor[5] having a pinched hysteresis loop at the origin.

[4]The *voltage–controlled extended memristor* can be defined by duality.

[5]While such current–controlled extended memristors has a series parasitic inductor, the dual case of voltage–controlled extended memristors has a parallel parasitic capacitor. A detailed model of memristor devices including four parasitic circuit elements is presented in [8].

All in all, the contribution $\mathbf{J} \cdot \dot{\mathbf{x}} + L(q, i, \mathbf{x})\dot{i}$ is an inherent emf of the memristor device (e.g. see for instance Na and K channels in Hodgkin–Huxley neuron model [9], [10]). This inherent emf may not appear in experiments, if the parasitic effects are (exactly) balanced by E, that is

$$E = \mathbf{J} \cdot \dot{\mathbf{x}} = -L(q, i, \mathbf{x})\dot{i}.$$

Furthermore, if *the parasitic inductive effects (i.e. the second–order derivative term in q) can be neglected respect to the memristor phenomenon (i.e. first–order derivative term in q)*, i.e $L(q, i, \mathbf{x})\ddot{q} \ll R(q, i, \mathbf{x})\dot{q}$, then the condition (6) reduces to

$$E = \mathbf{J}(q, i, \mathbf{x}) \cdot \dot{\mathbf{x}} = 0 \quad (16)$$

If (16) is not satisfied an extra voltage source in series with a memristor can be considered to model the nanoscale device[6]. The resistive switching memory cells (ReRAMs) proposed in [11] exhibits an inherent emf $E = \mathbf{J}(q, i, \mathbf{x}) \cdot \dot{\mathbf{x}}$ such that

$$v - E = R(q, i, \mathbf{x}) i$$

Remark 2: If the nonlinear dynamics of the internal vector is defined by

$$\dot{\mathbf{x}} = \mathbf{g}(q, i, \mathbf{x}) = \mathbf{h}(q, \mathbf{x}) i$$

then there are no constraints on $\mathbf{J}$ (because (16) is always satisfied when $i = 0$). In such a case the internal variables vector $\mathbf{x}$ is frozen $\forall t$ such that $i = 0$.

C. Subclasses of the extended memristor

From the previous Definition 2 and Corollary 1, the following classification is readily derived.

- An extended memristor with the static characteristic equation such that $\varphi = f(q, \mathbf{x})$, $\dot{\mathbf{x}}(t) = \mathbf{g}(q, i, \mathbf{x})$ and $\dot{q} = i$ is said to be a *generic memristor*, i.e. it is described in terms of the voltage v and the current i

$$v = R(q, \mathbf{x}) i \quad (17)$$
$$\dot{\mathbf{x}} = \mathbf{g}(q, i, \mathbf{x}) \quad (18)$$
$$\dot{q} = i \quad (19)$$

It is worth observing that the generic memristor includes (by definition) no parasitic inductive effect, i.e. *the generic memristor is an extended memristor with no parasitic effects*!

[6]The condition $E = \mathbf{J}(q, i, \mathbf{x}) \cdot \dot{\mathbf{x}} = \mathbf{J}(q, i, \mathbf{x}) \cdot \mathbf{g}(q, i, \mathbf{x})$ can be used to draw from the experiments the static characteristic equation $\varphi = f(q, i, \mathbf{x})$ and the dynamics of $\mathbf{x}(t)$.

An important class of generic memristors, used to model non–volatile memory devices (see the next section II-D), is described if the static characteristic equation can be factorized as $\varphi = q\, f_x(\mathbf{x})$ and $\dot{\mathbf{x}}(t) = g(i, \mathbf{x})$, $\dot{q} = i$, i.e. in terms of the voltage v and the current i

$$v = f_x(\mathbf{x})i = R(\mathbf{x})i \quad (20)$$
$$\dot{\mathbf{x}} = \mathbf{g}(i, \mathbf{x}) \quad (21)$$
$$\dot{q} = i \quad (22)$$

The *generic memristor* introduced in [5] belongs to this class.

- An extended memristor with a static characteristic equation such that $\varphi = f(q)$ and $\dot{q} = i$ is said to be *ideal memristor*, i.e. it is described in terms of the voltage v and the current i in the form

$$v = R(q)i \quad (23)$$
$$\dot{q} = i. \quad (24)$$

This representation is exactly the original Chua's definition of memristor introduced in 1971 [1]. It turns out that *the ideal memristor is an extended memristor with no parasitic effects and no internal state vector (i.e. no inherent emf E)*!

The following Tables I–III summarize the mathematical representations of (current–controlled) memristor devices both in terms of current–voltage momenta (φ, q) and current–voltage (i, v). The generic memristor is just referred to as *memristor* for simplicity.

It is worth observing that the definition of memristor provided in Tables I–III has the important advantage that it involves the voltage–momentum φ, and the current–momentum q, and NOT voltage v and current i, as in the original definition in [1]. This preserves the 4–element graph depicting the memristor.

For the sake of simplicity and to discuss the main properties of memristor devices according to those already available in literature [1], we mainly focus on the memristor (i.e. we neglect parasitic second–order effects that can be included, as shown in [8], to refine the memristor model).

Table I
EXTENDED MEMRISTOR

(φ, q)	(v, i)
$\varphi = f(q, i, \mathbf{x})$	$v = R(q, i, \mathbf{x})i$
$\dot{\mathbf{x}} = \mathbf{g}(q, i, \mathbf{x})$	$\dot{\mathbf{x}} = \mathbf{g}(q, i, \mathbf{x})$
$\dot{q} = i$	$\dot{q} = i$
$L(q, i, \mathbf{x})\dot{i} + \mathbf{J}(q, i, \mathbf{x}) \cdot \dot{\mathbf{x}} = 0$	

Table II
MEMRISTOR

(φ, q)	(v, i)
$\varphi = f(q, \mathbf{x})$	$v = R(q, \mathbf{x})i$
$\dot{\mathbf{x}} = \mathbf{g}(q, i, \mathbf{x})$	$\dot{\mathbf{x}} = \mathbf{g}(q, i, \mathbf{x})$
$\dot{q} = i$	$\dot{q} = i$
$\mathbf{J}(q, \mathbf{x}) \cdot \dot{\mathbf{x}} = 0$	

Table III
IDEAL MEMRISTOR

(φ, q)	(v, i)
$\varphi = f(q)$	$v = R(q)i$
$\dot{q} = i$	$\dot{q} = i$

D. Non–volatile memory property

One of the most important property of memristor is its capability to retain the value its resistance without a power supply.

Non-volatile memory (NVM) is a type of computer memory that has the capability to hold saved data even if the power is turned off. Unlike volatile memory, NVM does not require its memory data to be periodically refreshed.

Although non–volatile memory states correspond the memristance when electrical excitation is switched off, the non–volatile memory properties of memristor devices depends on the inherent nonlinear dynamic behavior described by the state equation (4), i.e the memristance play no role in the characterization of memristor's non–volatility.

It is worth to observe that, by definition, the ideal memristor has an entire continuum of non–volatile memory states (defined by any value $R(q = Q)$ with $Q \in \mathbf{R}$ constant for $i = 0$) (see also section 5.5 in [5]).

Using the definition given in the Table II with $i = 0$, the non–volatile memory states $R(Q, \bar{\mathbf{x}})$ are related to the *stable equilibrium points* $\bar{\mathbf{x}}$ of the state equation (i.e. $\bar{\mathbf{x}}$ are derived from the equation $\mathbf{g}(0, \mathbf{x}) = 0$). Similar considerations hold for the specific class of memristor defined by the equations (20)–(22). In such a case, the non–volatile memory states are given by $R(\bar{\mathbf{x}})$.

A continuum set of resistance values can be obtained if the state equation can be factorized as follows:

$$\dot{\mathbf{x}} = \mathbf{g}(q, i, \mathbf{x}) = \mathbf{h}(q, \mathbf{x})i. \quad (25)$$

It turns out that the whole continuum of non–volatile memory states correspond to the internal vector $\mathbf{x}$ frozen for all instants in which the current i is switched off.

The memristor with the state equation given by (25) acts as *analog non–volatile memory*.

On the other hand, if the state equation $\dot{\mathbf{x}} = \mathbf{g}(q, i, \mathbf{x})$ exhibits just equilibrium points, i.e. *closed orbits either periodic or aperiodic or even chaotic attractors are excluded*, then the memristor presents a finite number of memory states corresponding to the stable equilibrium points $\bar{\mathbf{x}}$. In such a case they are similar to conventional memory devices (e.g. a two state switching resistance is a memristor corresponding to a binary memory device), but each memristor memory state keeps the information related to the basin of attraction of $\bar{\mathbf{x}}$. The associated memristance $R(Q, \bar{\mathbf{x}})$ can be experimentally observed by means of a small sinusoidal excitation, once the state $\mathbf{x}(t)$ settles to $\bar{\mathbf{x}}$, i.e *the memristor is in a steady–state regime*.

The following Corollary 2 summarizes the non–volatile memory properties of memristor.

Corollary 2: A memristor such that $v = R(q, \mathbf{x})i$ ($v = R(\mathbf{x})i$), $\dot{\mathbf{x}} = \mathbf{h}(q, \mathbf{x})i$ and $\dot{q} = i$ is an *analog non–volatile memory* (i.e. it has a continuum of non–volatile memory states). Ideal memristor are (by definition) analog non–volatile memory.

From the application point of view, analog non–volatile memories are extremely important for mimicking synapses by means of pulses (e.g this permits to tune weights in memristor–based neural networks [12]).

Finally, the memristor has a volatile memory state if there exists just one stable equilibrium point $\bar{\mathbf{x}}$ or the state equation $\dot{\mathbf{x}} = \mathbf{g}(q, i, \mathbf{x})$ has a complex dynamic behavior, including periodic and/or chaotic trajectories.

E. DC curves

The dc V–I curves are obtained by considering the steady–state voltage due to a constant input applied to the memristor. Let us consider a dc current $i = I \in \mathbf{R}$ and let us assume that the static characteristic of the memristor in Tables I–III is asymptotic equivalent to αq for all $\mathbf{x}$, $\alpha \in \mathbf{R}$ and $|q| \to \infty$, that is we can explicitly write[7]:

- for the extended memristor

$$\varphi = f(q, i, \mathbf{x}) \sim \alpha q, \;\; \forall \mathbf{x}, \;\; |q| \to +\infty \tag{26}$$

- for the memristor

$$\varphi = f(q, \mathbf{x}) \sim \alpha q, \;\; \forall \mathbf{x}, \;\; |q| \to +\infty \tag{27}$$

- for the ideal memristor

$$\varphi = f(q) \sim \alpha q, \;\; |q| \to +\infty \tag{28}$$

It is readily derived that the dc current $i = I$ causes a time–varying current momentum, i.e. $q(t) = I\,t$ (assuming zero initial condition). As a consequence, if (26)–(28) hold then the steady–state voltage across the memristor device is constant, i.e. the memristor device admits of a dc V–I curve (see example 4.1 in section 4 in [5]). On the other hand, the memristor has no dc V–I curve (see example 4.2 in section 4 in [5]).

The condition (27) can be relaxed under the weak assumption that $\varphi = q f_x(\mathbf{x}) \sim \alpha q$, with $|q| \to +\infty$, i.e. for the specific class of memristor defined by the equations (20)–(22). In such a case the existence of dc V–I curves depend on the dynamics of state vector $\mathbf{x}$ governed by the state equation (21). In particular, the generic memristor has:

- as many dc V–I curves as the equilibrium points $\mathbf{x}^*$ of the equation $\dot{\mathbf{x}} = \mathbf{g}(I, \mathbf{x})$ for all I. It turn out that the dc V–I curves $V = R(\mathbf{x}^*)I$ that are observable at the steady–state correspond only to the stable equilibrium points (see section 7 in [5]).
- no dc V–I curve if there are values of I such that the equation $\dot{\mathbf{x}} = \mathbf{g}(I, \mathbf{x})$ exhibits complex dynamic behavior, including periodic and/or chaotic trajectories.

F. High frequency curves

It is well known that memristor devices subjected to a zero–mean periodic current $i(t)$ exhibit a pinched hysteresis loop that degenerates to a single–valued nonlinear V–I curve at sufficient high frequencies. The Fourier expansion of $i(t)$ can be written as

$$i(t) = \sum_{k=-\infty,\, k\neq 0}^{k=+\infty} a_k e^{jk\omega t}$$

As a consequence, the current momentum $q(t)$ results to be

$$q(t) = q_0 + \sum_{k=-\infty,\, k\neq 0}^{k=+\infty} \frac{a_k}{jk\omega} e^{jk\omega t}$$

that is, the sum of a constant q_0 and a zero–mean periodic function than can be expanded as a Fourier series with coefficients that tend to 0 as $\omega \to +\infty$. It is readily derived that $q(t) \to q_0$ as $\omega \to +\infty$. If the initial condition is zero then the constant $q_0 = 0$.

The ideal memristor acts as a linear resistor as $\omega \to +\infty$, i.e. $v = R(q_0)i$ with

$$R(q_0) = \frac{\mathrm{d}f(q)}{\mathrm{d}q}|_{q=q_0}.$$

[7]Two real–valued functions $f(x)$ and $g(x)$ are asymptotic equivalent when $x \to x_0$, i.e. $f(x) \sim g(x)$, $x \to x_0$, if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$$

The behavior of the memristor (extended memristor) at high frequency depends on $\mathbf{x}(t)$, i.e. it depends on the equation $\dot{\mathbf{x}} = \mathbf{g}(q_0, \mathbf{x})$ ($\dot{\mathbf{x}} = \mathbf{g}(q_0, i, \mathbf{x})$).

Following the proof of the Property 6 in [2], the bounded state vector $\mathbf{x}(t)$ approaches some constant vector $\mathbf{x}_0$ as the input frequency increases towards infinity. It follows that:

- the V–I curve of a memristor shrinks to a straigh line as $\omega \to +\infty$, i.e. $v = R(q_0, \mathbf{x}_0)i$
- the V–I curve of an extended memristor shrinks to a nonlinear curve as $\omega \to +\infty$, i.e. $v = R(q_0, i, \mathbf{x}_0)i$.

III. Ideal memristors

According to the assumptions given in the section II, the vector $\mathbf{x}$ contains only physical variables different from the input current (voltage) variable in current–controlled memristor (voltage–controlled memristor). We show that if there exists a relationship between each variable of the vector $\mathbf{x}$ and the current momentum q then the memristor reduces to an ideal memristor (i.e. the memristance depends only on the charge q).

The following Theorem states this result.

Theorem 2: The static characteristic equation $\varphi = f(q, \mathbf{x})$ of a memristor can be put in the form $\varphi = f(q)$ of the ideal memristor if a mapping $\mathbf{X} : \mathbf{R} \to \mathbf{R}^n$ exists such that $\mathbf{x} = \mathbf{X}(q)$.

Proof: The existence of a mapping $\mathbf{X} : \mathbf{R} \to \mathbf{R}^n$ such that $\mathbf{x} = X(q)$ implies that the static characteristic equation of a memristor is written in the form

$$\varphi = f(q, \mathbf{x}) = f(q, X(q)) = \bar{f}(q)$$

As a consequence, $v = R(q)i$ with $\dot{q} = i$. ■

Remark 3: The above result implies that, if the internal variables all depend on q, then the differential equation they satisfy reduces to the form

$$\dot{\mathbf{x}} = \frac{\mathrm{d}X(q)}{\mathrm{d}q} i$$

Note also that the existence of the mapping $\mathbf{X}$ is sufficient to guarantee that the memristor be ideal, but it is by no means necessary; in fact, it can happen that the static characteristic equation (3) has f that does not depend on $\mathbf{x}$ but only on q, in which case $R(q)$ also satisfies the same condition, and nevertheless the equation (4) is still present. This is the case in which the state $\mathbf{x}$ is *unobservable*.

A simple procedure to create ideal memristor sibling is provided in [5].

It turns out that the following corollary holds.

Corollary 3: A memristor such that

$$v = R(q, \mathbf{x})i$$

and

$$\dot{\mathbf{x}} = \frac{\mathrm{d}X(q)}{\mathrm{d}q} i$$

is an ideal memristor (see Table III).

On the other hand, if there is no mapping $\mathbf{X}$ (i.e. all variables the vector $\mathbf{x}$ are not dependent on q), the classification provided in the subsection II-C holds.

A. Examples

This Section presents pedagogical examples of memristor defined according to the representations given in the Tables I-III. The aim is to highlight the concept that *pinched hysteresis loops do not define completely a memristor model* (see also [4]), but they correspond to the dynamic characteristic defined in Corollary 1. An almost exhaustive collection of real memristor devices identified in different unrelated fields is reported in [5].

Without losing any generality, let us consider just a scalar internal variable vector, i.e. $x \in \mathbf{R}$. In addition, let us focus on ideal memristor, i.e. we assume that the equation governing the evolution of the internal variable x is $\dot{x} = x\, i$. It follows that there exists a mapping $X(\cdot)$ defined by

$$x = e^q.$$

It turns out that, given $q(0)$, $x(0) = e^{q(0)}$ and $q = q(0) + \int_0^t i(\tau)d\tau$. We consider $q(0) = 0$ for the sake of simplicity, but similar considerations hold for any initial condition.

The static characteristic $\varphi = f(q, x)$ is assumed to be $\varphi = e^q = x$ (see Figure 1), i.e. $v = x\, i$, with $R(q, x) = x$. The dynamic characteristic $\mathcal{C}$ obtained by considering a zero–mean periodic current $i = 1.5\,\sin(t) + 0.5\cos(3t)$ is superimposed to the the static characteristic in Figure 1. The projection of $\mathcal{C}$ on the plane φ–q is the flux–charge characteristic of the ideal memristor (see upper part of Figure 2) and the corresponding voltage–current pinched hysteresis loops for the given current is shown in Figure 2 (see bottom part).

If the current is simply a sinusoid $i = \cos(t)$ then just a part of the previous dynamic characteristic $\mathcal{C}$, lying on the static characteristic, is explored (see Figure 3). As a consequence, a different voltage–current pinched hysteresis loops is observed (see bottom part of Figure 4). The voltage momentum–current momentum characteristic of the ideal memristor is obviously the same as shown in the upper part of Figure 4.

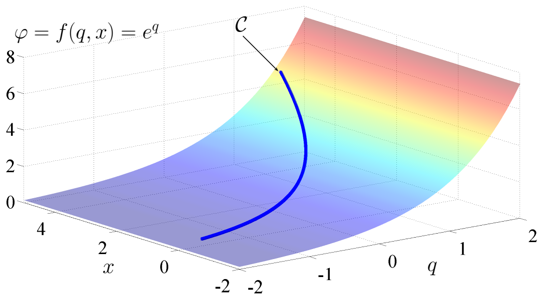

Figure 1. The static characteristic $\varphi = f(q, x) = e^q$ and the superimposed dynamic characteristic $\mathcal{C}$ under the input current $i = 1.5\,\sin(t) + 0.5\cos(3t)$. Initial condition is set to $q(0) = 0$.

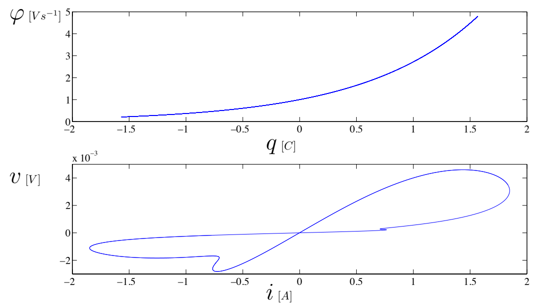

Figure 2. The projection of $\mathcal{C}$ in Figure 1 on the (φ, q) space (upper part) and the corresponding dynamic characteristic in the (v, i) space under the input current $i = 1.5\,\sin(t) + 0.5\cos(3t)$ (bottom part).

This simple example just points out that the most suitable variables to describe the ideal memristor, defined by $v = xi$ and $\dot{x} = xi$, are the voltage momentum φ or the current momentum q, i.e. the voltage–current curve can be misleading (different pinched loops can be obtained for the same ideal memristor subjected to different zero–mean periodic current). In addition, the

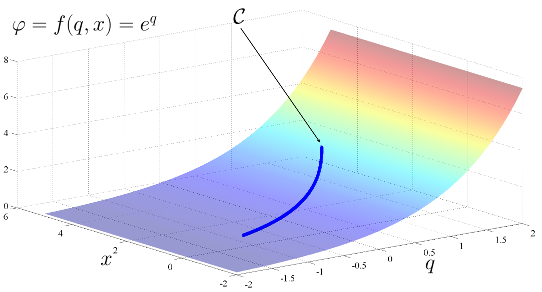

Figure 3. The static characteristic $\varphi = f(q, x) = e^q$ and the superimposed dynamic characteristic $\mathcal{C}$ under the input current $i = \cos(t)$. Initial condition is set to $q(0) = 0$.

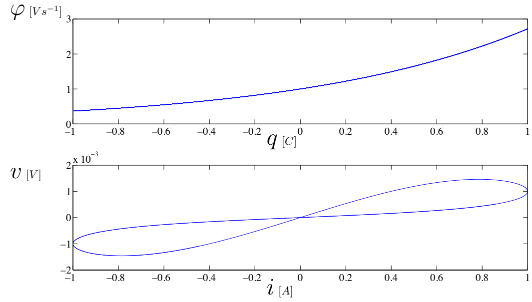

Figure 4. The projection of $\mathcal{C}$ in Figure 3 on the (φ, q) space (upper part) and the corresponding dynamic characteristic in the (v, i) space under the input current $i = \cos(t)$ (bottom part).

effect of a small bias is shown in Figure 5 in which the current is set to $i(t) = -0.1 + \sin(t)$. It follows that $v(t) = \exp(-0.1t)\exp(1 - \cos(t))i(t)$. The picture at the centre of Figure 5 shows that the pinched hysteresis loop tends to the straight-line $v = 0$ due to the small bias (see waveforms of $i(t)$ and $v(t)$ in the bottom part of Figure 5) even if the static and the dynamic characteristic is not changed.

B. *Passive or active ideal memristor?*

Let us consider an ideal memristor with a static characteristic $\varphi = x$ and a mapping between a scalar state x and the current momentum q such that $\dot{x} = -2\,q\,x\,i$, that is

$$\varphi = e^{-q^2} \Rightarrow v = \left(-2qe^{-q^2}\right) i \qquad (29)$$

The dynamic characteristic $\mathcal{C}$ lies on different side of the static characteristic if $i(t) = \sin(t)$ (see Figure 6) or $i(t) = \cos(t + \pi/2)$ (refer to Figure 8). The corresponding v–i curves are shown in Figures 7 and 9, respectively.

Although in the first case the ideal memristor acts as an active device (its i–v curve shown in Fig. 7 is in the second and fourth quadrants) whereas in the latter case results to be a passive device (its i–v curve shown in Fig. 9 is in the first and third quadrants), the passivity property of an ideal memristor must hold for any input, i.e. as shown in Teorem 1 of [1] the passivity of ideal memristor can be inferred only from the whole static characteristic. The upper part of Fig. 7 and Fig. 9 shows just the part of the static characteristic corresponding to the projection of $\mathcal{C}$ on the (φ, q) plane, i.e. the part of the static characteristic due to a specific input. If the ideal memristor is passive only for certain restricted inputs then the *restrict passivity*

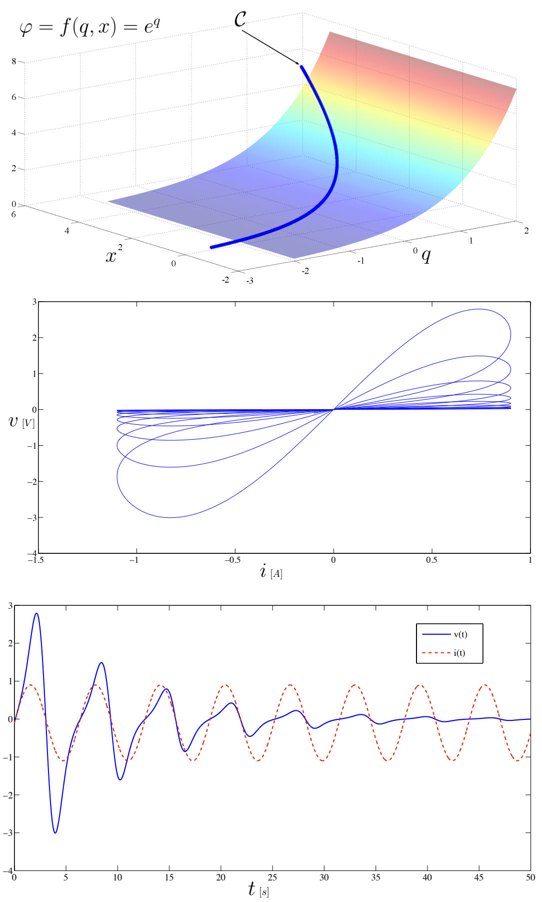

Figure 5. The static characteristic $\varphi = f(q,x) = e^q$ and the superimposed dynamic characteristic $\mathcal{C}$ under the input current $-0.1 + \sin(t)$. Initial condition is set to $q(0) = 0$ (upper part). The dynamic characteristic in the (v,i) plane (middle part). The waveforms of the (input) current $i = \cos(t)$ and the voltage $v(t) = \exp(-0.1t)\exp(1-\cos(t))i(t)$ (bottom part).

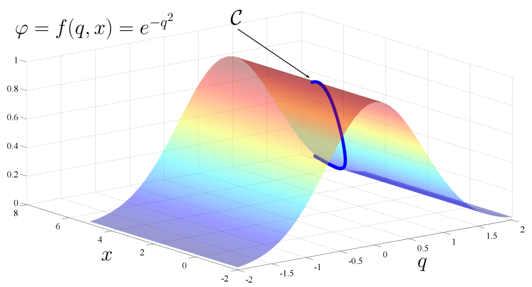

Figure 6. The static characteristic $\varphi = f(q,x) = e^{-q^2}$ and the superimposed dynamic characteristic $\mathcal{C}$ under the input current $i = \sin(t)$. Initial condition is set to $q(0) = 0$.

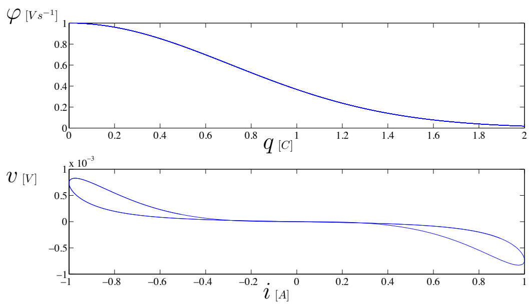

Figure 7. The projection of $\mathcal{C}$ in Figure 6 in the (φ, q) space (upper part) and the corresponding dynamic characteristic in the (v,i) space under the input current $i = \sin(t)$ (bottom part).

property holds[8]. The ideal memristor given in equation (29) is restricted passive for all the input current $i(t)$ such that $q(t) \leq 0$, whereas it is restricted active for all $i(t)$ such that $q(t) \geq 0$. It is worth to observe that the restricted passivity (activity) property is influenced by the initial condition as well. In particular, for the ideal memristor given in the equation (29) the difference between the two cases shown in Figures 6–7 and in Figures 8–9 is due to the assumption $q(0) = 0$ that implies a current momentum with mean–value different from zero. A zero–mean current momentum is obtained if $q(0) = 1$ when $i(t) = \cos(t + \pi/2)$ or $q(0) = -1$ when $i(t) = \sin(t)$. The latter case is presented in Figures 10 and 11.

It turns out the influence of initial conditions in the observed pinched hysteresis loops (see also the detailed study presented in [7]).

IV. Passive and lossless ideal memristors

Motivated from the case studied in the section III-B, the concepts of passivity and losslessness of ideal memristor are reviewed. The passivity and losslessness are defined in terms of the dynamic characteristic equation for an ideal memristor (see the previous Corollary 3)

$$\begin{aligned} v &= R(q, X(q))i = R(q)i \quad &(30) \\ x &= X(q). \quad &(31) \end{aligned}$$

A one–port is *passive* if and only if, for *any* admissible pair (v,i), the inequality holds

$$\int_{-\infty}^{t} v(\tau)i(\tau)\,\mathrm{d}\tau \geq 0, \qquad \forall t \qquad (32)$$

[8]The ideal memristor is *restricted active* if it is no restricted passive.

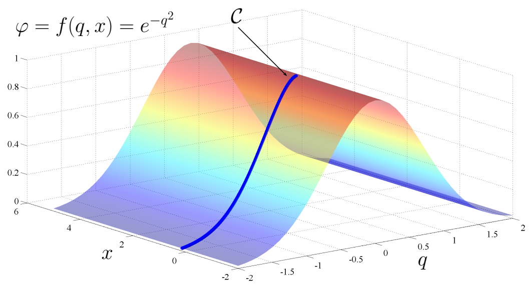

Figure 8. The static characteristic $\varphi = f(q,x) = e^{-q^2}$ and the superimposed dynamic characteristic $\mathcal{C}$ under the input current $i = \cos(t + \pi/2)$. Initial condition is set to $q(0) = 0$.

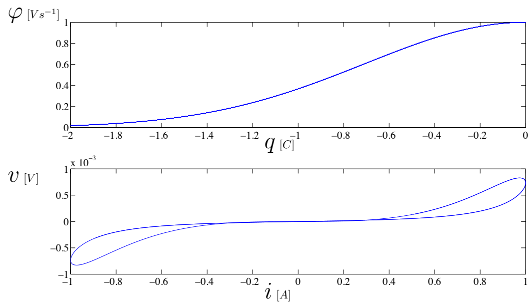

Figure 9. The projection of $\mathcal{C}$ in Figure 8 in the (φ, q) space (upper part) and the corresponding dynamic characteristic in the (v, i) space under the input current $i = \cos(t + \pi/2)$ (bottom part).

According to the first of equations (30), the previous one yields

$$\int_{-\infty}^{t} R(q(\tau))i^2(\tau)\,\mathrm{d}\tau \geq 0, \qquad \forall t \tag{33}$$

which implies

$$R(q) \geq 0, \qquad \forall q \tag{34}$$

so that the static characteristic in the (q, φ) plane must be monotone nondecreasing.

If the memristor has to be controllable both in current and voltage, strict monotonicity is required so that equations (33) and (34) are replaced by the following

$$\int_{-\infty}^{t} v(\tau)i(\tau)\,\mathrm{d}\tau > 0 \tag{35}$$

and $R(q) > 0 \qquad \forall q$.

The concept of *losslessness* is more difficult to apply because the further condition (beyond that of passivity)

$$\int_{-\infty}^{+\infty} v(\tau)i(\tau)\,\mathrm{d}\tau = 0 \tag{36}$$

would imply $R(q) = 0$, $\forall q$, i.e. the memristor would reduce to a short circuit!

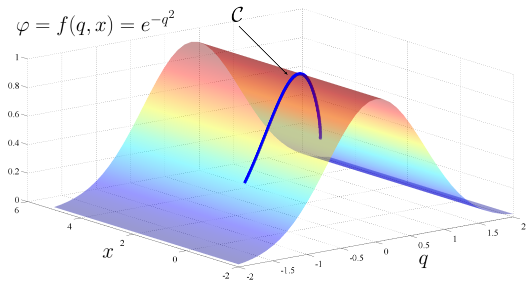

Figure 10. The static characteristic $\varphi = f(q,x) = e^{-q^2}$ and the superimposed dynamic characteristic $\mathcal{C}$ under the input current $i = \sin(t)$. Initial condition is set to $q(0) = -1$.

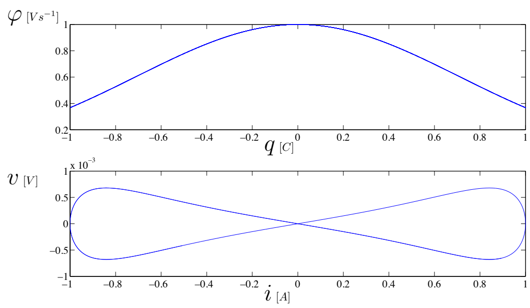

Figure 11. The projection of $\mathcal{C}$ in Figure 10 in the (φ, q) space (upper part) and the corresponding dynamic characteristic in the (v, i) space under the input current $i = \sin(t)$ (bottom part).

A concept of *restricted losslessness* can be introduced by removing the constraint of passivity and replacing equation (36) by

$$\int_{0}^{T} v(\tau)i(\tau)\,\mathrm{d}\tau = 0 \tag{37}$$

under the assumption that the initial energy (at $t = 0$)

$$\epsilon_0 = \int_{-\infty}^{0} v(\tau)i(\tau)\,\mathrm{d}\tau = 0 \tag{38}$$

and where $v(t)$ and $i(t)$ are zero–mean periodic over $[0, T]$, that is there exists at least one admissible pair (v, i) of zero mean periodic (of period T) functions such that equation (37) holds.

As an example consider the ideal memristor defined by the static characteristic $\varphi = q^2$ and therefore by the state equations

$$\begin{aligned} v &= 2qi \\ \dot{q} &= i \end{aligned} \tag{39}$$

Under the excitation $i = \cos(t)$ it is obtained $q = \sin(t)$

and $v = \sin(2t)$ so that

$$\int_0^T vi \,\mathrm{d}\tau = \int_0^{2\pi} \sin(2\tau)\cos(\tau)\,\mathrm{d}\tau = 0 \tag{40}$$

that is the above memristor is restricted lossless (for the specified (v, i) pair), but not passive. It is worth observing that this may not hold for other signals. For example, by choosing the input $i = -I + \cos(t)$ with $I > 0$ and calculating the integral of equation (33) for $T = 2\pi$, we obtain (assuming $\epsilon_0 = 0$)

$$\int_0^{2\pi} v(\tau)i(\tau)\,\mathrm{d}\tau < 0. \tag{41}$$

V. Conclusions

In this manuscritpt we have presented the first systematic theoretical approach to model any memristor device, regardless of the device material and physical operating mechanisms. We have introduced the concept of *restricted passivity* and *restricted losslessness*, that describe ideal memristors passive for certain restricted inputs and absorbing no average power under zero mean periodic excitations, respectively. It is worth noting that ideal memristors with static characteristic $\varphi = f(q, x)$ such that the mapping $x = X(q)$ is not a strictly monotonic relation can exhibit the restricted passivity property.

We conclude this paper by pointing out that:

> *The static characteristic of the extended memristor $\varphi = f(q, i, \mathbf{x})$ is the only model that completely describes a real two–terminal memristor device.*
>
> *The static characteristic of the memristor $\varphi = f(q, \mathbf{x})$ is the only model that completely describes a real two–terminal memristor device in which the second–order derivative term in q can be neglected.*
>
> *The static characteristic of the ideal memristor $\varphi = f(q)$ is the only model that completely describes a real two–terminal memristor device in which the second–order derivative term in q can be neglected and the state vector $\mathbf{x}$ is either completely expressed in function of q or is not existing.*

The main advantage of the proposed approach is that all memristor representations in Tables I-III are given in terms of the current–momentum q and voltage–momentum φ, i.e. the static characteristics completely describe memristor devices, whereas *pinched hysteresis i-v curves are just the specific response to a given input*! For example, figures 4 and 5 show that the i–v curves for a given memristor (e.g. $\varphi = e^q$) are different due to different currents.

Perhaps the most significant result from this paper is that we have unified our definitions of the "extended" memristor, the "generic" memristor, and the "ideal" memristor, under one umbrella based on the "voltage momentum" (aka "flux") φ, and the "current–momentum" (aka "charge") q, as depicted in the familiar "Four Basic Circuit Element Rectangle" given in many recent Review articles and Tutorials, such as [13] and [14]. Finally, the form of the state equation is also discussed according to the non–volatile memory property and the DC I-V curves of the memristor.

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