

Material Description of Fluxes in Terms of Differential Forms

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## Material Description of Fluxes in Terms of Differential Forms Dedicated to Prof. David Steigmann in Recognition of His Contributions

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**Abstract** The flux of a certain extensive physical quantity across a surface is often represented by the integral over the surface of the component of a pseudo-vector normal to the surface. A pseudo-vector is in fact a possible representation of a second-order differential form, i.e., a skew-symmetric second-order covariant tensor, which follows the regular transformation laws of tensors. However, because of the skew-symmetry of differential forms, the associated pseudo-vector follows a transformation law that is different from that of proper vectors, and is named after the Italian mathematical physicist Gabrio Piola (1794-1850). In this work, we employ the methods of Differential Geometry and the representation in terms of differential forms to demonstrate how the flux of an extensive quantity transforms from the spatial to the material point of view. After an introduction to the theory of differential forms, their transformation laws, and their role in Integration Theory, we apply them to the case of first-order transport laws such as Darcy's law and Ohm's law.

**Keywords** Differential Geometry · Differential Form · Flux · Material · Spatial · Darcy's Law · Ohm's Law

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## 1 Introduction

In the elementary treatment of Classical Physics and Continuum Mechanics, several quantities are referred to as pseudo-vectors, because the transformation laws they obey are different from those of vectors. Indeed, under any transformation other than a proper orthogonal transformation (proper means that the determinant is equal to 1), pseudo-vectors take a multiplicative factor equal to the determinant of the transformation, contrary to what vectors do. For instance, under reflection of one of the axes (a particular case of improper orthogonal transformation, for which the determinant is  $-1$ ), pseudo-vectors are not only reflected, but also see their sense reversed, unlike vectors, which are just reflected. The first examples that come in the study of elementary Mechanics are the moment of a force and the angular velocity. Pseudo-vectors also come into play in the calculation of the flux of an extensive physical quantity across a surface, which is the integral over the surface of the component of the pseudo-vector normal to the surface itself. Examples are the mass and charge density currents in Fluid Mechanics and Electromagnetism. Continuum Mechanics also presents us with objects that can be righteously called pseudo-scalars because, unlike scalars, which are invariant under any transformation, they transform with a multiplicative factor equal to the determinant of the transformation and, for instance, suffer a change in sign under an improper orthogonal transformation. Examples are the so-called “volume element” of the theory of integration, and the scalar product of a vector and a pseudo-vector.

In the Continuum Mechanics of solids, it is most often necessary to transform the various physical quantities from the spatial picture of Mechanics, in which the equations are naturally written, to the material picture, in which it is usually most convenient to build a constitutive framework. The original ideas on the transformation from spatial to material picture dates back to the Italian mathematical physicist Gabrio Piola (see [4], as well as [20]) and modern interpretations have been very often proposed in recent works in Continuum Mechanics (among many others, see, e.g., [3, 2]). In the modern language of Continuum Mechanics, the transformation from the spatial to the material picture of Mechanics is called a pull-back, a terminology that is shared by Differential Geometry.

In Continuum Mechanics one always works with a non-orthogonal transformation, i.e., the deformation gradient. Therefore, the pull-back laws of pseudo-vectors and pseudo-scalars differ from those of vectors and scalars (the latter being invariant), respectively, by a multiplicative factor equal to the determinant of the deformation gradient. In fact, in the customary three-dimensional case, a pseudo-vector represents a second-order differential form (or two-form) and a pseudo-scalar represents a (non-vanishing) third-order differential form (or non-vanishing three-form, or volume form) [7, 24]. Differential forms are skew-symmetric covariant tensors, and follow the regular transformation rules of tensors. The form of that these rules take in the representation in terms of pseudo-vectors and pseudo-scalars is that of a Piola transformation. In the case of pseudo-vectors and pseudo-scalars, the Piola transformation preserves the invariance of fluxes across deforming oriented surfaces and of the extent of physical quantities over volumes, respectively, when passing from the current configuration to the body manifold (or the reference configuration).

In this work, we first briefly recall the tensor algebra notation, show the definitions of  $r$ -forms, the particular cases of  $n$ -forms and  $(n - 1)$ -forms and the associated pseudo-vectors and pseudo-scalars (Section 2). Then, after introducing the basic definitions of Continuum Kinematics (Section 3), we introduce differential forms on general manifolds, their relationship to the Theory of Integration, including how fluxes are calculated as integrals of  $(n - 1)$ -forms, and demonstrate how the pull-backs of pseudo-vectors and pseudo-scalars are obtained from those of the corresponding  $(n - 1)$ - and  $n$ -forms (Section 4). Finally, we

49 apply this method to first-order transport laws such as Darcy's law in the Theory of Porous  
 50 Media and Ohm's law in Electromagnetism, and show how the Piola transformation on the  
 51 flux quantity induces another Piola transformation on the first leg of the tensor providing the  
 52 constitutive relation between the flux and the generalised force density, e.g., on the first leg  
 53 of the permeability tensor, which constitutively relates filtration velocity with the gradient  
 54 of the pore pressure in a porous medium (Section 5).

55 The purpose of this work is to move a step toward a unified formalism, which might  
 56 help to account for phenomena characterised by a seemingly different Physics, which nev-  
 57 ertheless is described by the very same Mathematics.

58 **Remark.** Throughout this work, for the sake of generality, both the body  $\mathcal{B}$  and the space  
 59  $\mathcal{S}$  are treated as differentiable manifolds or, when the metric structure is required, as Rie-  
 60 mannian manifolds (differentiable manifolds are treated exhaustively, e.g., in the treatise  
 61 by Epstein [7]). However, if one prefers, the physical space  $\mathcal{S}$  can be regarded as a three-  
 62 dimensional affine space, and the body  $\mathcal{B}$ , or any arbitrary reference configuration  $\mathcal{B}_R$ , as an  
 63 open subset of  $\mathcal{S}$ . A very exhaustive introduction to affine spaces is out of the scope of this  
 64 work (again, see [7]) but, roughly speaking, an affine space consists of a set  $\mathcal{A}$ , called the  
 65 point space, and a function that maps a pair  $(x, y)$  of points  $x, y$  of  $\mathcal{A}$  into an element  $\mathbf{u} = y - x$   
 66 of a vector space  $\mathcal{V}$ , called the supporting or modelling space. In the affine space  $\mathcal{A}$  one can  
 67 thus attach a vector  $\mathbf{u} = y - x$  at every point  $x$ . The vector space of all vectors emanating  
 68 from a point  $x$  is called the tangent space,  $T_x\mathcal{A}$ , at  $x$ . The dual space of  $T_x\mathcal{A}$ , i.e., the set of al  
 69 linear maps from  $T_x\mathcal{A}$  to the real numbers  $\mathbb{R}$ , is called the cotangent space,  $T_x^*\mathcal{A}$ , at  $x$ . The  
 70 the disjoint unions of all tangent spaces and of all cotangent spaces for all points  $x \in \mathcal{A}$  are  
 71 the tangent bundle  $T\mathcal{A}$  and the cotangent bundle  $T^*\mathcal{A}$ , respectively. When the point space  $\mathcal{A}$   
 72 and the supporting space  $\mathcal{V}$  are both  $\mathbb{R}^3$ , one obtains the familiar affine space  $\mathbb{E}^3$  of Classical  
 73 Mechanics.

## 74 2 Tensor Algebra

75 In this section, we briefly illustrate the tensor algebra notation [13, 10, 11] employed in this  
 76 work, then introduce  $r$ -forms in an  $n$ -dimensional vector space, and subsequently present  
 77 the important cases of  $n$ -forms and  $(n - 1)$ -forms. Note that, for the sake of a succinct, yet  
 78 reasonably self-contained presentation of these concepts, we avoid introducing the wedge  
 79 product  $\wedge$ , skew-symmetrisation of the tensor product  $\otimes$ , and instead rely on use of the  
 80 Ricci/Levi-Civita symbol (Equation (6)). Exhaustive introductions to  $r$ -forms and spaces of  
 81  $r$ -forms can be found, e.g., in the works by Epstein [7] and Segev [24].

### 82 2.1 Tensors on a Vector Space

83 Given a vector space  $\mathcal{V}$  on the real numbers  $\mathbb{R}$ , its dual space, i.e., the space of all linear  
 84 forms  $\boldsymbol{\pi} : \mathcal{V} \rightarrow \mathbb{R}$ , is denoted  $\mathcal{V}^*$ . The space of all multilinear forms

$$\mathbb{A} : \underbrace{\mathcal{V}^* \times \dots \times \mathcal{V}^*}_{r \text{ times}} \times \underbrace{\mathcal{V} \times \dots \times \mathcal{V}}_{s \text{ times}} \rightarrow \mathbb{R}, \quad (1)$$

85 i.e., all tensors of order  $r + s$  with the first  $r$  legs being vectorial and the last  $s$  legs being  
 86 covectorial, is denoted

$$\mathcal{V}^r_s = \underbrace{\mathcal{V} \otimes \dots \otimes \mathcal{V}}_{r \text{ times}} \otimes \underbrace{\mathcal{V}^* \otimes \dots \otimes \mathcal{V}^*}_{s \text{ times}}. \quad (2)$$

87 Note that the identifications  $\mathcal{V}_0^0 \equiv \mathbb{R}$ ,  $\mathcal{V}_0^1 \equiv \mathcal{V}$ ,  $\mathcal{V}_1^0 \equiv \mathcal{V}^*$  hold, and that, if  $\mathcal{V}$  has dimension  $n$ ,  
 88 the dimension of  $\mathcal{V}_s^r$  is  $n^{r+s}$ .

89 If  $\mathcal{V}$  has dimension  $n$ , considering a basis  $\{\mathbf{e}_i\}_{i=1}^n$  of  $\mathcal{V}$ , together with the associated  
 90 basis  $\{\mathbf{e}^i\}_{i=1}^n$  of  $\mathcal{V}^*$ , the components of a tensor  $\mathbb{A} \in \mathcal{V}_s^r$  with respect to the given bases are  
 91  $A^{i_1 \dots i_r}_{j_1 \dots j_s}$ , with the first  $r$  indices being contravariant and the last  $s$  indices being covariant,  
 92 i.e.,

$$\mathbb{A} = A^{i_1 \dots i_r}_{j_1 \dots j_s} \mathbf{e}_{i_1} \otimes \dots \otimes \mathbf{e}_{i_r} \otimes \mathbf{e}^{j_1} \otimes \dots \otimes \mathbf{e}^{j_s}, \quad (3)$$

93 where the components  $A^{i_1 \dots i_r}_{j_1 \dots j_s}$  are, by definition,

$$A^{i_1 \dots i_r}_{j_1 \dots j_s} = \mathbb{A}(\mathbf{e}^{i_1}, \dots, \mathbf{e}^{i_r}, \mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_s}). \quad (4)$$

## 94 2.2 $r$ -Forms on a Vector Space

95 Given a vector space  $\mathcal{V}$  of dimension  $n$ , and  $r \leq n$ , an  $r$ -form (or form of order  $r$ , or multi-  
 96 covector of order  $r$ ) is a tensor  $\boldsymbol{\beta} \in \mathcal{V}_r^0$  that is skew-symmetric, i.e., it is invariant for even  
 97 permutations of the arguments and changes sign for odd permutations of the arguments.  
 98 Therefore, for every set of  $r$  vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_r\} \subset \mathcal{V}$ ,

$$\boldsymbol{\beta}(\mathbf{v}_{i_1}, \dots, \mathbf{v}_{i_r}) = \varepsilon_{i_1 \dots i_r} \boldsymbol{\beta}(\mathbf{v}_1, \dots, \mathbf{v}_r), \quad (5)$$

99 where  $\varepsilon_{i_1 \dots i_r}$  is the Ricci/Levi-Civita permutation symbol, defined as

$$\varepsilon_{i_1 \dots i_r} = \begin{cases} +1 & \text{for } \{i_1, \dots, i_r\} \text{ even w.r.t. } \{1, \dots, r\}, \\ -1 & \text{for } \{i_1, \dots, i_r\} \text{ odd w.r.t. } \{1, \dots, r\}. \end{cases} \quad (6)$$

100 All  $r$ -forms in  $\mathcal{V}_r^0$  constitute a subspace denoted  $\Lambda_r(\mathcal{V})$ , whose dimension can be shown  
 101 to be equal to the binomial coefficient  $\binom{n}{r} = \frac{n!}{(n-r)!r!}$ , where  $n = \dim \mathcal{V}$  [7, 24]. With the help  
 102 of the Tartaglia-Pascal triangle, the scheme below reports the dimension of  $\Lambda_r(\mathcal{V})$  for every  
 103 order  $r$  and for every dimension  $n$  of the “mother” space  $\mathcal{V}$ .

						dim $\mathcal{V}$ :		
			1			0		
			1	1		1		
			1	2	1	2		
			1	3	3	1	3	
104			1	4	6	4	1	4
			...	...	...	...	...	...
	dim $\Lambda_r(\mathcal{V})$ :	1	$n$	$\binom{n}{r}$	$n$	1	$n$	$n$
	order $r$ :	0	1	$r$	$n-1$	$n$		

105 For a given  $n$ , the spaces  $\Lambda_r(\mathcal{V})$  of  $r$ -forms constitute a “fusiform” structure [7], with  $r = n$   
 106 being the maximum possible order for an  $r$ -form, and with the pairs of spaces of the forms  
 107 of order  $r$  and  $n - r$  having the same dimension. It is immediate to make the conventional  
 108 identifications  $\Lambda_0(\mathcal{V}) \equiv \mathbb{R}$  between zero-forms and scalars, and  $\Lambda_1(\mathcal{V}) \equiv \mathcal{V}^*$  between one-  
 109 forms and covectors. The spaces  $\Lambda_n(\mathcal{V})$  and  $\Lambda_{n-1}(\mathcal{V})$  “look a lot” like  $\mathbb{R}$  and  $\mathcal{V}$ , respectively,  
 110 but they do not quite coincide with these. Indeed, we shall show that  $n$ -forms and  $(n -$   
 111  $1)$ -forms obey transformation laws that are *different* from those which scalars (which are  
 112 invariant) and vectors obey. For this reason, they are often referred to as pseudo-scalars and  
 113 pseudo-vectors.

114 We close this section with an important definition that will be employed later. Given a  
 115 vector  $\mathbf{u} \in \mathcal{V}$  and an  $r$ -form  $\boldsymbol{\beta} \in \Lambda_r(\mathcal{V})$ , their interior product  $\mathbf{i}_\mathbf{u}\boldsymbol{\beta} \in \Lambda_{r-1}(\mathcal{V})$  is the  $(r-1)$ -  
 116 form given by the contraction of  $\mathbf{u}$  with the first leg of  $\boldsymbol{\beta}$ , i.e., for every system of  $r-1$   
 117 vectors  $\{\mathbf{v}_2, \dots, \mathbf{v}_r\} \subset \mathcal{V}$ ,

$$(\mathbf{i}_\mathbf{u}\boldsymbol{\beta})(\mathbf{v}_2, \dots, \mathbf{v}_r) = \boldsymbol{\beta}(\mathbf{u}, \mathbf{v}_2, \dots, \mathbf{v}_r). \quad (7)$$

### 118 2.3 $n$ -Forms and Their Transformation Laws

119 In a vector space  $\mathcal{V}$  of dimension  $n$ , the space  $\Lambda_n(\mathcal{V})$  has dimension one, and thus an  $n$ -form  
 120 has only one independent component with respect to a given basis  $\{\mathbf{e}_i\}_{i=1}^n$ , as it can be seen  
 121 by using the definition of skew-symmetry, which implies

$$\boldsymbol{\mu}(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_n}) = \varepsilon_{i_1 \dots i_n} \boldsymbol{\mu}(\mathbf{e}_1, \dots, \mathbf{e}_n) = h \varepsilon_{i_1 \dots i_n}, \quad (8)$$

122 where the well-defined scalar  $h = \boldsymbol{\mu}(\mathbf{e}_1, \dots, \mathbf{e}_n)$  is the only independent component of  $\boldsymbol{\mu}$ . The  
 123  $n$ -form  $\boldsymbol{\mu}$  can therefore be expressed, in components, as

$$\boldsymbol{\mu} = h \varepsilon_{i_1 \dots i_n} \mathbf{e}^{i_1} \otimes \dots \otimes \mathbf{e}^{i_n}. \quad (9)$$

124 Given the basis  $\{\mathbf{e}_i\}_{i=1}^n$  of  $\mathcal{V}$ , the unique  $n$ -form with independent component equal to  
 125 one is called determinant with respect to  $\{\mathbf{e}_i\}_{i=1}^n$  and is denoted by  $\det$ , i.e.,

$$\det(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_n}) = \varepsilon_{i_1 \dots i_n} \Rightarrow \det(\mathbf{e}_1, \dots, \mathbf{e}_n) = 1, \quad (10)$$

126 and, via the definitions of multi linearity and skew-symmetry, defines the determinant of the  
 127 system of vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  with respect to the basis  $\{\mathbf{e}_i\}_{i=1}^n$  as the scalar

$$\begin{aligned} \det(\mathbf{v}_1, \dots, \mathbf{v}_n) &= \det(v_1^{i_1} \mathbf{e}_{i_1}, \dots, v_n^{i_n} \mathbf{e}_{i_n}) \\ &= v_1^{i_1} \dots v_n^{i_n} \det(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_n}) \\ &= \varepsilon_{i_1 \dots i_n} v_1^{i_1} \dots v_n^{i_n}. \end{aligned} \quad (11)$$

128 The determinant of a matrix  $\llbracket a^i_j \rrbracket$  is defined as the determinant of the system of vectors  
 129  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  such that  $\llbracket a^i_j \rrbracket = \llbracket v^i_j \rrbracket$ . A discussion on the definition of determinant for the case  
 130 of second-order tensors belonging to  $\mathcal{V}^1_1$  (“mixed” tensors),  $\mathcal{V}^2_0$  (“contravariant” tensors)  
 131 and  $\mathcal{V}^0_2$  (“covariant” tensors) is given in [12].

132 As mentioned before, both spaces  $\Lambda_0(\mathcal{V}) \equiv \mathbb{R}$  and  $\Lambda_n(\mathcal{V})$  have dimension one. However,  
 133 whereas a scalar of  $\mathbb{R}$  is invariant under a change of basis in  $\mathcal{V}$ , an  $n$ -form is not. Indeed, if  
 134  $\{\mathbf{e}_i\}_{i=1}^n$  and  $\{\mathbf{e}'_j\}_{j=1}^n$  are two bases of  $\mathcal{V}$  related by

$$\mathbf{e}'_j = a^i_j \mathbf{e}_i, \quad \mathbf{e}_i = b^j_i \mathbf{e}'_j, \quad (12)$$

135 where the matrices  $\llbracket a^i_j \rrbracket$  and  $\llbracket b^j_i \rrbracket$  are one the inverse of the other, then the component of an  
 136  $n$ -form  $\boldsymbol{\mu}$  transforms according to

$$\begin{aligned} h' &= \boldsymbol{\mu}(\mathbf{e}'_1, \dots, \mathbf{e}'_n) = \boldsymbol{\mu}(a^{i_1}_1 \mathbf{e}_{i_1}, \dots, a^{i_n}_n \mathbf{e}_{i_n}) \\ &= a^{i_1}_1 \dots a^{i_n}_n \boldsymbol{\mu}(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_n}) \\ &= a^{i_1}_1 \dots a^{i_n}_n \varepsilon_{i_1 \dots i_n} \boldsymbol{\mu}(\mathbf{e}_1, \dots, \mathbf{e}_n) = \det \llbracket a^i_j \rrbracket h. \end{aligned} \quad (13)$$

137 Therefore, the component of an  $n$ -form transforms with a coefficient equal to the determinant  
 138 of the change of basis, and this is why it is often called a pseudo-scalar. Note that, under a  
 139 change of basis for which the determinant is equal to one (e.g. a proper orthogonal change of  
 140 basis), an  $n$ -form remains invariant, i.e., the difference between the component of an  $n$ -form  
 141 and the corresponding scalar is unnoticeable.

## 142 2.4 $(n-1)$ -Forms and Axial Vectors

143 In a vector space  $\mathcal{V}$  of dimension  $n$ , the space  $\Lambda_{n-1}(\mathcal{V})$  has dimension  $n$ , like  $\mathcal{V}^*$  and  $\mathcal{V}$  itself.  
 144 Because of this, with every  $(n-1)$ -form  $\boldsymbol{\omega} \in \Lambda_{n-1}(\mathcal{V})$ , it is possible to univocally associate  
 145 a vector  $\boldsymbol{u} \in \mathcal{V}$  with respect to a non-vanishing  $n$ -form  $\boldsymbol{\mu} \in \Lambda_n(\mathcal{V})$ , via the interior product

$$\mathbf{l}_u \boldsymbol{\mu} = \boldsymbol{\omega}. \quad (14)$$

146 The vector  $\boldsymbol{u}$  is called the axial vector of  $\boldsymbol{\omega}$  with respect to  $\boldsymbol{\mu}$  and, in elementary Algebra  
 147 and Mechanics, is called a pseudo-vector as it follows a transformation law that is different  
 148 from that of vectors, and it is obtained via the transformation followed by the corresponding  
 149  $(n-1)$ -form  $\boldsymbol{\omega}$ , which will be shown later in Section 4.

## 150 3 Kinematics of the Deformation

151 Here we report some definitions about the kinematics of deformation in Continuum Me-  
 152 chanics. The notation follows generally that of the treatise by Marsden and Hughes [18] and  
 153 that used in some previous works [12, 11]. The presentation is fairly standard, except for  
 154 the fact that, in order to illustrate the transformations in a more general case, we keep the  
 155 dimension  $n$  rather than going to the customary dimension 3.

### 156 3.1 Deformation and Configuration Map

157 We work on two  $n$ -dimensional manifolds: the body  $\mathcal{B}$  and the physical space  $\mathcal{S}$ . The con-  
 158 figuration map describing the deformation is, for the moment, assumed time-independent,  
 159 and is defined as an embedding (i.e., a differentiable map on whose image the inverse map  
 160 is defined and differentiable; with a subtle abuse of terminology, the configuration map is  
 161 often said to be a diffeomorphism)

$$\boldsymbol{\chi} : \mathcal{B} \rightarrow \mathcal{S} : X \mapsto x = \boldsymbol{\chi}(X). \quad (15)$$

162 Note that, if one wishes, it is possible to choose a reference configuration  $\boldsymbol{\chi}_R : \mathcal{B} \rightarrow \mathcal{B}_R \subset \mathcal{S}$   
 163 and then refer to the map  $\boldsymbol{\chi} \circ \boldsymbol{\chi}_R^{-1} : \mathcal{B}_R \rightarrow \mathcal{S}$ , which differs from (15) by a mere change of  
 164 coordinates [8]. Here, we prefer to follow the modern approach to Continuum Mechanics,  
 165 in which no particular reference configuration is chosen.

### 166 3.2 Physical Quantities

167 Material and spatial physical quantities are tensor fields of the type

$$\mathbb{A} : \mathcal{B} \rightarrow [T\mathcal{B}]'_s : X \mapsto \mathbb{A}(X), \quad (16)$$

$$\mathbb{A} : \boldsymbol{\chi}(\mathcal{B}) \rightarrow [T\mathcal{S}]'_s : x \mapsto \mathbb{A}(x), \quad (17)$$

168 respectively, where  $T\mathcal{B}$  and  $T\mathcal{S}$  are the tangent bundles of  $\mathcal{B}$  and  $\mathcal{S}$ , disjoint union of all  
 169 tangent spaces  $T_X\mathcal{B}$  and  $T_x\mathcal{S}$  at all points  $X \in \mathcal{B}$  and  $x \in \mathcal{S}$ , respectively. We also recall that  
 170 the dual spaces of the tangent spaces  $T_X\mathcal{B}$  and  $T_x\mathcal{S}$  are the cotangent spaces  $T_X^*\mathcal{B}$  and  $T_x^*\mathcal{S}$ ,  
 171 the disjoint unions of which are the cotangent bundles  $T^*\mathcal{B}$  and  $T^*\mathcal{S}$ .

## 172 3.3 Deformation Gradient, Push-Forward, Pull-Back

173 The deformation gradient  $\mathbf{F}$  is defined as the tangent map of  $\chi$ ,

$$T\chi = \mathbf{F} : T\mathcal{B} \rightarrow T\mathcal{S}, \quad (18)$$

174 such that, at each point  $X \in \mathcal{B}$ , the two-point tensor

$$(T\chi)(X) = \mathbf{F}(X) : T_X\mathcal{B} \rightarrow T_X\mathcal{S}, \quad (19)$$

175 is the the Frechét differential of  $\chi$  at  $X$ . Thus, in the coordinate charts  $\{\hat{X}^A\}$  in  $\mathcal{B}$  and  $\{\hat{x}^a\}$   
176 in  $\mathcal{S}$ , we have

$$F^a_{\ A}(X) = \chi^a_{\ ,A}(X). \quad (20)$$

177 Note that, as two-point tensor fields, the deformation gradient, its inverse, its transpose and  
178 its inverse transpose are defined as

$$\mathbf{F} : \mathcal{B} \rightarrow T\mathcal{S} \otimes T^*\mathcal{B}, \quad (21)$$

$$\mathbf{F}^{-1} : \chi(\mathcal{B}) \rightarrow T\mathcal{B} \otimes T^*\mathcal{S}, \quad (22)$$

$$\mathbf{F}^T : \chi(\mathcal{B}) \rightarrow T^*\mathcal{B} \otimes T\mathcal{S}, \quad (23)$$

$$\mathbf{F}^{-T} : \mathcal{B} \rightarrow T^*\mathcal{S} \otimes T\mathcal{B}, \quad (24)$$

179 respectively.

180 Given a material vector field  $\mathbf{U} : \mathcal{B} \rightarrow T\mathcal{B}$ , its push-forward is the spatial vector field  
181  $\mathbf{u} = \chi_*\mathbf{U} : \chi(\mathcal{B}) \rightarrow T\mathcal{S}$ , defined as

$$\mathbf{u} = \chi_*\mathbf{U} = (\mathbf{F}\mathbf{U}) \circ \chi^{-1}. \quad (25)$$

182 The inverse operation is called pull-back:

$$\mathbf{U} = \chi^*\mathbf{u} = (\mathbf{F}^{-1}\mathbf{u}) \circ \chi. \quad (26)$$

183 The push-forward of a material covector field (i.e., a one-form)  $\mathbf{\Pi} : \mathcal{B} \rightarrow T^*\mathcal{B}$  is the spatial  
184 covector field  $\boldsymbol{\pi} = \chi_*\mathbf{\Pi} : \chi(\mathcal{B}) \rightarrow T^*\mathcal{S}$ , defined via the pull-back of vector fields of Equation  
185 (26),

$$\begin{aligned} (\chi_*\mathbf{\Pi})\mathbf{u} &= [\mathbf{\Pi}(\chi^*\mathbf{u})] \circ \chi^{-1} \Rightarrow \\ (\boldsymbol{\pi} \circ \chi^{-1})(\mathbf{F}^{-1}\mathbf{u}) &= [(\mathbf{F}^{-T}\mathbf{\Pi}) \circ \chi^{-1}]\mathbf{u}, \end{aligned} \quad (27)$$

186 from which the push-forward rule is

$$\boldsymbol{\pi} = \chi_*\mathbf{\Pi} = (\mathbf{F}^{-T}\mathbf{\Pi}) \circ \chi^{-1}, \quad (28)$$

187 and therefore the pull-back rule is

$$\mathbf{\Pi} = \chi^*\boldsymbol{\pi} = (\mathbf{F}^T\boldsymbol{\pi}) \circ \chi. \quad (29)$$

188 The push-forward of tensor fields valued in  $[T\mathcal{B}]^r_s$  and the pull-back of tensors in  $[T\mathcal{S}]^r_s$  are  
189 obtained by performing pull-backs and push-forwards of each vector or covector leg of the  
190 tensor. The next section specialises the pull-back transformation laws to differential forms,  
191 i.e., fields valued in spaces of  $r$ -forms and, in particular, to the cases of differential  $n$ -forms  
192 and  $(n-1)$ -forms in the context of the Theory of Integration.

## 193 4 Differential Forms on Manifolds

194 A spatial differential form of order  $r \leq n$  in the  $n$ -dimensional manifold  $\mathcal{S}$  is an  $r$ -form-valued  
195 field on  $\mathcal{S}$ , i.e., a mapping

$$\boldsymbol{\beta} : \mathcal{S} \rightarrow \Lambda_r(T\mathcal{S}) : x \mapsto \boldsymbol{\beta}(x). \quad (30)$$

196 The definition of material forms (on  $\mathcal{B}$ ) is analogous. An  $r$ -differential form can be called,  
197 with a slight abuse of terminology, an  $r$ -form, whenever there is no danger of confusion  
198 between the field  $\boldsymbol{\beta}$  and its value  $\boldsymbol{\beta}(x)$ .

199 In this section we describe the role of  $r$ -forms in the Theory of Integration on manifolds  
200 and enunciate the theorem of the change of variables in integrals, before going to the crucial  
201 point of this work: the transformation rules of integrals of  $n$ -forms and  $(n-1)$ -forms.

### 202 4.1 Differential Forms and Integration

203 Differential  $r$ -forms on an  $n$ -dimensional manifold  $\mathcal{S}$  are intimately connected with the the-  
204 ory of integration in that they induce a measure on sub-manifolds (i.e., subsets of  $\mathcal{S}$  possess-  
205 ing the structure of manifold by themselves) of the same order  $r$ . Here we are interested in  
206 the case of  $n$ -forms and  $(n-1)$ -forms on an  $n$ -dimensional manifold.

207 A non-vanishing  $n$ -form  $\boldsymbol{\theta} : \mathcal{S} \rightarrow \Lambda_n(T\mathcal{S})$ , also called a volume form, is a volume in-  
208 tegrand on the manifold itself and represents the density of a certain extensive quantity  $q$ .  
209 Therefore, the integral of  $\boldsymbol{\theta}$  is the extent of  $q$  over an  $n$ -dimensional submanifold  $\mathcal{C} \subset \mathcal{S}$ :

$$\text{Extent}(q, \mathcal{C}) = \int_{\mathcal{C}} \boldsymbol{\theta}. \quad (31)$$

210 In a given chart, an  $n$ -form is uniquely determined by its single scalar component. Therefore,  
211 often, one takes a suitable volume form  $\boldsymbol{\mu}$  to calculate the physical volume of  $\mathcal{C}$ , and then  
212 derives any other volume form via multiplication by a non-vanishing function  $\rho$ , exactly  
213 like in measure theory (see, e.g., [22]), so that

$$\text{Volume}(\mathcal{C}) = \int_{\mathcal{C}} \boldsymbol{\mu}, \quad \text{Extent}(q, \mathcal{C}) = \int_{\mathcal{C}} \rho \boldsymbol{\mu}, \quad (32)$$

214 which would read  $\int_{\mathcal{C}} d\mathbf{v}$  and  $\int_{\mathcal{C}} \rho d\mathbf{v}$  in the traditional formalism.

215 Similarly, an  $(n-1)$ -form  $\boldsymbol{\omega} : \mathcal{S} \rightarrow \Lambda_{n-1}(T\mathcal{S})$  is an integrand on a hypersurface  $s \subset \mathcal{S}$   
216 and its integral represents the flux of an extensive quantity  $q$  across the hypersurface  $s$ :

$$\text{Flux}(q, s) = \int_s \boldsymbol{\omega}. \quad (33)$$

217 When a metric tensor  $\mathbf{g}$  (i.e., a symmetric and positive-definite tensor field valued in  
218  $[T\mathcal{S}]_2^0$ ) is available in  $\mathcal{S}$ , the integral of an  $(n-1)$ -form  $\boldsymbol{\omega}$  on a surface  $s$  can be expressed  
219 in terms of the axial vector field  $\mathbf{w}$  of  $\boldsymbol{\omega}$  with respect to the volume form  $\boldsymbol{\mu}$ , i.e.,  $\mathbf{w}$  is such  
220 that  $\boldsymbol{\iota}_{\mathbf{w}}\boldsymbol{\mu} = \boldsymbol{\omega}$ . Indeed, the metric  $\mathbf{g}$  allows for the definition of the normal covector  $\mathbf{n}$  to  $s$   
221 (such that its squared norm is  $\|\mathbf{n}\|^2 = \mathbf{n} \cdot \mathbf{n} = \langle \mathbf{n}, \mathbf{n} \rangle = n_a g^{ab} n_b = 1$ ) and of the associated  
222 normal vector  $\mathbf{n}^\sharp = \mathbf{g}^{-1}\mathbf{n}$  (with components  $n^a = g^{ab} n_b$ ). Exploiting the identity  $\mathbf{i}^T = \mathbf{n} \mathbf{n}^\sharp$  (in  
223 components,  $n_a n^b = \delta_a^b$ , which are the components of the transpose of the spatial identity  
224 tensor  $\mathbf{i}$ ), the  $(n-1)$ -form  $\boldsymbol{\omega}$  can be written

$$\begin{aligned} \boldsymbol{\omega} &= \boldsymbol{\iota}_{\mathbf{w}}\boldsymbol{\mu} = \boldsymbol{\iota}_{[\mathbf{w}\mathbf{i}^T]}\boldsymbol{\mu} = \boldsymbol{\iota}_{[\mathbf{w}\mathbf{n}\mathbf{n}^\sharp]}\boldsymbol{\mu} \\ &= (\mathbf{w}\mathbf{n}) \boldsymbol{\iota}_{\mathbf{n}^\sharp}\boldsymbol{\mu} = (\mathbf{w}\mathbf{n}) \boldsymbol{\alpha} \end{aligned} \quad (34)$$

225 where  $\mathbf{wn} \equiv \langle \mathbf{w} | \mathbf{n} \rangle = w^a n_a$  is the contraction of the vector  $\mathbf{w}$  and the covector  $\mathbf{n}$ , and

$$\boldsymbol{\alpha} = \mathbf{l}_{n^\sharp} \boldsymbol{\mu} \quad (35)$$

226 is the  $(n - 1)$ -form induced by the volume form  $\boldsymbol{\mu}$  and the metric  $\mathbf{g}$  on the hypersurface  
227  $s$ . With this definition, the flux of an extensive quantity  $q$  across  $s$  can be expressed in the  
228 alternative notation

$$\text{Flux}(q, s) = \int_s (\mathbf{wn}) \boldsymbol{\alpha} \equiv \int_s \mathbf{wn}. \quad (36)$$

229 In the traditional notation, the flux reads  $\int_s \mathbf{wn} da$  or  $\int_s \mathbf{w} da$ , where  $da$  is the “element of  
230 area” inclusive of the normal  $\mathbf{n}$ .

## 231 4.2 Theorem of the Change of Variables

232 In the context of Continuum Mechanics, the theorem of the change of variables in integrals  
233 is used to transform integrals from the spatial to the material picture. If  $\chi : \mathcal{B} \rightarrow \mathcal{S}$  is a  
234 configuration,  $\mathcal{D}$  is an  $r$ -dimensional sub-manifold of the  $n$ -dimensional body manifold  $\mathcal{B}$ ,  
235 and  $\chi(\mathcal{D})$  is its image through  $\chi$ , the spatial integral of the  $r$ -form  $\boldsymbol{\beta}$  transforms according  
236 to

$$\int_{\chi(\mathcal{D})} \boldsymbol{\beta} = \int_{\mathcal{D}} \chi^* \boldsymbol{\beta}, \quad (37)$$

237 where the “change of variables” is precisely the configuration  $\chi$ . Therefore, the need arises  
238 to calculate the pull-backs of differential forms. In the jargon of Continuum Mechanics,  
239 the pull-backs of volume forms and  $(n - 1)$ -forms are called Piola transformations, a ter-  
240 minology that actually refers to the pseudo-scalar and the pseudo-vector quantities they are  
241 associated with, respectively.

## 242 4.3 Change of Variables: Volume Forms

243 Let  $\boldsymbol{\mu} : \mathcal{S} \rightarrow \Lambda_n(T\mathcal{S})$  and  $\boldsymbol{\mathcal{M}} : \mathcal{B} \rightarrow \Lambda_n(T\mathcal{B})$  be a spatial and a material volume form. If  
244  $\{\mathbf{e}_a\}_{a=1}^n$  and  $\{\mathbf{E}_A\}_{A=1}^n$  are the bases induced by the coordinate charts  $\{\hat{x}^a\}$  and  $\{\hat{X}^A\}$ , re-  
245 spectively, the component forms of  $\boldsymbol{\mu}$  and  $\boldsymbol{\mathcal{M}}$  read

$$\boldsymbol{\mu} = h \varepsilon_{a_1 \dots a_n} \mathbf{e}^{a_1} \otimes \dots \otimes \mathbf{e}^{a_n}, \quad (38)$$

$$\boldsymbol{\mathcal{M}} = H \varepsilon_{A_1 \dots A_n} \mathbf{E}^{A_1} \otimes \dots \otimes \mathbf{E}^{A_n}. \quad (39)$$

246 The pull-back  $\chi^* \boldsymbol{\mu}$  can be calculated explicitly, as

$$\begin{aligned} \chi^* \boldsymbol{\mu} &= (h \circ \chi) \varepsilon_{a_1 \dots a_n} (\mathbf{F}^T \mathbf{e}^{a_1}) \circ \chi \otimes \dots \otimes (\mathbf{F}^T \mathbf{e}^{a_n}) \circ \chi \\ &= (h \circ \chi) \varepsilon_{a_1 \dots a_n} F^{a_1}_{A_1} \dots F^{a_n}_{A_n} \mathbf{E}^{A_1} \otimes \dots \otimes \mathbf{E}^{A_n} \\ &= (h \circ \chi) \det[[F^a_A]] \varepsilon_{A_1 \dots A_n} \mathbf{E}^{A_1} \otimes \dots \otimes \mathbf{E}^{A_n}, \end{aligned} \quad (40)$$

247 i.e.,  $\chi^* \boldsymbol{\mu}$  is the volume form on  $\mathcal{B}$  with independent component  $(h \circ \chi) \det[[F^a_A]]$ . We remark  
248 that the determinant  $\det[[F^a_A]]$  is *not* a scalar invariant of the deformation gradient  $\mathbf{F}$  [18], as

249 it depends on the choice of the coordinate charts  $\{\hat{x}^a\}$  and  $\{\hat{X}^a\}$ . Therefore, it is convenient  
250 to express the pull-back  $\chi^*\boldsymbol{\mu}$  in terms of the material volume form  $\mathcal{M}$  [12], as

$$\begin{aligned}\chi^*\boldsymbol{\mu} &= (h \circ \chi) \det[[F^a_A]] \frac{1}{H} H \varepsilon_{A_1 \dots A_n} \mathbf{E}^{A_1} \otimes \dots \otimes \mathbf{E}^{A_n} \\ &= (h \circ \chi) \det[[F^a_A]] \frac{1}{H} \mathcal{M} \\ &= J \mathcal{M},\end{aligned}\tag{41}$$

251 where

$$J = \det \mathbf{F} \equiv (h \circ \chi) \det[[F^a_A]] \frac{1}{H}\tag{42}$$

252 is indeed a scalar invariant, which is *defined* as the determinant of the two-point tensor  $\mathbf{F}$   
253 with respect to the volume form  $\boldsymbol{\mu}$  on  $\mathcal{S}$  and the volume form  $\mathcal{M}$  on  $\mathcal{B}$  (see [12], which also  
254 reports an expression of Equation (42) for the case in which metric tensors are available  
255 and the corresponding induced volume forms are employed). In practice, it can be fairly  
256 easily shown that, since  $h$  and  $H$  transform according to Equation (13), the factor  $(h \circ \chi)/H$   
257 makes  $J$  an invariant. Therefore, for the case of volume forms, the theorem of the change of  
258 variables can be expressed as

$$\int_{\chi(\mathcal{B})} \boldsymbol{\mu} = \int_{\mathcal{B}} \chi^*\boldsymbol{\mu} = \int_{\mathcal{B}} J \mathcal{M},\tag{43}$$

259 which in the traditional notation reads  $dv = J dV$ .

#### 260 4.4 Change of Variables: $(n-1)$ -Forms

261 Let  $\boldsymbol{\mu}$  and  $\mathcal{M}$  be a spatial and a material volume form as above,  $S$  a hypersurface in  $\mathcal{B}$ ,  
262  $s = \chi(S)$  its image in  $\mathcal{S}$  through the configuration  $\chi$ , and  $\boldsymbol{\omega} : s \rightarrow \Lambda_{n-1}(T\mathcal{S})$  a spatial  $(n-1)$ -  
263 form, with axial vector  $\mathbf{w}$  with respect to the volume form  $\boldsymbol{\mu}$ , i.e.,  $\mathbf{w}$  is such that  $\mathbf{i}_{\mathbf{w}}\boldsymbol{\mu} = \boldsymbol{\omega}$ .  
264 The pull-back of  $\boldsymbol{\omega}$  is obtained in terms of its axial vector  $\mathbf{w}$ , by exploiting the distributivity  
265 of the pull-back operation and the fact that the interior product of a vector and an  $r$ -form is  
266 merely the contraction of the vector with the first leg of the  $r$ -form. Indeed,

$$\begin{aligned}\chi^*\boldsymbol{\omega} &= \chi^*[\mathbf{i}_{\mathbf{w}}\boldsymbol{\mu}] = \mathbf{i}_{[\chi^*\mathbf{w}]} \chi^*\boldsymbol{\mu} = \mathbf{i}_{[(F^{-1}\mathbf{w}) \circ \chi]} J \mathcal{M} \\ &= \mathbf{i}_{[J(\mathbf{w} \circ \chi) \mathbf{F}^{-T}]} \mathcal{M} = \mathbf{i}_{\mathbf{W}} \mathcal{M} = \boldsymbol{\Omega},\end{aligned}\tag{44}$$

267 where

$$\mathbf{W} = J(\mathbf{w} \circ \chi) \mathbf{F}^{-T} = J(\mathbf{F}^{-1}\mathbf{w}) \circ \chi\tag{45}$$

268 is called the Piola transform of  $\mathbf{w}$ , and is the axial vector of the pulled-back  $(n-1)$ -form  
269  $\boldsymbol{\Omega} = \chi^*\boldsymbol{\omega} : S \rightarrow \Lambda_{n-1}(T\mathcal{B})$  with respect to the material volume form  $\mathcal{M}$ .

270 If metric tensors  $\mathbf{g}$  and  $\mathbf{G}$  are available in  $\mathcal{S}$  and  $\mathcal{B}$ , the normal covectors  $\mathbf{n}$  and  $\mathbf{N}$  and  
271 the associated normal vectors  $\mathbf{n}^\sharp$  and  $\mathbf{N}^\sharp$  can be defined on  $s$  and  $S$ . Therefore, following  
272 Equation (35), it is possible to define the  $(n-1)$ -forms  $\boldsymbol{\alpha} = \mathbf{i}_{\mathbf{n}^\sharp}\boldsymbol{\mu}$  and  $\boldsymbol{\mathcal{A}} = \mathbf{i}_{\mathbf{N}^\sharp}\mathcal{M}$  induced  
273 on  $s$  by  $\boldsymbol{\mu}$  and on  $S$  by  $\mathcal{M}$ . Therefore, using the pull-back rules for  $(n-1)$ -forms in Equation

274 (44) and for their axial vectors in Equation (45), the theorem of the change of variables (37)  
275 takes the form

$$\begin{aligned} \int_{\chi(S)} \boldsymbol{\omega} &= \int_{\chi(S)} \mathbf{l}_w \boldsymbol{\mu} = \int_{\chi(S)} (\mathbf{w} \mathbf{n}) \boldsymbol{\alpha} = \int_S (\mathbf{W} \mathbf{N}) \mathcal{A} \\ &= \int_S [J(\mathbf{w} \circ \chi) \mathbf{F}^{-T} \mathbf{N}] \mathcal{A} = \int_S \mathbf{l}_w \mathcal{M} = \int_S \boldsymbol{\Omega} = \int_S \chi^* \boldsymbol{\omega} \end{aligned} \quad (46)$$

276 from which, omitting the  $(n-1)$ -forms  $\boldsymbol{\alpha}$  and  $\mathcal{A}$ , as in Equation (36),

$$\int_{\chi(S)} \mathbf{w} \mathbf{n} = \int_S J(\mathbf{w} \circ \chi) \mathbf{F}^{-T} \mathbf{N} = \int_S \mathbf{W} \mathbf{N}. \quad (47)$$

277 In the traditional notation, this becomes Nanson's formula, which can be alternatively writ-  
278 ten  $\mathbf{n} \, da = J \mathbf{F}^{-T} \mathbf{N} \, dA$  or, by including the normal corresponding to each "area element",  
279  $d\mathbf{a} = J \mathbf{F}^{-T} d\mathbf{A}$ .

## 280 5 Application to First-Order Transport Laws

281 A first-order transport law is most often used in Physics and Mechanics to formalise a con-  
282 stitutive relation between a flux density and a generalised force density. Naturally, higher-  
283 order laws are always possible in principle, and sometimes necessary to represent certain  
284 phenomena. However, a first-order law is sufficient for a vast range of phenomena, and has  
285 the objective advantage of a simpler mathematical structure. The general structure of a first-  
286 order linear transport law is usually written

$$\mathbf{w} = \mathbf{k} \mathbf{h}, \quad (48)$$

287 where the spatial vector field  $\mathbf{w}$ , valued in  $T\mathcal{S}$ , is the flux density or current density of a  
288 certain extensive quantity  $q$ , the second-order tensor field  $\mathbf{k}$ , valued in  $[T\mathcal{S}]_0^2$  (i.e., a "con-  
289 travariant" tensor) is a permittivity, and the covector field  $\mathbf{h}$ , valued in  $T^*\mathcal{S}$ , is a generalised  
290 force density. The flux density  $\mathbf{w}$  may, or may not, be given by the product of a (pseudos-  
291 calar) density and a vector field.

292 As seen in Section 4.1, in the three-dimensional space  $\mathcal{S}$ , the flux density  $\mathbf{w}$  is nothing  
293 but the axial vector of a two-form  $\boldsymbol{\omega}$  with respect to a volume form  $\boldsymbol{\mu}$ , via the interior product  
294  $\boldsymbol{\omega} = \mathbf{l}_w \boldsymbol{\mu}$ . Therefore, in terms of forms, Equation (48) reads

$$\boldsymbol{\omega} = \mathbf{k} \mathbf{h}, \quad (49)$$

295 where the  $[T\mathcal{S}]_0^2$ -valued tensor field  $\mathbf{k}$  has been replaced by the tensor field  $\mathbf{k}$ , valued in  
296  $\Lambda_2(T\mathcal{S}) \otimes T\mathcal{S}$ , i.e., the first vector leg of  $\mathbf{k}$  has been replaced by a two-form in the definition of  
297  $\mathbf{k}$ . If we define the third-order tensor field  $\mathbf{l}_\mu$ , valued in  $\Lambda_2(T\mathcal{S}) \otimes T^*\mathcal{S}$ , via the isomorphism  
298  $\mathbf{l}_\mu \mathbf{w} = \mathbf{l}_w \boldsymbol{\mu}$ , then  $\mathbf{k}$  is given by contracting the last leg (the covector leg) of  $\mathbf{l}_\mu$  with the first  
299 leg of  $\mathbf{k}$ , i.e.,

$$\mathbf{k} = \mathbf{l}_\mu \mathbf{k}. \quad (50)$$

300 In components, if  $\mathbf{k} = k^{ab} \mathbf{e}_a \otimes \mathbf{e}_b$  and  $\mathbf{h} = h_b \mathbf{e}^b$ ,

$$\boldsymbol{\omega} = \mathbf{l}_w \boldsymbol{\mu} = \mathbf{l}_{[\mathbf{k} \mathbf{h}]} \boldsymbol{\mu} = \mathbf{l}_{[k^{ab} h_b \mathbf{e}_a]} \boldsymbol{\mu} = k^{ab} (\mathbf{l}_{\mathbf{e}_a} \boldsymbol{\mu}) h_b = k^{ab} (\mathbf{l}_{\mathbf{e}_a} \boldsymbol{\mu}) \mathbf{e}_b(\mathbf{h}) = [k^{ab} (\mathbf{l}_{\mathbf{e}_a} \boldsymbol{\mu}) \otimes \mathbf{e}_b] \mathbf{h}, \quad (51)$$

301 from which

$$\mathbf{k} = k^{ab} (\mathbf{l}_{\mathbf{e}_a} \boldsymbol{\mu}) \otimes \mathbf{e}_b. \quad (52)$$

Therefore, not only  $\boldsymbol{\omega}$ , but also the first leg of the tensor  $\mathbf{k}$  transforms like a two-form. In the corresponding vectorial equation (48), the Piola transformation reads

$$J(\mathbf{w} \circ \chi) \mathbf{F}^{-T} = [J(\mathbf{F}^{-1} \mathbf{k}) \circ \chi] \mathbf{F}^{-T} (\mathbf{F}^T \mathbf{h}) \circ \chi, \quad (53)$$

where

$$\mathbf{W} = J(\mathbf{w} \circ \chi) \mathbf{F}^{-T}, \quad \mathbf{K} = [J(\mathbf{F}^{-1} \mathbf{k}) \circ \chi] \mathbf{F}^{-T}, \quad \mathbf{H} = (\mathbf{F}^T \mathbf{h}) \circ \chi, \quad (54)$$

are the material flux density (Piola transform of the spatial flux density), the material permittivity (Piola transform on the first leg and pull-back on the second leg of the spatial permittivity) and the material generalised force density (pull-back of the spatial generalised force density). The material equations (54) have been shown for the case of Darcy's law [23, 17, 3, 15, 1, 16, 25, 11], as well as for the analogous case of the polarisation of a dielectric [9, 27, 26, 5, 19, 6], in the traditional manner, without the use of differential forms.

In some cases, the flux density pseudo-vector  $\mathbf{w}$  may be given as the product of a pseudo-scalar density  $\rho$  times a proper vector field  $\mathbf{v}$ , i.e.,

$$\mathbf{w} = \rho \mathbf{v}, \quad (55)$$

which, in terms of the associated two- and three-forms, reads

$$\boldsymbol{\omega} = \mathbf{l}_w \boldsymbol{\mu} = \mathbf{l}_{[\rho \mathbf{v}]} \boldsymbol{\mu} = \mathbf{l}_v (\rho \boldsymbol{\mu}). \quad (56)$$

Therefore, if the volume form  $\boldsymbol{\mu}$  is thought to be associated with a measure of physical volume, the new three-form  $\rho \boldsymbol{\mu}$  is the density of a certain extensive quantity  $q$ . The extent of the quantity  $q$  in the body  $\mathcal{B}$  is, thus,

$$\text{Extent}(q, \chi(\mathcal{B})) = \int_{\chi(\mathcal{B})} \rho \boldsymbol{\mu} = \int_{\mathcal{B}} \chi^* [\rho \boldsymbol{\mu}] = \int_{\mathcal{B}} (\rho \circ \chi) (J \mathcal{M}) = \int_{\mathcal{B}} J(\rho \circ \chi) \mathcal{M}, \quad (57)$$

which, in the traditional notation, reads  $\int_{\chi(\mathcal{B})} \rho d\mathbf{v} = \int_{\mathcal{B}} J(\rho \circ \chi) dV$ . Furthermore, the flux of the extensive quantity  $q$  across a material surface  $S$ , with image  $s = \chi(S)$ , can be written in terms of the two-form  $\boldsymbol{\omega}$ , as in the standard case described by Equation (33), with the pull-back shown in Equation (46),

$$\text{Flux}(q, \chi(S)) = \int_{\chi(S)} \boldsymbol{\omega} = \int_S \chi^* \boldsymbol{\omega} = \int_S \boldsymbol{\Omega}, \quad (58)$$

or, whenever metric tensors  $\mathbf{G}$  and  $\mathbf{g}$  are available in  $\mathcal{B}$  and  $S$ , and the normal covectors to  $S$  and  $s = \chi(S)$  can be defined, the flux can be written in terms of pseudo-vectors as in Equation (46), i.e.,

$$\text{Flux}(q, \chi(S)) = \int_{\chi(S)} \mathbf{w} \mathbf{n} = \int_S J(\mathbf{w} \circ \chi) \mathbf{F}^{-T} \mathbf{N} = \int_S \mathbf{W} \mathbf{N}, \quad (59)$$

where we have omitted writing the two-forms  $\mathcal{A}$  and  $\boldsymbol{\alpha}$ , induced by  $\mathbf{G}$  and  $\mathbf{g}$  on  $S$  and  $s = \chi(S)$ .

The two cases that we report as an example are Darcy's law for fluid filtration in a porous medium and Ohm's law for the conduction of charges in an electrical conductor. We chose these two cases because their flux densities both have pseudo-vectors expressible as the product of a pseudo-scalar density and a proper vector. Because of this analogy, and of the fact that it is well-established to study electromagnetism in terms of forms (see, e.g., [14, 18, 24]), it is interesting to report a treatment of Darcy's law too in this geometric formalism.

332 In Darcy's law,

$$\mathbf{w} = \phi_f (\mathbf{v}_f - \mathbf{v}_s) = \mathbf{k} \mathbf{h} = -\mathbf{k} (\text{grad } p - \rho_{fT} \mathbf{f}), \quad (60)$$

333 the flux density  $\mathbf{w} = \phi_f (\mathbf{v}_f - \mathbf{v}_s)$  is the filtration velocity, obtained as the product of the fluid  
 334 volumetric fraction  $\phi_f$  (pseudo-scalar density) and the velocity  $\mathbf{v}_f - \mathbf{v}_s$  of the fluid relative  
 335 to the solid (proper vector),  $\mathbf{k}$  is the permeability tensor (function of fluid viscosity and fluid  
 336 volumetric fraction), and the generalised force density  $\mathbf{h} = -(\text{grad } p - \rho_{fT} \mathbf{f})$  is comprised  
 337 of the negative of the gradient of the pore pressure  $p$  and the body force term (usually  
 338 gravity).

339 In Ohm's law,

$$\mathbf{j} = \rho \mathbf{v} = \boldsymbol{\kappa} \mathbf{e} = -\boldsymbol{\kappa} \text{grad } \varphi, \quad (61)$$

340 the flux density  $\mathbf{w} \equiv \mathbf{j} = \rho \mathbf{v}$  is the current density, obtained as the product of the charge  
 341 density  $\rho$  (pseudo-scalar density) and the velocity  $\mathbf{v}$  of the charges,  $\mathbf{k} \equiv \boldsymbol{\kappa}$  is the conductivity  
 342 tensor, and the generalised force density  $\mathbf{h} \equiv \mathbf{e}$  is the electric field, given by the negative of  
 343 the gradient of the scalar potential  $\varphi$ .

344 **Remark.** We observe that, in the work of Noll (e.g., [21]), a body is viewed as a differenti-  
 345 able manifold, the physical space is modelled as a Euclidean space (i.e., an affine space  
 346 with a metric), and configurations are viewed as charts of the body manifold. Thus, there is  
 347 no preferred reference configuration for the body in space. In this context, the flux field is  
 348 represented by a single  $(n - 1)$ -form on the body manifold independently of any configura-  
 349 tion, and the various spatial fields associated with that form are simply different representa-  
 350 tions of a single geometric object.

## 351 6 Summary

352 In this work, we gave an overview of the tools of Differential Geometry needed for the  
 353 description of those quantities that, in elementary Physics and Mechanics, are called pseudo-  
 354 vectors and pseudo-scalars. The nature of these objects and the transformation laws they  
 355 obey is completely unveiled if they are described as two-forms and three-forms, respectively,  
 356 in the three-dimensional space of Classical Mechanics. In particular, we studied the case in  
 357 which the integration of a two-form over a surface represents the flux of a certain extensive  
 358 physical quantity across that surface. As an example of application to first-order transport  
 359 laws, we reported Darcy's law for fluid filtration in a porous medium and Ohm's law for the  
 360 conduction of a current in a conductor. This work contributes to the path towards a unified  
 361 geometrical formalism in Continuum Mechanics, within which it is possible to represent  
 362 different physical phenomena sharing the same mathematical structure, and to rigorously  
 363 describe the transformation laws which the various physical quantities at play must obey.

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## References

- 369 1. Ateshian, G.A., Weiss, J.A.: Anisotropic hydraulic permeability under finite deformation. *J. Biomech. Eng.* **132**, 111,004 (2010)
- 370 2. Auffray, N., dell'Isola, F., Eremeyev, V.A., Madeo, A., Rosi, G.: Analytical continuum mechanics à la Hamilton-Piola: least action principle for second gradient continua and capillary fluids. *Math. Mech. Solids* **20**, 375–417 (2015)
- 371 3. dell'Isola, F., Madeo, A., Seppecher, P.: Boundary conditions at fluid-permeable interfaces in porous media: A variational approach. *Int. J. Solids Struct.* **46**, 3150–3164 (2009)
- 372 4. dell'Isola, F., Maier, G., Perego, U., Andreaus, U., Esposito, R., Forest, S. (eds.): *The Complete Works of Gabrio Piola: Volume I*. Springer, Berlin, Germany (2014)
- 373 5. Dorfmann, A., Ogden, R.W.: Nonlinear electroelasticity. *Acta Mech.* **174**, 167–183 (2005)
- 374 6. Edmiston, J., Steigmann, D.J.: Analysis of nonlinear electroelastic membranes. In: R.W. Ogden, D.J. Steigmann (eds.) *Mechanics and Electrodynamics of Magneto- and Electro-Elastic Materials*, pp. 153–180. Springer, Berlin (2011). CISM Courses and Lectures No. 527, International Centre for Mechanical Sciences
- 375 7. Epstein, M.: *The Geometrical Language of Continuum Mechanics*. Cambridge University Press, Cambridge, UK (2010)
- 376 8. Epstein, M., Elżanowski, M.: *Material Inhomogeneities and Their Evolution*. Springer, Berlin, Germany (2007)
- 377 9. Eringen, A.C., Maugin, G.A.: *Electrodynamics of Continua I*. Springer, Berlin, Germany (1990)
- 378 10. Federico, S.: Covariant formulation of the tensor algebra of non-linear elasticity. *Int. J. Non-Linear Mech.* **47**, 273–284 (2012)
- 379 11. Federico, S.: Porous materials with statistically oriented reinforcing fibres. In: L. Dorfmann, R.W. Ogden (eds.) *Nonlinear Mechanics of Soft Fibrous Materials*, pp. 49–120. Springer, Berlin (2015). CISM Courses and Lectures No. 559, International Centre for Mechanical Sciences
- 380 12. Federico, S.: Some remarks on metric and deformation. *Math. Mech. Solids* **20**, 522–539 (2015)
- 381 13. Federico, S., Grillo, A.: Elasticity and permeability of porous fibre-reinforced materials under large deformations. *Mech. Mat.* **44**, 58–71 (2012)
- 382 14. Felsager, B.: *Geometry, Particles and Fields*. Springer, Berlin, Germany (1989)
- 383 15. Grillo, A., Zingali, G., Borrello, G., Federico, S., Herzog, W., Giaquinta, G.: A multiscale description of growth and transport in biological tissues. *Theor. Appl. Mech.* **34**, 51–87 (2007)
- 384 16. Grillo, A., Federico, S., Wittum, G.: Growth, mass transfer, and remodeling in fiber-reinforced, multi-constituent materials. *Int. J. Non-Linear Mech.* **47**, 388–401 (2012)
- 385 17. Madeo, A., dell'Isola, F., Ianiro, N., Sciarra, G.: A variational deduction of second gradient poroelasticity II: an application to the consolidation problem. *J. Mech. Mater. Struct.* **3**, 607–625 (2008)
- 386 18. Marsden, J.E., Hughes, T.J.R.: *Mathematical Foundations of Elasticity*. Prentice-Hall, Englewood Cliff, NJ, USA (1983)
- 387 19. Maugin, G.A.: *Configurational Forces: Thermomechanics, Physics, Mathematics, and Numerics*. CRC Press, Boca Raton, FL, USA (2011)
- 388 20. Maugin, G.A.: *Continuum Mechanics Through the Twentieth Century: A Concise Historical Perspective*. Springer, Berlin, Germany (2013)
- 389 21. Noll, W.: Materially uniform simple bodies with inhomogeneities. *Arch. Rat. Mech. Anal.* **27**, 1–32 (1967)
- 390 22. Rudin, W.: *Principles of Mathematical Analysis*, 3rd edn. McGraw-Hill, New York, USA (1976)
- 391 23. Sciarra, G., dell'Isola, F., Ianiro, N., Madeo, A.: A variational deduction of second gradient poroelasticity I: General theory, journal of mechanics of materials and structures. *J. Mech. Mater. Struct.* **3**, 507–526 (2008)
- 392 24. Segev, R.: Notes on metric independent analysis of classical fields. *Math. Meth. Appl. Sci.* **36**, 497–566 (2013)
- 393 25. Tomic, A., Grillo, A., Federico, S.: Poroelastic materials reinforced by statistically oriented fibres - numerical implementation and application to articular cartilage. *IMA J. Appl. Math.* **79**, 1027–1059 (2014)
- 394 26. Trimarco, C.: A Lagrangian approach to electromagnetic bodies. *Tech. Mech.* **22**, 175–180 (2002)
- 395 27. Trimarco, C., Maugin, G.A.: Material mechanics of electromagnetic bodies. In: R. Kienzler, G.A. Maugin (eds.) *Configurational Mechanics of Materials*, pp. 129–172. Springer, Berlin (2001). CISM Courses and Lectures No. 427, International Centre for Mechanical Sciences