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# The Linear Elasticity Tensor of Incompressible Materials

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*Dedicated to Prof. Raymond W. Ogden, on occasion of his 70th birthday*

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## Abstract

With a universally accepted abuse of terminology, materials having much larger stiffness for volumetric than for shear deformations are called *incompressible*. This work proposes two approaches to the evaluation of the correct form of the linear elasticity tensor of the so-called incompressible materials, both stemming from the non-linear theory. In the approach of strict incompressibility, one imposes the kinematical constraint of isochoric deformation. In the approach of quasi-incompressibility, which is often employed to enforce incompressibility in numerical applications such as the Finite Element Method, one instead assumes a decoupled form of the elastic potential (or strain energy), which is written as the sum of a function of the volumetric deformation only and a function of the distortional deformation only, and then imposes that the bulk modulus be much larger than all other moduli. The conditions which the elasticity tensor has to obey for both strict incompressibility and quasi-incompressibility have been derived, regardless of the material symmetry. The representation of the linear elasticity tensor for the quasi-incompressible case differs from that of the strictly incompressible case by one parameter, which can be conveniently chosen to be the bulk modulus. Some important symmetries have been studied in detail, showing that the linear elasticity tensor for the cases of isotropy, transverse isotropy and orthotropy is characterised by 1, 3, 6 independent parameters, respectively, for the case of strict incompressibility, and 2, 4, 7 independent parameters, respectively, for the case of quasi-incompressibility, as opposed to the 2, 5, 9 parameters, respectively, of the general compressible case.

**Keywords:** covariant representation, Elasticity, elasticity tensor, incompressibility, quasi-incompressibility, incompressible, quasi-incompressible, nearly-incompressible, material symmetry, anisotropy.

# 27 1 Introduction

28 In several contexts of Continuum Mechanics, and particularly for materials such as elas-  
29 tomers and soft biological tissues, whose stiffness under volumetric compression is usually  
30 several orders of magnitude higher than the stiffness in shear, the mechanical behaviour  
31 of materials is studied under the assumption of either strict incompressibility or quasi-  
32 incompressibility. The constraint of isochoric (i.e., volume-preserving) deformation is often  
33 employed to approximate the behaviour of incompressible materials. To be more precise,  
34 we recall here that an idealised material body is said to be strictly incompressible when  
35 the substantial derivative of its mass density vanishes identically [1], i.e., when  $D_t \varrho = 0$ ,  
36 with  $\varrho$  being the mass density of the body, and  $D_t$  the substantial derivative operator. In  
37 the case in which the mass of the body is locally conserved, the mass balance law of the  
38 body reads

$$D_t \varrho + \varrho \operatorname{div}(\mathbf{v}) = 0, \quad (1)$$

39 where  $\mathbf{v}$  is the velocity. Thus, setting  $D_t \varrho$  equal to zero implies that Equation (1) reduces  
40 to

$$\operatorname{div}(\mathbf{v}) = 0, \quad (2)$$

41 in which case the velocity field is said to be divergence-free. Since the divergence of the  
42 velocity field is related to the time derivative of the volume ratio  $J = \det \mathbf{F}$  (where  $\mathbf{F}$  is  
43 the deformation gradient) by

$$\dot{J} = J \operatorname{div}(\mathbf{v}), \quad (3)$$

44 the vanishing of  $\operatorname{div}(\mathbf{v})$  implies that the volume ratio  $J$  is constant in time, and therefore  
45 isochoric motions are compatible with the requirement of incompressibility.

46 The assumption of strict incompressibility, however, yields both theoretical and com-  
47 putational issues. Within the framework of the Finite Element Method, it requires the de-  
48 velopment of robust and efficient numerical schemes that prevent from mesh locking (e.g.,  
49 the Lagrange multiplier method, penalty methods, the Hu-Washizu variational principle,  
50 methods based on higher-order shape functions), while ensuring flexibility and containing  
51 computational cost [2, 3, 4]. Granted hyperelastic material behaviour from some given  
52 natural configuration of a body, these schemes generally express the elastic potential into  
53 a part depending solely on the volumetric deformation and a part depending solely on the  
54 distortional deformation. Whereas the former one depends solely on parameters that, in  
55 the linear theory, reduce to the bulk modulus (this can be either the true one or some  
56 suitably chosen constant, as in penalty methods), the latter one strongly depends on the  
57 body's material symmetries (see, e.g., [5, 6]) and, in non-linear theories, is either obtained  
58 by fitting experimental data or given from the outset. In any case, the total non-linear  
59 elastic potential should lead to an elasticity tensor that is consistent with its linear coun-  
60 terpart [7], which is, in principle, always measurable experimentally. In linear elasticity,  
61 the usual approach to study incompressibility is based on the compliance elasticity tensor,  
62 the inverse of the (stiffness) elasticity tensor  $\mathbb{L}$ , and on making the bulk modulus diverge  
63 (e.g., [8]). In this way, the (stiffness) elasticity tensor diverges and is thus not defined.

64 Here, we propose a rigorous framework for determining the correct form of the lin-  
65 ear elasticity tensor of incompressible and quasi-incompressible materials, starting from  
66 the theory of Non-Linear Elasticity. We shall start by performing a full inverse Piola  
67 transform of the standard material elasticity tensor, so to derive the standard spatial elas-  
68 ticity tensor, and then evaluate the latter at zero strain, to finally obtain the (spatial)

69 linear elasticity tensor. This approach can be exploited to enforce that the non-linear  
70 elastic material is consistent with its linearised counterpart [7]. We had previously [9]  
71 worked out the calculations for the case of isotropic quasi-incompressible materials and  
72 now aim at giving the general expression of the elasticity tensor for the strictly incom-  
73 pressible and the quasi-incompressible cases, regardless of material symmetry, and then  
74 retrieve the important particular cases of isotropy, transverse isotropy and orthotropy. For  
75 the case of strict incompressibility, we show that the number of independent elastic con-  
76 stants decreases from 2 to 1 for isotropy, from 5 to 3 for transverse isotropy, and from 9  
77 to 6 for orthotropy. For the case of quasi-incompressibility, the bulk modulus is an ad-  
78 ditional independent elastic constant in all cases. The framework we propose also allows  
79 to conveniently check for the positive semi-definiteness (strictly incompressible case) or  
80 definiteness (quasi-incompressible case) of the elasticity tensor. Positive definiteness or  
81 semi-definiteness determine the strict convexity or convexity, respectively, of the quadratic  
82 potential of the linear theory, and influence the mathematical properties of the solutions,  
83 such as existence, uniqueness, smoothness, etc. (see, e.g., [10]).

84 This work is motivated by the importance that the elasticity tensor has in general, in  
85 both linear and non-linear Elasticity. Indeed, the elasticity tensor plays an essential role in  
86 Computational Mechanics, as it is the main “ingredient” defining the large stiffness matrices  
87 that are then employed by the solver modules of Finite Element packages. In Non-Linear  
88 Elasticity, the choice of the form of the elasticity tensor is various, depending on the choice  
89 of objective stress rate and measure of rate of deformation (see, e.g., [11, 12]). In contrast,  
90 in the small-strain theory, since all measures of stress converge to the Cauchy stress, and  
91 all measures of strain converge to the infinitesimal strain, also all possible elasticity tensors  
92 converge to the “classical” elasticity tensor of Linear Elasticity.

93 The paper is structured in six sections (including Introduction and Discussion). Section  
94 2 introduces the notation, reports some results from Tensor Algebra that are relevant to  
95 our purposes, and recalls the expressions of the elasticity tensor of the non-linear and linear  
96 theory for the general compressible case when the volumetric-distortional decomposition  
97 of the deformation [13, 14, 15] is used. Section 3 deals with incompressible elasticity and  
98 includes our results on the representation of the elasticity tensor in the non-linear and the  
99 linear theory, *regardless* of material symmetry. Section 4 consists of the study of the linear  
100 elasticity tensor for incompressible materials for the case of isotropy, transverse isotropy  
101 and orthotropy. Section 5 is devoted to a discussion about the issues of invertibility and  
102 positive definiteness of the linear elasticity tensor for the cases of strict incompressibility  
103 and quasi-incompressibility.

## 104 2 Theoretical Background

105 Here, we briefly introduce the general notation employed in this work, report some re-  
106 sults from Tensor Algebra that are related to the Theory of Elasticity [16], and recall the  
107 representation of the material, spatial and linear elasticity tensors when the volumetric-  
108 distortional decomposition of the deformation is used [17, 16]. Furthermore, we also refer  
109 to Walpole’s formalism for the representation of fourth-order tensors [18] in all possible  
110 symmetries, although here we limit ourselves to the most common cases: isotropy, trans-  
111 verse isotropy, orthotropy. Walpole had introduced this formalism in an earlier work [19],  
112 which we used extensively in the past (see, e.g., [20]). The newer representation devised by  
113 Walpole [18], which we employ here, introduces a very convenient matrix-based formalism.  
114 With respect to, e.g., Spencer’s [21] representation (which has several convenient features  
115 on which we shall not elaborate here), one of the greatest advantages of Walpole’s repre-

116 sentation [18] is that it makes it extremely easy to check for the positive definiteness or  
117 invertibility of a fourth-order tensor, seen as an operator between spaces of second-order  
118 tensors.

119 Although differentiable manifolds are the most general and appropriate theatre for the  
120 description of Mechanics [22, 23], we restrict ourselves to the (much) simpler case of a three-  
121 dimensional affine space, which avoids the long series of theoretical intricacies brought  
122 about by high-level Differential Geometry. Roughly speaking, an affine space is a vector  
123 space in which any point can be a “local origin”, thereby allowing vectors to be attached  
124 at any point. More rigorously, an affine space is given by a set  $\mathcal{S}$ , called the point space,  
125 considered together with a vector space  $\mathcal{V}$ , called the modelling space, and a map  $\mathcal{F} : \mathcal{S} \times$   
126  $\mathcal{S} \rightarrow \mathcal{V}$  that, for every pair of points  $x, y$  of  $\mathcal{S}$ , yields a vector of  $\mathcal{V}$  denoted  $\mathcal{F}(x, y) = y - x =$   
127  $\mathbf{v}$ , called the oriented segment from  $x$  to  $y$ . This map has to satisfy anti-commutativity,  
128 i.e.,  $[x - y] = -[y - x]$ , the triangle rule, i.e.,  $y - x = [y - z] + [z - x]$ , and the axiom of  
129 arbitrary origin, i.e., for every  $x \in \mathcal{S}$  and  $\mathbf{v} \in \mathcal{V}$  there exists one, and only one,  $y \in \mathcal{S}$ , such  
130 that  $y - x = \mathbf{v}$ . Given any point  $x \in \mathcal{S}$ , the axiom of arbitrary origin permits to define the  
131 set  $T_x\mathcal{S} = \{\mathbf{v}_x = y - x : y \in \mathcal{S}\}$  of all the vectors emanating from  $x$ . The space  $T_x\mathcal{S}$  and  
132 its dual space  $T_x^*\mathcal{S}$  are called the tangent space and the cotangent space, respectively, at  
133 point  $x$ , and their elements are called tangent vectors and tangent covectors, respectively,  
134 at point  $x$ . The disjoint union of all tangent spaces  $T_x\mathcal{S}$  for all  $x \in \mathcal{S}$  is called the tangent  
135 bundle of  $\mathcal{S}$ , and is denoted by  $T\mathcal{S}$ ; the cotangent bundle  $T^*\mathcal{S}$  is defined analogously. A  
136 thorough introduction to affine spaces is given, e.g., by Epstein [23].

137 The structure of affine space is the minimal structure needed for Differential Calculus,  
138 since a derivative is in fact a tangent vector. This is immediately reflected in the descrip-  
139 tion of Classical Physics, where the structure of affine space allows for attaching a vector  
140 representing a given physical quantity at any point of space. The prime example is that  
141 of the velocity, which, being the time derivative of a trajectory, is in fact a tangent vector  
142 in the sense of affine spaces, aside from being also tangent to the trajectory of the partic-  
143 cle. The modelling space used in the definition of the physical affine space  $\mathcal{S}$  of Classical  
144 Physics is the familiar  $\mathbb{R}^3$ . This space  $\mathcal{S}$  is indeed very similar to  $\mathbb{R}^3$  and one barely sees  
145 the difference, as long as vectors from the same tangent space are involved. Therefore, in  
146 many works in the literature (among which some of our past works), the affine space  $\mathcal{S}$   
147 of Classical Physics is simply denoted  $\mathbb{R}^3$ . However, following a didactical approach, we  
148 prefer to keep the distinction between the affine space  $\mathcal{S}$  and its modelling space  $\mathbb{R}^3$ .

149 Throughout this work, we employ the covariant formalism, i.e., we keep the distinction  
150 between a vector space and its dual space or, equivalently, between vectors and covectors.  
151 Aside from the fact that this allows for introducing general curvilinear coordinates, and  
152 for accounting for geometrical non-linearities, it is of fundamental importance to clarify  
153 the transformation laws that each physical quantity obeys. Indeed, vectors and covectors  
154 obey different transformation laws, and therefore the pull-back and push-forward opera-  
155 tions, crucial in Continuum Mechanics, are performed in a different way (see Section 2.1).  
156 Furthermore, as has also been remarked by Marsden and Hughes [22], the operations of  
157 pull-back/push-forward and of index raising/lowering *do not* commute, which means that  
158 even extra care must be taken when transforming vectorial or covectorial objects. The  
159 covariant formalism helps avoid errors, since it makes this non-commutativity evident.

160 In conclusion, we deem the small additional pain of using the structure of affine space  
161 and the covariant formalism worth it for the exposition of our results. The notation in this  
162 and in some previous works [16, 24], to which we shall extensively refer, mostly follows the  
163 classical treatise by Marsden and Hughes [22], with some relatively small variations.

164 **2.1 General Notation**

165 Lowercase symbols and indices are reserved to spatial quantities in the natural three-  
 166 dimensional space  $\mathcal{S}$  of Classical Mechanics. Uppercase symbols and indices denote ma-  
 167 terial quantities in the reference configuration  $\mathcal{B}_R \subset \mathcal{S}$  (or in the body manifold  $\mathcal{B}$ , if no  
 168 particular reference configuration is chosen [22, 25, 23]). At each point  $x \in \mathcal{S}$ , the tangent  
 169 and cotangent spaces are denoted  $T_x\mathcal{S}$  and  $T_x^*\mathcal{S}$ , respectively. The tangent and cotangent  
 170 bundles are denoted  $T\mathcal{S}$  and  $T^*\mathcal{S}$ , respectively. Similarly, one defines the tangent and  
 171 cotangent spaces  $T_X\mathcal{B}_R$  and  $T_X^*\mathcal{B}_R$  at  $X \in \mathcal{B}_R$ , and the tangent and cotangent bundles  
 172  $T\mathcal{B}_R$  and  $T^*\mathcal{B}_R$ . The spaces of spatial and material tensors of order  $m = r + s$ , with  $r$   
 173 vector feet and  $s$  covector feet (i.e., with  $r$  contravariant indices and  $s$  covariant indices)  
 174 are denoted  $[T\mathcal{S}]_s^r$  and  $[T\mathcal{B}_R]_s^r$ , respectively. The simple contraction of two tensors such  
 175 that the last foot of the first tensor is a vector and the first foot of the second tensor  
 176 is a covector (or vice versa) is indicated by simply juxtaposing the two tensors, e.g., for  
 177  $\mathbf{a} \in [T\mathcal{S}]_0^2$  and  $\mathbf{c} \in [T\mathcal{S}]_2^0$ , the contraction  $\mathbf{a}\mathbf{c}$  has components  $a^{ab}c_{bc}$ . The double contrac-  
 178 tion of two tensors is similar to the simple contraction, except that the last two feet of the  
 179 first tensor and the first two feet of the second tensor contract, and is denoted by a colon,  
 180 e.g., for  $\mathbb{T} \in [T\mathcal{S}]_2^2$  and  $\mathbf{a} \in [T\mathcal{S}]_0^2$ , the contraction  $\mathbb{T} : \mathbf{a}$  has components  $\mathbb{T}^{ab}_{cd}a^{cd}$ .

181 The spaces  $T\mathcal{S}$  and  $T\mathcal{B}_R$  are assumed to be equipped with metric tensors  $\mathbf{g}$  and  $\mathbf{G}$ ,  
 182 respectively. The scalar products induced by the metric tensors  $\mathbf{g}$  and  $\mathbf{G}$  are denoted by  
 183 the symbol  $\langle \cdot, \cdot \rangle$  for tensors of any order. For vectors or covectors, this is replaced by a  
 184 simple low dot, e.g., for the case of spatial vectors,  $\mathbf{g}(\mathbf{u}, \mathbf{v}) = \mathbf{u}\mathbf{g}\mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$ . For the  
 185 case of higher-order tensors (of the same type), each couple of homologous indices has to be  
 186 contracted with the appropriate metric tensor, e.g., for the case of spatial ‘‘contravariant’’  
 187 fourth-order tensors (i.e., tensors in  $[T\mathcal{S}]_0^4$ ), we have  $\langle \mathbb{A}, \mathbb{B} \rangle = \mathbb{A}^{abcd}g_{ai}g_{bj}g_{ck}g_{dl}\mathbb{B}^{ijkl}$ . Note  
 188 that we employ the usual identification  $g^{ab} \equiv (\mathbf{g}^{-1})^{ab}$  throughout. The metric tensor  $\mathbf{g}$   
 189 lowers contravariant indices, e.g., for the case of a vector  $\mathbf{v}$ , it gives the associated covector  
 190  $\mathbf{v}^b = \mathbf{g}\mathbf{v}$ , with components  $v_a = g_{ab}v^b$ . Analogously, the inverse metric tensor  $\mathbf{g}^{-1}$  raises  
 191 covariant indices, e.g., for the case of a covector  $\varphi$ , it gives the associated vector  $\varphi^\sharp = \mathbf{g}^{-1}\varphi$ ,  
 192 with components  $\varphi^a = g^{ab}\varphi_b$ . Moreover, we use a single low dot to indicate that the metric  
 193 tensor (or its inverse) is involved in the contraction of two tensors such that the last foot of  
 194 the first tensor and the first foot of the second tensor are of the same type. For instance,  
 195 if  $\mathbf{a}, \mathbf{b} \in [T\mathcal{S}]_0^2$ , the expression  $\mathbf{a} \cdot \mathbf{b}$  stands for  $\mathbf{a}\mathbf{g}\mathbf{b}$ , which has components  $a^{ab}g_{bc}b^{cd}$ .

196 The deformation,  $\chi : \mathcal{B}_R \rightarrow \mathcal{S}$ , maps material points  $X \in \mathcal{B}_R$  into spatial points  
 197  $x = \chi(X) \in \mathcal{S}$ , and its tangent map, the deformation gradient  $\mathbf{F} : T\mathcal{B}_R \rightarrow T\mathcal{S}$ , maps  
 198 material tangent vectors  $\mathbf{W} \in T\mathcal{B}_R$  into spatial tangent vectors  $\mathbf{w} = \mathbf{F}\mathbf{W} \in T\mathcal{S}$ , such  
 199 that the directional derivative of  $\chi$  with respect to  $\mathbf{W}$  at point  $X$  is  $(\partial_{\mathbf{W}}\chi)(X) = \mathbf{F}(X)\mathbf{W}$ ,  
 200 and the components of  $\mathbf{F}$  are  $F^a_A = \chi^a_{,A}$ . Given a material tensor field  $\mathbb{P}$  valued  
 201 in  $[T\mathcal{B}_R]_s^r$ , its push-forward  $\chi_*[\mathbb{P}] = \mathbb{P}$  is the tensor field valued in  $[T\mathcal{S}]_s^r$  obtained  
 202 by contracting each contravariant index with  $\mathbf{F}$  and each covariant index with  $\mathbf{F}^{-T}$ ,  
 203 which in components reads  $\mathbb{P}^{a\dots b} = F^a_A \dots (\mathbf{F}^{-T})_b^B \mathbb{P}^{A\dots B}$ . Analogously, given a spa-  
 204 tial tensor field  $\mathbb{Q}$  valued in  $[T\mathcal{S}]_s^r$ , its pull-back  $\chi^*[\mathbb{Q}] = \mathbb{Q}$  is the tensor field valued  
 205 in  $[T\mathcal{B}_R]_s^r$  obtained by contracting each contravariant index by  $\mathbf{F}^{-1}$  and each covari-  
 206 ant index by  $\mathbf{F}^T$ , i.e.,  $\mathbb{Q}^{A\dots B} = (\mathbf{F}^{-1})^A_a \dots (\mathbf{F}^T)_B^b \mathbb{Q}^{a\dots b}$ . Note that the operations  
 207 of pull-back/push-forward and of index raising/lowering *do not* commute: indeed, e.g.,  
 208  $\chi^*[\mathbf{v}^b] = \mathbf{F}^T[\mathbf{g}\mathbf{v}] \neq \mathbf{G}[\mathbf{F}^{-1}\mathbf{v}] = [\chi^*[\mathbf{v}]]^b$  (see, e.g., [22]).

209 The right and left Cauchy-Green deformation tensors are the pull-back  $\mathbf{C} = \mathbf{F}^T\mathbf{F} =$   
 210  $\mathbf{F}^T\mathbf{g}\mathbf{F}$  of the spatial metric  $\mathbf{g}$ , and the push-forward  $\mathbf{b} = \mathbf{F}\mathbf{F}^T = \mathbf{F}\mathbf{G}^{-1}\mathbf{F}^T$  of the inverse  
 211 material metric  $\mathbf{G}^{-1}$ , respectively. Their inverses  $\mathbf{B} = \mathbf{C}^{-1} = \mathbf{F}^{-1}\mathbf{F}^{-T} = \mathbf{F}^{-1}\mathbf{g}^{-1}\mathbf{F}^{-T}$   
 212 and  $\mathbf{c} = \mathbf{b}^{-1} = \mathbf{F}^{-T}\mathbf{F}^{-1} = \mathbf{F}^{-T}\mathbf{G}\mathbf{F}^{-1}$  are the pull-back of the inverse spatial metric  $\mathbf{g}^{-1}$

213 and the push-forward of the material metric  $\mathbf{G}$ , respectively. The Green-Lagrange strain,  
 214 comparing the pull-back  $\mathbf{C}$  of the spatial metric  $\mathbf{g}$  to the material metric  $\mathbf{G}$ , is given by  
 215  $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{G})$ . The volume ratio can be defined as  $J = \det \mathbf{F} \equiv \sqrt{\det \mathbf{C}} = \sqrt{\det \mathbf{b}}$  [24]  
 216 and its time derivative is  $\dot{J} = J \operatorname{div}(\mathbf{v}) = J \mathbf{B} : \dot{\mathbf{E}} = \frac{1}{2} J \mathbf{B} : \dot{\mathbf{C}}$  [3]. In the volumetric-  
 217 distortional decomposition of the deformation [13, 14, 15], we have  $\mathbf{F} = J^{1/3} \bar{\mathbf{F}}$ ,  $\mathbf{C} = J^{2/3} \bar{\mathbf{C}}$ ,  
 218 where  $\bar{\mathbf{C}} = \bar{\mathbf{F}}^T \cdot \bar{\mathbf{F}}$ , and  $\mathbf{E} = J^{2/3} \bar{\mathbf{E}} + \frac{1}{2}(J^{2/3} - 1)\mathbf{G}$ , where  $\bar{\mathbf{E}} = \frac{1}{2}(\bar{\mathbf{C}} - \mathbf{G})$ .

## 219 2.2 Identity, Spherical, Deviatoric Operators; Tensor Basis for Isotropy

220 In the space  $[TS]_2^2$  of symmetric fourth-order tensors (symmetric in the sense of metric  
 221 transposition [16]) with the first two feet being vectorial and the second two being cov-  
 222 ectorial (in terms of components, with the first two indices being contravariant and the  
 223 second two being covariant), the symmetric identity, spherical, and deviatoric operators  
 224 [16], defined by using the special tensor products  $\underline{\otimes}$  and  $\overline{\otimes}$  introduced by Curnier et al.  
 225 [26], read

$$\mathbb{I} = \frac{1}{2}(\mathbf{i} \underline{\otimes} \mathbf{i} + \mathbf{i} \overline{\otimes} \mathbf{i}), \quad (4a)$$

$$\mathbb{K} = \frac{1}{3} \mathbf{g}^{-1} \underline{\otimes} \mathbf{g}, \quad (4b)$$

$$\mathbb{M} = \mathbb{I} - \mathbb{K}, \quad (4c)$$

226 and have components

$$I^{ab}_{cd} = \frac{1}{2}(\delta^a_c \delta^b_d + \delta^a_d \delta^b_c), \quad (5a)$$

$$K^{ab}_{cd} = \frac{1}{3} g^{ab} g_{cd}, \quad (5b)$$

$$M^{ab}_{cd} = \frac{1}{2}(\delta^a_c \delta^b_d + \delta^a_d \delta^b_c) - \frac{1}{3} g^{ab} g_{cd}. \quad (5c)$$

227 When applied to a symmetric second-order tensor  $\mathbf{a} \in [TS]_0^2$ ,  $\mathbb{K}$  and  $\mathbb{M}$  yield the spherical  
 228 and deviatoric parts of  $\mathbf{a}$ , respectively, i.e.,

$$\operatorname{sph}(\mathbf{a}) = \mathbb{K} : \mathbf{a} = \frac{1}{3} \operatorname{tr}(\mathbf{a}) \mathbf{g}^{-1}, \quad \operatorname{dev}(\mathbf{a}) = \mathbb{M} : \mathbf{a} = \mathbf{a} - \frac{1}{3} \operatorname{tr}(\mathbf{a}) \mathbf{g}^{-1}, \quad (6)$$

229 where  $\operatorname{tr}(\cdot)$  is the natural trace operator, such that  $\operatorname{tr}(\mathbf{a}) = \mathbf{g}^{-1} : \mathbf{a} = g^{ab} a_{ab}$ . Further-  
 230 more,  $\{\mathbb{K}, \mathbb{M}\}$  is the canonical basis of the subspace of symmetric isotropic tensors in  $[TS]_2^2$ ,  
 231 where isotropy is defined as the symmetry (i.e., invariance) with respect to arbitrary ro-  
 232 tations. The spherical and deviatoric operators enjoy the properties of idempotence and  
 233 orthogonality [19, 18, 16], i.e.,

$$\mathbb{K} : \mathbb{K} = \mathbb{K}, \quad \mathbb{M} : \mathbb{M} = \mathbb{M}, \quad (7a)$$

$$\mathbb{K} : \mathbb{M} = \mathbb{O}, \quad \mathbb{M} : \mathbb{K} = \mathbb{O}, \quad (7b)$$

234 where  $\mathbb{O}$  is the null fourth-order tensor in  $[TS]_2^2$ .

235 Stiffness and compliance elasticity tensors belong to  $[TS]_0^4$  and  $[TS]_4^0$ , respectively and,  
 236 for our purposes, it is important to recall the expressions of the identity, spherical and  
 237 deviatoric operators in these spaces. These are obtained by raising and lowering all indices  
 238 of the tensors in Equation (4), respectively, to obtain [16]

$$\mathbb{I}^\sharp = \frac{1}{2}(\mathbf{g}^{-1} \underline{\otimes} \mathbf{g}^{-1} + \mathbf{g}^{-1} \overline{\otimes} \mathbf{g}^{-1}), \quad (8a)$$

$$\mathbb{K}^\sharp = \frac{1}{3} \mathbf{g}^{-1} \underline{\otimes} \mathbf{g}^{-1}, \quad (8b)$$

$$\mathbb{M}^\sharp = \mathbb{I}^\sharp - \mathbb{K}^\sharp, \quad (8c)$$

239 and

$$\mathbb{I}^b = \frac{1}{2}(\mathbf{g} \otimes \mathbf{g} + \mathbf{g} \bar{\otimes} \mathbf{g}), \quad (9a)$$

$$\mathbb{K}^b = \frac{1}{3} \mathbf{g} \otimes \mathbf{g}, \quad (9b)$$

$$\mathbb{M}^b = \mathbb{I}^b - \mathbb{K}^b, \quad (9c)$$

240 which have component expressions

$$\mathbb{I}^{abcd} = \frac{1}{2}(g^{ac}g^{bd} + g^{ad}g^{bc}), \quad (10a)$$

$$\mathbb{K}^{abcd} = \frac{1}{3}g^{ab}g^{cd}, \quad (10b)$$

$$\mathbb{M}^{abcd} = \frac{1}{2}(g^{ac}g^{bd} + g^{ad}g^{bc}) - \frac{1}{3}g^{ab}g^{cd}, \quad (10c)$$

241 and

$$\mathbb{I}_{abcd} = \frac{1}{2}(g_{ac}g_{bd} + g_{ad}g_{bc}), \quad (11a)$$

$$\mathbb{K}_{abcd} = \frac{1}{3}g_{ab}g_{cd}, \quad (11b)$$

$$\mathbb{M}_{abcd} = \frac{1}{2}(g_{ac}g_{bd} + g_{ad}g_{bc}) - \frac{1}{3}g_{ab}g_{cd}. \quad (11c)$$

242 Again,  $\{\mathbb{K}^\sharp, \mathbb{M}^\sharp\}$  and  $\{\mathbb{K}^b, \mathbb{M}^b\}$  are the canonical bases of the subspaces of symmetric isotropic  
 243 tensors in  $[\mathcal{TS}]_0^4$  and  $[\mathcal{TS}]_4^0$ , respectively. Also the tensors  $\mathbb{K}^\sharp$  and  $\mathbb{M}^\sharp$ , and the tensors  $\mathbb{K}^b$   
 244 and  $\mathbb{M}^b$  enjoy idempotence and orthogonality, and a thorough analysis can be found in a  
 245 previous work [16], which reports the results obtained by Walpole [19, 18] in the covariant  
 246 formalism also adopted here.

247 Note that a symmetric isotropic fourth-order tensor in  $[\mathcal{TS}]^2_2$  (or  $[\mathcal{TS}]_0^4$  or  $[\mathcal{TS}]_4^0$ ) is  
 248 positive definite if, and only if, its components in the basis  $\{\mathbb{K}, \mathbb{M}\}$  (or  $\{\mathbb{K}^\sharp, \mathbb{M}^\sharp\}$  or  $\{\mathbb{K}^b, \mathbb{M}^b\}$ ,  
 249 respectively), are strictly positive, and invertible if, and only if, its components are both  
 250 different from zero.

251 In the Theory of Elasticity, the pulled-back material counterparts of the spatial opera-  
 252 tors in  $[\mathcal{TS}]^2_2$ ,  $[\mathcal{TS}]_0^4$  and  $[\mathcal{TS}]_4^0$  are of particular relevance, and we recall them here [16],  
 253 omitting the component forms, which can be deduced by analogy with those of the spatial  
 254 operators [16]. The pull-back of the operators in  $[\mathcal{TS}]^2_2$  yields the operators in  $[\mathcal{TB}_R]^2_2$ ,

$$\mathbb{I} = \frac{1}{2}(\mathbf{I} \otimes \mathbf{I} + \mathbf{I} \bar{\otimes} \mathbf{I}), \quad (12a)$$

$$\mathbb{K}^* = \frac{1}{3} \mathbf{B} \otimes \mathbf{C}, \quad (12b)$$

$$\mathbb{M}^* = \mathbb{I}^* - \mathbb{K}^*, \quad (12c)$$

255 where we note that the pull-back  $\mathbb{I}^*$  coincides with the material identity  $\mathbb{I}$ . When applied  
 256 to a symmetric second-order tensor  $\mathbf{A} \in [\mathcal{TB}_R]_0^2$ ,  $\mathbb{K}^*$  and  $\mathbb{M}^*$  yield the pulled-back spherical  
 257 and deviatoric parts of  $\mathbf{A}$ , respectively, evaluated with respect to the pulled-back metric  
 258  $\mathbf{C} = \chi^*[\mathbf{g}]$ , i.e.,

$$\text{Sph}^*(\mathbf{A}) = \mathbb{K}^* : \mathbf{A} = \frac{1}{3} \text{Tr}^*(\mathbf{A}) \mathbf{B}, \quad \text{Dev}^*(\mathbf{A}) = \mathbb{M}^* : \mathbf{A} = \mathbf{A} - \frac{1}{3} \text{Tr}^*(\mathbf{A}) \mathbf{B}, \quad (13)$$

259 where  $\text{Tr}^*(\cdot)$  is the material pulled-back trace operator [16], i.e., the trace evaluated with  
 260 respect to the pulled-back metric  $\mathbf{C} = \chi^*[\mathbf{g}]$ , such that  $\text{Tr}^*(\mathbf{A}) = \mathbf{C} : \mathbf{A} = C_{AB}A^{AB}$ . The  
 261 pull-back of the operators in  $[\mathcal{TS}]_0^4$  yields the operators in  $[\mathcal{TB}_R]_0^4$

$$\mathbb{I}^{\sharp*} = \frac{1}{2}(\mathbf{B} \otimes \mathbf{B} + \mathbf{B} \bar{\otimes} \mathbf{B}), \quad (14a)$$

$$\mathbb{K}^{\sharp*} = \frac{1}{3} \mathbf{B} \otimes \mathbf{B}, \quad (14b)$$

$$\mathbb{M}^{\sharp*} = \mathbb{I}^{\sharp*} - \mathbb{K}^{\sharp*}, \quad (14c)$$

262 and the pull-back of the operators in  $[TS]_4^0$  yields the operators in  $[TB_R]_4^0$

$$\mathbb{I}^{b*} = \frac{1}{2}(\mathbf{C} \otimes \mathbf{C} + \mathbf{C} \bar{\otimes} \mathbf{C}), \quad (15a)$$

$$\mathbb{K}^{b*} = \frac{1}{3}\mathbf{C} \otimes \mathbf{C}, \quad (15b)$$

$$\mathbb{M}^{b*} = \mathbb{I}^{b*} - \mathbb{K}^{b*}. \quad (15c)$$

263 We recall that  $\mathbf{C}$  is the right Cauchy-Green deformation and  $\mathbf{B} = \mathbf{C}^{-1}$  is its inverse.

### 264 2.3 Tensor Basis For Transverse Isotropy

265 Let  $\mathbf{m} \in TS$  be a unit vector with respect to the metric  $\mathbf{g}$ , i.e., such that its Euclidean  
266 norm is unitary:

$$\|\mathbf{m}\|^2 = \mathbf{m} \cdot \mathbf{m} = \mathbf{m} \mathbf{g} \mathbf{m} = 1. \quad (16)$$

267 Transverse isotropy with respect to  $\mathbf{m}$  is defined as the symmetry (i.e., the invariance)  
268 with respect to rotations about  $\mathbf{m}$ . The direction identified by  $\mathbf{m}$  is called symmetry axis  
269 and the class of equivalence of the planes orthogonal to  $\mathbf{m}$  is called transverse plane.

270 The basis of all second-order tensors in  $[TS]_0^2$  with transverse isotropy with respect to  
271 direction  $\mathbf{m}$  is given by

$$\mathbf{a} = \mathbf{m} \otimes \mathbf{m}, \quad (17a)$$

$$\mathbf{t} = \mathbf{g}^{-1} - \mathbf{a}. \quad (17b)$$

272 Note that  $\mathbf{t}$  is the complement of tensor  $\mathbf{a}$  to  $\mathbf{g}^{-1}$  (the ‘‘contravariant identity’’, i.e., the  
273 identity in the tensor space  $[TS]_0^2$ ), and that both  $\mathbf{a}$  and  $\mathbf{t}$  are invariant under the trans-  
274 formation mapping  $\mathbf{m}$  into  $-\mathbf{m}$ , i.e., the sense of  $\mathbf{m}$  is irrelevant. Tensor  $\mathbf{a}$  is often called  
275 structure tensor or fabric tensor of direction  $\mathbf{m}$ . By means of the metric tensor  $\mathbf{g}$ , it is  
276 possible to contract tensors  $\mathbf{a}$  and  $\mathbf{t}$  with a vector  $\mathbf{v}$ , and to obtain the axial and transverse  
277 components of  $\mathbf{v}$ :

$$v_{\parallel} = \mathbf{a} \cdot \mathbf{v}, \quad (18a)$$

$$v_{\perp} = \mathbf{t} \cdot \mathbf{v}. \quad (18b)$$

278 By means of suitable tensor products, Walpole [18] derived a basis for fourth-order tensors  
279 with transverse isotropy with respect to  $\mathbf{m}$ , which we report for tensors in  $[TS]_0^4$ :

$$\mathbb{U}_{11} = \mathbf{a} \otimes \mathbf{a}, \quad (19a)$$

$$\mathbb{U}_{22} = \frac{1}{2}\mathbf{t} \otimes \mathbf{t}, \quad (19b)$$

$$\mathbb{U}_{12} = \frac{\sqrt{2}}{2}\mathbf{a} \otimes \mathbf{t}, \quad (19c)$$

$$\mathbb{U}_{21} = \frac{\sqrt{2}}{2}\mathbf{t} \otimes \mathbf{a}, \quad (19d)$$

$$\mathbb{V}_1 = \frac{1}{2}(\mathbf{t} \otimes \mathbf{t} + \mathbf{t} \bar{\otimes} \mathbf{t} - \mathbf{t} \otimes \mathbf{t}), \quad (19e)$$

$$\mathbb{V}_2 = \frac{1}{2}(\mathbf{a} \otimes \mathbf{t} + \mathbf{a} \bar{\otimes} \mathbf{t} + \mathbf{t} \otimes \mathbf{a} + \mathbf{t} \bar{\otimes} \mathbf{a}). \quad (19f)$$

280 In this basis, a tensor  $\mathbb{T} \in [TS]_0^4$ , transversely isotropic with respect to  $\mathbf{m}$ , is expressed as

$$\mathbb{T} = \tilde{\mathbb{T}}^{pr} \mathbb{U}_{pr} + \tilde{\mathbb{T}}^{\alpha} \mathbb{V}_{\alpha}, \quad (20)$$

281 where we call the collection  $\{\tilde{\mathbb{T}}\}$  of Walpole’s components  $\tilde{\mathbb{T}}^{pr}$  and  $\tilde{\mathbb{T}}^{\alpha}$  *Walpole’s represen-*  
282 *tation* of  $\mathbb{T}$  [20]. Since the tensors  $\mathbb{U}_{pr}$  constitute an algebra isomorphic to that of  $2 \times 2$   
283 matrices, Walpole’s components can be grouped as [18]

$$\{\tilde{\mathbb{T}}\} = \left\{ \begin{bmatrix} \tilde{\mathbb{T}}^{11} & \tilde{\mathbb{T}}^{12} \\ \tilde{\mathbb{T}}^{21} & \tilde{\mathbb{T}}^{22} \end{bmatrix}, \tilde{\mathbb{T}}^1, \tilde{\mathbb{T}}^2 \right\}, \quad (21)$$

284 and all operations on transversely isotropic tensors  $\mathbb{T}$  can be performed by working on their  
 285 representations  $\{\tilde{\mathbb{T}}\}$ . The four  $\tilde{\mathbb{T}}^{pr}$  and the two  $\tilde{\mathbb{T}}^\alpha$  are obtained by the scalar product of  $\mathbb{T}$   
 286 with each of the basis tensors, with some normalisation constants:

$$\tilde{\mathbb{T}}^{pr} = \langle \mathbb{T}, \mathbb{U}_{pr} \rangle, \quad \tilde{\mathbb{T}}^\alpha = \frac{1}{2} \langle \mathbb{T}, \mathbb{V}_\alpha \rangle. \quad (22)$$

287 Since  $\mathbb{U}_{12}^T = \mathbb{U}_{21}$ , tensor  $\mathbb{T}$  possesses diagonal symmetry if, and only if,  $\tilde{\mathbb{T}}^{12} = \tilde{\mathbb{T}}^{21}$ , in which  
 288 case it has only 5, rather than 6, independent components.

289 Given an orthonormal basis  $\{\mathbf{e}_a\}_{a=1}^3$ , such that  $\mathbf{e}_1 = \mathbf{m}$ , the components  $\{\tilde{\mathbb{T}}\}$  are related  
 290 to the conventional components  $\mathbb{T}^{abcd}$  of  $\mathbb{T}$  by

$$\{\tilde{\mathbb{T}}\} = \left\{ \left[ \begin{array}{cc} \mathbb{T}^{1111} & \sqrt{2} \mathbb{T}^{1122} \\ \sqrt{2} \mathbb{T}^{2211} & 2 \mathbb{T}^{2222} - 2 \mathbb{T}^{2323} \end{array} \right], 2 \mathbb{T}^{2323}, 2 \mathbb{T}^{1212} \right\}. \quad (23)$$

291 Note that the full-symmetric ‘‘contravariant’’ fourth-order identity, spherical and deviatoric  
 292 operators in  $[TS]_0^4$ , defined in Equation (8), have Walpole’s representations

$$\{\tilde{\mathbb{I}}^\#\} = \left\{ \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], 1, 1 \right\}, \quad (24a)$$

$$\{\tilde{\mathbb{K}}^\#\} = \left\{ \left[ \begin{array}{cc} \frac{1}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} \end{array} \right], 0, 0 \right\}, \quad (24b)$$

$$\{\tilde{\mathbb{M}}^\#\} = \left\{ \left[ \begin{array}{cc} \frac{2}{3} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} \end{array} \right], 1, 1 \right\}. \quad (24c)$$

293 It is very important to notice that *all* the associated tensors obtained from a tensor in  
 294  $[TS]_0^4$  by lowering any of its indices (i.e., by transforming any of its vector feet into covector  
 295 feet by means of the metric tensor  $\mathbf{g}$ ) share *the same* Walpole representation, as it becomes  
 296 clear by looking at the scalar products (22) in components (e.g.,  $\mathbb{T}^{abcd} g_{ai} g_{bj} g_{ck} g_{dl} [\mathbb{U}_{pr}]^{ijkl}$ ),  
 297 manipulating the metric tensors, and exploiting identities of the type  $g^{hm} g_{mn} = \delta^h_n$ . In  
 298 practice, the transformation is entirely shifted onto the basis tensors, leaving Walpole’s  
 299 components untouched. This allows for exploiting the isomorphism between transversely  
 300 isotropic fourth-order tensors and their Walpole’s representation to perform any operation.  
 301 For example, the double contraction of a tensor in  $[TS]_2^2$  and one in  $[TS]_0^4$  can be performed  
 302 by multiplying the matrix of the former with the matrix of the latter, and the individual  
 303 scalars of the former with those of the latter, without worrying about which indices are  
 304 contravariant and which covariant, as this is all taken into account by the basis tensors.

305 For the case of transverse isotropy, a tensor  $\mathbb{T}$  is positive definite if its Walpole’s repre-  
 306 sentation  $\{\tilde{\mathbb{T}}\}$  is such that the  $2 \times 2$  matrix  $[\tilde{\mathbb{T}}^{pq}]$  is positive definite, and the two scalars  $\tilde{\mathbb{T}}^\alpha$   
 307 are strictly positive. Similarly,  $\mathbb{T}$  is invertible if  $[\tilde{\mathbb{T}}^{pq}]$  is invertible and the two scalars  $\tilde{\mathbb{T}}^\alpha$   
 308 are different from zero, and the inverse  $\mathbb{T}^{-1}$  (which belongs to  $[TS]_4^0$ , if  $\mathbb{T}$  belongs to  $[TS]_0^4$ )  
 309 has Walpole’s representation

$$\{\tilde{\mathbb{T}}^{-1}\} = \left\{ \left[ \begin{array}{cc} \tilde{\mathbb{T}}^{11} & \tilde{\mathbb{T}}^{12} \\ \tilde{\mathbb{T}}^{21} & \tilde{\mathbb{T}}^{22} \end{array} \right]^{-1}, 1/\tilde{\mathbb{T}}^1, 1/\tilde{\mathbb{T}}^2 \right\}. \quad (25)$$

## 310 2.4 Tensor Basis For Orthotropy

311 Let  $\{\mathbf{m}_p\}_{p=1}^3$  be a basis for  $TS$ , satisfying the condition of orthonormality with respect  
 312 to the metric  $\mathbf{g}$ , i.e.,

$$\mathbf{m}_p \cdot \mathbf{m}_q = \mathbf{m}_p \mathbf{g} \mathbf{m}_q = \delta_{pq}, \quad (26)$$

313 Given such a basis, the inverse metric tensor can be expressed as

$$\mathbf{g}^{-1} = \sum_{p=1}^3 \mathbf{m}_p \otimes \mathbf{m}_p. \quad (27)$$

314 Orthotropy with respect to the basis  $\{\mathbf{m}_p\}_{p=1}^3$  is defined as the symmetry (i.e., invariance)  
315 under reflection of any of the three  $\mathbf{m}_p$ .

316 The orthonormal basis  $\{\mathbf{m}_p\}_{p=1}^3$  can be used to construct the basis for the space of  
317 second-order tensors in  $[T\mathcal{S}]_0^2$  as

$$\mathbf{z}_{pq} = \mathbf{m}_p \otimes \mathbf{m}_q, \quad (28)$$

318 and the basis for the space of fourth-order tensors in  $[T\mathcal{S}]_0^4$  as

$$\mathbb{Z}_{pqrs} = \mathbf{z}_{pq} \otimes \mathbf{z}_{rs} = \mathbf{m}_p \otimes \mathbf{m}_q \otimes \mathbf{m}_r \otimes \mathbf{m}_s. \quad (29)$$

319 The basis for the subspace of the space of  $[T\mathcal{S}]_0^2$  with orthotropy with respect to  $\{\mathbf{m}_p\}_{p=1}^3$   
320 is obtained by defining the three tensors [18]

$$\mathbf{a}_p = \mathbf{z}_{pp} = \mathbf{m}_p \otimes \mathbf{m}_p, \quad \text{no sum on } p, \quad (30)$$

321 which are often called structure tensors or fabric tensors of the directions  $\mathbf{m}_p$ . It is im-  
322 mediate to verify that the tensors (30) are invariant for reflections of the  $\mathbf{m}_p$  (transfor-  
323 mations mapping  $\mathbf{m}_p$  into  $-\mathbf{m}_p$ ), i.e., are orthotropic with respect to  $\{\mathbf{m}_p\}_{p=1}^3$ , linearly  
324 independent, and generate the space of orthotropic tensors with respect to  $\{\mathbf{m}_p\}_{p=1}^3$ . The  
325 corresponding basis for the subspace of the space of fourth-order tensors in  $[T\mathcal{S}]_0^4$  with  
326 orthotropy with respect to  $\{\mathbf{m}_p\}_{p=1}^3$  was obtained by Walpole [18] as

$$\mathbb{U}_{pr} = \mathbb{Z}_{pprr}, \quad \forall p, r \in \{1, 2, 3\}, \quad \text{no sum on } p \text{ and } r, \quad (31a)$$

$$\mathbb{V}_1 = \frac{1}{2} [\mathbb{Z}_{2323} + \mathbb{Z}_{3232}], \quad (31b)$$

$$\mathbb{V}_2 = \frac{1}{2} [\mathbb{Z}_{1313} + \mathbb{Z}_{3131}], \quad (31c)$$

$$\mathbb{V}_3 = \frac{1}{2} [\mathbb{Z}_{1212} + \mathbb{Z}_{2121}]. \quad (31d)$$

327 A fourth-order tensor  $\mathbb{T} \in [T\mathcal{S}]_0^4$ , orthotropic with respect to  $\{\mathbf{m}_p\}_{p=1}^3$ , can be thus written  
328 as

$$\mathbb{T} = \tilde{\mathbb{T}}^{pr} \mathbb{U}_{pr} + \tilde{\mathbb{T}}^\alpha \mathbb{V}_\alpha, \quad (32)$$

329 where we call the collection  $\{\tilde{\mathbb{T}}\}$  of Walpole's components  $\tilde{\mathbb{T}}^{pr}$  and  $\tilde{\mathbb{T}}^\alpha$  *Walpole's representa-*  
330 *tion* of the tensor  $\mathbb{T}$ . Similarly to the case of transverse isotropy, Walpole [18] showed that  
331 the basis tensors  $\mathbb{U}_{pr}$  constitute an algebra isomorphic to that of  $3 \times 3$  matrices and that  
332 the components  $\tilde{\mathbb{T}}^{pr}$  and  $\tilde{\mathbb{T}}^\alpha$  can be grouped as

$$\{\tilde{\mathbb{T}}\} = \{[\tilde{\mathbb{T}}^{pr}], \tilde{\mathbb{T}}^1, \tilde{\mathbb{T}}^2, \tilde{\mathbb{T}}^3\}. \quad (33)$$

333 The nine  $\tilde{\mathbb{T}}^{pr}$  and the three  $\tilde{\mathbb{T}}^\alpha$  are obtained as the scalar product of  $\mathbb{T}$  with each of the  
334 basis tensors:

$$\tilde{\mathbb{T}}^{pr} = \langle \mathbb{T}, \mathbb{U}_{pr} \rangle, \quad \tilde{\mathbb{T}}^\alpha = \frac{1}{2} \langle \mathbb{T}, \mathbb{V}_\alpha \rangle. \quad (34)$$

335 Since  $\mathbb{U}_{pr} = \mathbb{U}_{rp}^T$ , diagonal symmetry of  $\mathbb{T}$  is attained if, and only if, the matrix  $[\tilde{\mathbb{T}}^{pr}]$  is  
336 symmetric. In this case,  $\mathbb{T}$  has 9, rather than 12, independent components.

337 Note that the relation of Walpole's components  $\tilde{\mathbb{T}}^{pr}$  and  $\tilde{\mathbb{T}}^\alpha$  with the conventional com-  
338 ponents  $\mathbb{T}^{abcd}$  of  $\mathbb{T}$  is quite more straightforward in the case of orthotropy compared to the  
339 case of transverse isotropy, indeed:

$$\{\tilde{\mathbb{T}}\} = \left\{ \begin{bmatrix} \mathbb{T}^{1111} & \mathbb{T}^{1122} & \mathbb{T}^{1133} \\ \mathbb{T}^{2211} & \mathbb{T}^{2222} & \mathbb{T}^{2233} \\ \mathbb{T}^{3311} & \mathbb{T}^{3322} & \mathbb{T}^{3333} \end{bmatrix}, 2\mathbb{T}^{2323}, 2\mathbb{T}^{1313}, 2\mathbb{T}^{1212} \right\}. \quad (35)$$

340 The full-symmetric ‘‘contravariant’’ fourth-order identity, spherical and deviatoric operators  
 341 in  $[TS]_0^4$  of Equation (8) have Walpole’s representations

$$\{\tilde{\mathbb{I}}^\#\} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, 1, 1, 1 \right\}, \quad (36a)$$

$$\{\tilde{\mathbb{K}}^\#\} = \left\{ \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}, 0, 0, 0 \right\}, \quad (36b)$$

$$\{\tilde{\mathbb{M}}^\#\} = \left\{ \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}, 1, 1, 1 \right\}. \quad (36c)$$

342 Analogously to the case of transverse isotropy, *all* the associated tensors obtained from  
 343 a tensor in  $[TS]_0^4$  by lowering any of its indices by means of the metric tensor  $\mathbf{g}$  share  
 344 *the same* Walpole representation, as the transformation is entirely ascribed to the basis  
 345 tensors. Again, this allows for orthotropic fourth-order tensors to be inverted, summed  
 346 and double-contracted by working on their Walpole’s representations.

347 positive definiteness and invertibility of an orthotropic fourth-order tensor  $\mathbb{T}$  are checked  
 348 analogously to the case of transverse isotropy.  $\mathbb{T}$  is positive definite if the  $3 \times 3$  matrix  $[\tilde{\mathbb{T}}^{pq}]$   
 349 in its Walpole’s representation  $\tilde{\mathbb{T}}$  is positive definite, and the three scalars  $\tilde{\mathbb{T}}^\alpha$  are strictly  
 350 positive, and invertible if  $[\tilde{\mathbb{T}}^{pq}]$  is invertible and the three scalars  $\tilde{\mathbb{T}}^\alpha$  are different from zero.  
 351 The Walpole representation of the inverse is analogous to that of the transversely isotropic  
 352 case seen in Equation (25).

## 353 2.5 Hyperelasticity and Volumetric-Distortional Decomposition

354 Within a purely mechanical framework, the dissipation density  $D$  per unit volume of  
 355 the undeformed configuration of a body comprised of a simple material is defined by [27]

$$D = -\dot{W} + \mathbf{S} : \dot{\mathbf{E}} \geq 0. \quad (37)$$

356 In the inequality (37), which has to hold at all points  $X$  of  $\mathcal{B}_R$  and at all times,  $W$  is  
 357 the stored energy function per unit volume of  $\mathcal{B}_R$ ,  $\mathbf{S}$  is the second Piola-Kirchhoff stress  
 358 tensor, and  $\mathbf{E}$  is the Green-Lagrange strain tensor. For the case of a hyperelastic material,  
 359  $W$  and  $\mathbf{S}$  are expressed as constitutive functions of  $\mathbf{E}$ , such that

$$W = \hat{W}(\mathbf{E}), \quad \mathbf{S} = \hat{\mathbf{S}}(\mathbf{E}), \quad (38)$$

360 and  $W$  is referred to as the elastic potential (or strain energy) density. We remark that the  
 361 constitutive functions may depend explicitly on the position  $X$ , in which case the material  
 362 is inhomogeneous, but we omit indicating this dependence for the sake of a lighter notation.  
 363 Substituting (38) into (37) yields

$$\begin{aligned} D &= -\frac{\partial \hat{W}}{\partial \mathbf{E}}(\mathbf{E}) : \dot{\mathbf{E}} + \hat{\mathbf{S}}(\mathbf{E}) : \dot{\mathbf{E}} \\ &= \left[ \hat{\mathbf{S}}(\mathbf{E}) - \frac{\partial \hat{W}}{\partial \mathbf{E}}(\mathbf{E}) \right] : \dot{\mathbf{E}} \geq 0. \end{aligned} \quad (39)$$

364 The inequality (39) implies that  $D$  is a function of  $\mathbf{E}$  and  $\dot{\mathbf{E}}$ , i.e.  $D = \hat{D}(\mathbf{E}, \dot{\mathbf{E}})$ . Since  $\dot{\mathbf{E}}$   
 365 is neither an independent nor a dependent constitutive variable,  $\hat{D}$  depends linearly on  $\dot{\mathbf{E}}$

366 (in particular,  $\hat{D}(\mathbf{E}, \mathbf{O}) = 0$ ), which therefore can be varied arbitrarily. Consequently, in  
 367 order to ensure that the inequality is always respected, it must hold that

$$\left[ \hat{\mathbf{S}}(\mathbf{E}) - \frac{\partial \hat{W}}{\partial \mathbf{E}}(\mathbf{E}) \right] : \dot{\mathbf{E}} = 0, \quad (40)$$

368 which implies that the second Piola-Kirchhoff stress is given by the derivative of the elastic  
 369 potential with respect to the Green-Lagrange strain:

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{E}) = \frac{\partial \hat{W}}{\partial \mathbf{E}}(\mathbf{E}). \quad (41)$$

370 The second derivative of the elastic potential is the material elasticity tensor

$$\mathbb{C} = \hat{\mathbb{C}}(\mathbf{E}) = \frac{\partial^2 \hat{W}}{\partial \mathbf{E}^2}(\mathbf{E}), \quad (42)$$

371 which, evaluated at zero strain, yields the material linear elasticity tensor

$$\mathbb{L} = \hat{\mathbb{C}}(\mathbf{O}) = \frac{\partial^2 \hat{W}}{\partial \mathbf{E}^2}(\mathbf{O}). \quad (43)$$

372 The inverse Piola transform of the material elasticity tensor  $\mathbb{C}$  is the spatial elasticity  
 373 tensor

$$\mathbb{c} = J^{-1} \chi_*[\mathbb{C}], \quad c^{abcd} = J^{-1} F^a{}_A F^b{}_B F^c{}_C F^d{}_D C^{ABCD}, \quad (44)$$

374 which, evaluated at zero strain, yields the spatial linear elasticity tensor  $\mathbb{L}$ . Equivalently,  
 375 the spatial linear elasticity tensor  $\mathbb{L}$  can be obtained as the inverse Piola transform of the  
 376 material linear elasticity tensor  $\mathbb{L}$  performed in the undeformed state, when  $J = 1$  and  
 377  $\mathbf{F} = \mathbf{1}$ , where  $\mathbf{1}$  is the shifter [1, 22], i.e., in components,

$$\mathbb{L}^{abcd} = \mathbf{1}^a{}_A \mathbf{1}^b{}_B \mathbf{1}^c{}_C \mathbf{1}^d{}_D \mathbb{L}^{ABCD}. \quad (45)$$

378 Physically, the shifter parallel transports tangent vectors from a material point to a spatial  
 379 point and, in the most general case, its representing matrix is orthogonal, which means that  
 380 the components of  $\mathbb{L}$  and  $\mathbb{L}$  differ merely by a rigid rotation. Moreover, for the particular  
 381 case of collinear Cartesian coordinates in  $\mathcal{B}_R$  and  $\mathcal{S}$ , the components of the shifter  $\mathbf{1}$  are  
 382 simply  $\mathbf{1}^a{}_A = \delta^a{}_A$ , and therefore the components of the material and spatial linear elasticity  
 383 tensors coincide. For this reason, in Linear Elasticity, it is practically equivalent to speak  
 384 about the material or the spatial linear elasticity tensor. Therefore, it is indifferent to  
 385 speak about material symmetries in the material or in the spatial picture, and this is why,  
 386 in Sections 2.2, 2.3, 2.4, we reported the tensor bases in the spatial picture only. We remark  
 387 that, in the general non-linear case, the material symmetries of a body are studied in the  
 388 *material* picture of Mechanics (e.g., [28, 1, 22, 15]).

389 When the volumetric-distortional decomposition of the deformation [13, 14] is employed,  
 390 the elastic potential is written as a function

$$\hat{W}(\mathbf{E}) = \hat{\Psi}(J(\mathbf{E}), \bar{\mathbf{E}}(\mathbf{E})) \quad (46)$$

391 of the determinant  $J$  of the deformation gradient  $\mathbf{F}$  and the distortional Green-Lagrange  
 392 strain  $\bar{\mathbf{E}}$ , which are both regarded as explicit functions of the “full” Green-Lagrange strain  
 393  $\mathbf{E}$ . Note the slight abuse of notation in writing  $J = J(\mathbf{E}) = \sqrt{\det(2\mathbf{E} + \mathbf{G})} = \sqrt{\det \mathbf{C}}$

394 and  $\bar{\mathbf{E}} = \bar{\mathbf{E}}(\mathbf{E})$ . It has been shown [17, 16] that, with the decomposition (46), the material  
 395 elasticity tensor reads

$$\begin{aligned}
 \mathbb{C} = & -Jp[3\mathbb{K}^{\sharp*} - 2\mathbb{I}^{\sharp*}] + 3J^2\mathbb{K}\mathbb{K}^{\sharp*} + \\
 & + J^{1/3}[\mathbf{B} \otimes (\mathbb{M}^* : \mathbf{Y}) + (\mathbb{M}^* : \mathbf{Y}) \otimes \mathbf{B}] + \\
 & + J^{-4/3}\mathbb{M}^* : \tilde{\mathbb{C}} : \mathbb{M}^{*T} + \\
 & + \frac{2}{3}J^{-2/3}\text{Tr}^*(\tilde{\mathbf{S}})\mathbb{M}^{\sharp*} - \\
 & - \frac{2}{3}[\mathbf{B} \otimes \text{Dev}^*(\mathbf{S}) + \text{Dev}^*(\mathbf{S}) \otimes \mathbf{B}], \tag{47}
 \end{aligned}$$

396 where  $p = -\partial\hat{\Psi}/\partial J$  is the hydrostatic pressure,  $\tilde{\mathbf{S}} = \partial\hat{\Psi}/\partial\bar{\mathbf{E}}$  is the second Piola-Kirchhoff  
 397 pseudo stress,  $\mathbb{K} = \partial^2\hat{\Psi}/\partial J^2$  is the (large strain) bulk modulus,  $\mathbf{Y} = \partial^2\hat{\Psi}/\partial J\partial\bar{\mathbf{E}}$  is the  
 398 coupling tensor (or interaction tensor),  $\tilde{\mathbb{C}} = \partial^2\hat{\Psi}/\partial\bar{\mathbf{E}}^2$  is the pseudo elasticity tensor,  
 399  $\text{Tr}^*(\tilde{\mathbf{S}}) = \mathbf{C} : \tilde{\mathbf{S}}$  is the pulled-back trace of  $\tilde{\mathbf{S}}$ ,  $\text{Dev}^*(\mathbf{S}) = \mathbb{M}^* : \mathbf{S} = J^{-2/3}\mathbb{M}^* : \tilde{\mathbf{S}}$  is the  
 400 pulled-back deviatoric part of  $\mathbf{S}$ . The spatial elasticity tensor reads

$$\begin{aligned}
 \mathbb{c} = & -p[3\mathbb{K}^{\sharp} - 2\mathbb{I}^{\sharp}] + 3J\mathbb{K}\mathbb{K}^{\sharp} + \\
 & + J^{1/3}[\mathbf{g}^{-1} \otimes (\mathbb{M} : \mathbf{y}) + (\mathbb{M} : \mathbf{y}) \otimes \mathbf{g}^{-1}] + \\
 & + J^{-4/3}\mathbb{M} : \tilde{\mathbb{C}} : \mathbb{M}^T + \\
 & + \frac{2}{3}J^{-2/3}\text{tr}(\tilde{\boldsymbol{\sigma}})\mathbb{M}^{\sharp} - \\
 & - \frac{2}{3}[\mathbf{g}^{-1} \otimes \text{dev}(\boldsymbol{\sigma}) + \text{dev}(\boldsymbol{\sigma}) \otimes \mathbf{g}^{-1}], \tag{48}
 \end{aligned}$$

401 where  $\tilde{\mathbb{C}} = J^{-1}\chi_*[\tilde{\mathbb{C}}]$ ,  $\tilde{\boldsymbol{\sigma}} = J^{-1}\chi_*[\tilde{\mathbf{S}}]$ ,  $\text{dev}(\boldsymbol{\sigma}) = J^{-1}\chi_*[\text{Dev}^*(\mathbf{S})]$ , and  $\mathbf{y} = J^{-1}\chi_*[\mathbf{Y}]$  are the  
 402 inverse Piola transforms of  $\tilde{\mathbb{C}}$ ,  $\tilde{\mathbf{S}}$ ,  $\text{Dev}^*(\mathbf{S})$ , and  $\mathbf{Y}$ , respectively, and  $\text{tr}(\tilde{\boldsymbol{\sigma}}) = J^{-1}\text{Tr}^*(\tilde{\mathbf{S}})$ .

403 If the undeformed configuration, achieved when  $\mathbf{E}$  vanishes and  $J$  is identically one,  
 404 is also stress-free, then both  $p$  and  $\boldsymbol{\sigma}$  vanish identically, and the linear elasticity tensor is  
 405 obtained from Equation (48) as

$$\mathbb{L} = 3\kappa\mathbb{K}^{\sharp} + \mathbf{g}^{-1} \otimes [\mathbb{M} : \boldsymbol{\alpha}] + [\mathbb{M} : \boldsymbol{\alpha}] \otimes \mathbf{g}^{-1} + \mathbb{M} : [\tilde{\mathbb{L}} + 2\beta\mathbb{M}^{\sharp}] : \mathbb{M}^T, \tag{49}$$

406 where the (linear elasticity) bulk modulus  $\kappa$ ,  $\boldsymbol{\alpha}$ ,  $\beta$  and  $\tilde{\mathbb{L}}$  are the values of  $\mathbb{K}$ ,  $\mathbf{y}$ ,  $\frac{1}{3}\text{tr}(\tilde{\boldsymbol{\sigma}})$  and  
 407  $\tilde{\mathbb{C}}$ , respectively, in the undeformed configuration.

408 It has also been shown [17] that, in the purely algebraic decomposition of the linear  
 409 elasticity tensor, obtained by premultiplying  $\mathbb{L}$  by the identity  $\mathbb{I}$ , post multiplying by  $\mathbb{I}^T$ ,  
 410 and decomposing the identity into  $\mathbb{K} + \mathbb{M}$ , i.e.,

$$\begin{aligned}
 \mathbb{L} = \mathbb{I} : \mathbb{L} : \mathbb{I}^T & = (\mathbb{K} + \mathbb{M}) : \mathbb{L} : (\mathbb{K} + \mathbb{M})^T \\
 & = \mathbb{K} : \mathbb{L} : \mathbb{K}^T + \mathbb{K} : \mathbb{L} : \mathbb{M}^T + \mathbb{M} : \mathbb{L} : \mathbb{K}^T + \mathbb{M} : \mathbb{L} : \mathbb{M}^T, \tag{50}
 \end{aligned}$$

411 the identities

$$\mathbb{K} : \mathbb{L} : \mathbb{K}^T = 3\kappa\mathbb{K}^{\sharp}, \tag{51a}$$

$$\mathbb{K} : \mathbb{L} : \mathbb{M}^T = \mathbf{g}^{-1} \otimes [\mathbb{M} : \boldsymbol{\alpha}], \tag{51b}$$

$$\mathbb{M} : \mathbb{L} : \mathbb{K}^T = [\mathbb{M} : \boldsymbol{\alpha}] \otimes \mathbf{g}^{-1}, \tag{51c}$$

$$\mathbb{M} : \mathbb{L} : \mathbb{M}^T = \mathbb{M} : [\tilde{\mathbb{L}} + 2\beta\mathbb{M}^{\sharp}] : \mathbb{M}^T, \tag{51d}$$

412 hold, implying that the expression (49) of the linear elasticity tensor, obtained by use of  
 413 the decomposition of the deformation, is term-by-term equivalent to the purely algebraic  
 414 decomposition (50). In Equations (51), the term (51a) is *purely spherical*, the terms (51b)

415 and (51c) are *mixed*, and the term (51d) is *purely deviatoric*. Equations (51) are the  
 416 key result in the evaluation of the linear elasticity tensor of strictly incompressible and  
 417 quasi-incompressible materials.

418 It is very important to note that, because of the orthogonality of the spherical and  
 419 deviatoric operators, each of the four terms (51) is orthogonal to the other three in the  
 420 scalar product induced by the metric  $\mathbf{g}$  in the space  $[T\mathcal{S}]_0^4$  of fourth-order “contravariant”  
 421 tensors. In particular, we note that, since  $\mathbb{K} : \mathbb{L} : \mathbb{K}^T = 3\kappa \mathbb{K}^\sharp$  is orthogonal to the other  
 422 three terms, and  $\langle \mathbb{K}^\sharp, \mathbb{K}^\sharp \rangle = 1$ , it is possible to obtain the bulk modulus as

$$\kappa = \frac{1}{3} \langle \mathbb{K}^\sharp, \mathbb{L} \rangle = \frac{1}{3} \langle \mathbb{K}^\sharp, \mathbb{K} : \mathbb{L} : \mathbb{K}^T \rangle = \frac{1}{3} \langle \mathbb{K}^\sharp, 3\kappa \mathbb{K}^\sharp \rangle = \frac{1}{9} g_{ab} g_{cd} L^{abcd}. \quad (52)$$

### 423 3 Incompressible Hyperelasticity

424 This section is dedicated to the derivation of the conditions that the linear elasticity  
 425 tensor must obey for the cases of strict incompressibility and quasi-incompressibility. Strict  
 426 incompressibility is a kinematical constraint on the volumetric deformation  $J = \det \mathbf{F}$ ,  
 427 whereas quasi-incompressibility is obtained by requiring that a very large elastic energy is  
 428 needed to make the volumetric deformation  $J$  change from its initial value of 1.

#### 429 3.1 Strict Incompressibility

430 When the deformation is isochoric (strict incompressibility),  $\dot{\mathbf{E}}$  in Equation (40) is no  
 431 longer arbitrary. Rather, it is subjected to the constraint

$$\dot{J} = J \operatorname{div}(\mathbf{v}) = J \mathbf{B} : \dot{\mathbf{E}} = \frac{1}{2} J \mathbf{B} : \dot{\mathbf{C}} = 0, \quad (53)$$

432 which states that the only admissible deformations are those such that  $\mathbf{B} = \mathbf{C}^{-1}$  is or-  
 433 thogonal to  $\dot{\mathbf{C}}$  in the sense of (53), i.e.,  $\mathbf{B} : \dot{\mathbf{C}} = B^{AB} \dot{C}_{AB} = 0$ . Since the constraint  
 434 (53) is holonomic, it can be put into algebraic form by direct integration with respect to  
 435 time. Setting, with the usual abuse of notation,  $J = J(\mathbf{E})$ , and performing the integra-  
 436 tion under the condition that  $J$  is equal to one in the undeformed configuration leads to  
 437  $J = J(\mathbf{E}) = 1$ .

438 Combining Equations (40) and (53) one obtains

$$\mathbf{S} - \frac{\partial \hat{W}}{\partial \mathbf{E}}(\mathbf{E}) = \lambda J \mathbf{B}, \quad (54)$$

439 where  $\lambda$  is an arbitrary scalar, Lagrange multiplier arising from the kinematical constraint  
 440 of isochoric motion. If we denote the hydrostatic pressure by  $\pi$ , in order to distinguish  
 441 it from the “constitutive” hydrostatic pressure  $p = -\partial \hat{\Psi} / \partial J$  introduced in the previous  
 442 section, and recall the definition of hydrostatic pressure as the scalar of the spherical part  
 443 (hydrostatic stress) of the Cauchy stress  $\boldsymbol{\sigma}$ ,

$$-\pi \mathbf{g}^{-1} = \mathbb{K} : \boldsymbol{\sigma} = \frac{1}{3} \operatorname{tr}(\boldsymbol{\sigma}) \mathbf{g}^{-1} = \frac{1}{3} (\mathbf{g} : \boldsymbol{\sigma}) \mathbf{g}^{-1}, \quad (55)$$

444 and its full Piola transform,

$$-J \pi \mathbf{B} = J \mathbb{K}^* : \mathbf{S} = \frac{1}{3} J \operatorname{Tr}^*(\mathbf{S}) \mathbf{B} = \frac{1}{3} J (\mathbf{C} : \mathbf{S}) \mathbf{B}, \quad (56)$$

445 involving the second Piola-Kirchhoff stress  $\mathbf{S}$ , it can be shown that  $\lambda = -\pi$  if, and only if,

$$\operatorname{Tr}^* \left( \frac{\partial \hat{W}}{\partial \mathbf{E}}(\mathbf{E}) \right) = \frac{\partial \hat{W}}{\partial \mathbf{E}}(\mathbf{E}) : \mathbf{C} = 0, \quad (57)$$

446 where we recall that  $\text{Tr}^*$  is the pulled-back trace operator [16] described in Section 2.2. If  
 447  $\hat{W}$  is regarded a function of  $\mathbf{C}$  rather than of  $\mathbf{E}$ , Equation (57) means that the potential  
 448 has to be a homogeneous function of order zero in  $\mathbf{C}$ . Exploiting Euler's theorem on  
 449 homogeneous functions (see [3]) and going back to the argument  $\mathbf{E}$ , one shows that the  
 450 potential  $\hat{W}$  must have the form

$$\hat{W}(\mathbf{E}) = \hat{W}_d(\bar{\mathbf{E}}(\mathbf{E})), \quad (58)$$

451 i.e.,  $\hat{W}$  must be given by an explicit function  $\hat{W}_d$  of the distortional strain  $\bar{\mathbf{E}}$ , called  
 452 distortional potential. The second Piola-Kirchhoff stress reads

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{E}) = \text{Sph}^*(\mathbf{S}) + \text{Dev}^*(\mathbf{S}) = -J\pi \mathbf{B} + \frac{\partial \hat{W}}{\partial \bar{\mathbf{E}}}(\mathbf{E}), \quad (59)$$

453 where we recall that  $\text{Sph}^*$  and  $\text{Dev}^*$  are the pulled-back spherical and deviatoric operators  
 454 associated with  $\mathbb{K}^*$  and  $\mathbb{M}^*$ , respectively (Equation (12)). The material elasticity tensor is  
 455 evaluated as in Equation (60), keeping in mind that all derivatives of  $\hat{W}(\mathbf{E}) = \hat{W}_d(\bar{\mathbf{E}}(\mathbf{E}))$   
 456 with respect to  $J$  vanish identically:

$$\begin{aligned} \mathbb{C} = & -J\pi [3\mathbb{K}^{\sharp*} - 2\mathbb{I}^{\sharp*}] + \\ & + J^{-4/3} \mathbb{M}^* : \tilde{\mathbb{C}} : \mathbb{M}^{*T} + \\ & + \frac{2}{3} J^{-2/3} \text{Tr}^*(\tilde{\mathbf{S}}) \mathbb{M}^{\sharp*} - \\ & - \frac{2}{3} [\mathbf{B} \otimes \text{Dev}^*(\mathbf{S}) + \text{Dev}^*(\mathbf{S}) \otimes \mathbf{B}]. \end{aligned} \quad (60)$$

457 The spatial elasticity tensor is therefore

$$\begin{aligned} \mathbb{C} = & -\pi [3\mathbb{K}^{\sharp} - 2\mathbb{I}^{\sharp}] + \\ & + J^{-4/3} \mathbb{M} : \tilde{\mathbb{C}} : \mathbb{M}^T + \\ & + \frac{2}{3} J^{-2/3} \text{tr}(\tilde{\boldsymbol{\sigma}}) \mathbb{M}^{\sharp} - \\ & - \frac{2}{3} [\mathbf{g}^{-1} \otimes \text{dev}(\boldsymbol{\sigma}) + \text{dev}(\boldsymbol{\sigma}) \otimes \mathbf{g}^{-1}], \end{aligned} \quad (61)$$

458 and the linear elasticity tensor reduces to

$$\mathbb{L} = \mathbb{M} : [\tilde{\mathbb{L}} + 2\beta \mathbb{M}^{\sharp}] : \mathbb{M}^T. \quad (62)$$

459 Comparing Equations (62) and (51) we conclude that the linear elasticity tensor  $\mathbb{L}$  of a  
 460 strictly incompressible material must obey the three conditions

$$\mathbb{K} : \mathbb{L} : \mathbb{K}^T = \mathbb{O}, \quad \mathbb{K} : \mathbb{L} : \mathbb{M}^T = \mathbb{O}, \quad \mathbb{M} : \mathbb{L} : \mathbb{K}^T = \mathbb{O}, \quad (63)$$

461 i.e., it must not contain spherical or mixed terms, but exclusively the deviatoric one. This  
 462 result is valid in general, *regardless* of the material symmetry.

### 463 3.2 Quasi-Incompressibility

464 In this case, the elastic potential admits the particular *decoupled* form

$$\hat{W}(\mathbf{E}) = \hat{\Psi}(J(\mathbf{E}), \bar{\mathbf{E}}(\mathbf{E})) = \hat{U}(J(\mathbf{E})) + \hat{W}_d(\bar{\mathbf{E}}(\mathbf{E})). \quad (64)$$

The mixed derivative  $\mathbf{Y} = \partial^2 \hat{\Psi} / \partial J \partial \bar{\mathbf{E}}$  vanishes identically, which yields the material  
 elasticity tensor

$$\begin{aligned} \mathbb{C} = & -Jp [3\mathbb{K}^{\sharp*} - 2\mathbb{I}^{\sharp*}] + 3J^2 \mathbb{K} \mathbb{K}^{\sharp*} + \\ & + J^{-4/3} \mathbb{M}^* : \tilde{\mathbb{C}} : \mathbb{M}^{*T} + \\ & + \frac{2}{3} J^{-2/3} \text{Tr}^*(\tilde{\mathbf{S}}) \mathbb{M}^{\sharp*} - \\ & - \frac{2}{3} [\mathbf{B} \otimes \text{Dev}^*(\mathbf{S}) + \text{Dev}^*(\mathbf{S}) \otimes \mathbf{B}], \end{aligned} \quad (65)$$

the spatial elasticity tensor

$$\begin{aligned}
\mathbb{C} = & -p [3\mathbb{K}^\sharp - 2\mathbb{I}^\sharp] + 3JK\mathbb{K}^\sharp + \\
& + J^{-4/3}\mathbb{M} : \tilde{\mathbb{C}} : \mathbb{M}^T + \\
& + \frac{2}{3}J^{-2/3}\text{tr}(\tilde{\boldsymbol{\sigma}})\mathbb{M}^\sharp - \\
& - \frac{2}{3}[\mathbf{g}^{-1} \otimes \text{dev}(\boldsymbol{\sigma}) + \text{dev}(\boldsymbol{\sigma}) \otimes \mathbf{g}^{-1}], \tag{66}
\end{aligned}$$

465 and the linear elasticity tensor

$$\mathbb{L} = 3\kappa\mathbb{K}^\sharp + \mathbb{M} : [\tilde{\mathbb{L}} + 2\beta\mathbb{M}^\sharp] : \mathbb{M}^T. \tag{67}$$

466 Comparing Equations (67) and (51), we deduce that the linear elasticity tensor of a quasi-  
467 incompressible material must obey the two conditions

$$\mathbb{K} : \mathbb{L} : \mathbb{M}^T = \mathbb{O}, \quad \mathbb{M} : \mathbb{L} : \mathbb{K}^T = \mathbb{O}, \tag{68}$$

468 i.e., it must contain no mixed terms, but only the spherical and the deviatoric ones. By  
469 comparing Equations (62) and (67) and recalling (end of Section 2.5) that the term  $\mathbb{K} :$   
470  $\mathbb{L} : \mathbb{K}^T = 3\kappa\mathbb{K}^\sharp$  is orthogonal to the other three terms in Equation (51), it is evident that,  
471 as one would expect, the linear elasticity tensor for the quasi-incompressible case has one  
472 additional parameter with respect to the strictly incompressible case. It is convenient to  
473 identify this one additional parameter with the bulk modulus  $\kappa$ , obtained in Equation (52).  
474 We emphasise again that this is valid *regardless* of the material symmetry.

475 **Remark 3.1.** We take this chance to remark that the decoupled potential (64) can be used  
476 *exclusively* for quasi-incompressible materials, and yields inconsistent material behaviour  
477 in the general compressible case: this has been reported a few decades ago by Musgrave  
478 [29] in the context of crystal elasticity, and demonstrated in a previous work [17] with  
479 the same methodology used here, i.e., by linearising the spatial elasticity tensor of the  
480 non-linear theory. Indeed, if the potential (64) were used for a compressible material,  
481 the linear elasticity tensor would be subjected to the conditions (68), which would reduce  
482 the number of independent elastic constants with respect to the general case. Therefore,  
483 one would find a compressible material with a given symmetry having *less* independent  
484 constants than expected (e.g., 4 rather than 5 for transverse isotropy, and 7, rather than 9,  
485 for orthotropy, as we shall show in Section 4 for quasi-incompressible materials). Whereas  
486 nothing, in principle, prevents conditions (68) from occurring for a compressible material,  
487 such material cannot certainly be considered a general case. Indeed, a first consequence  
488 of the adoption of (64) for the compressible case would be that an anisotropic material  
489 would not undergo distortional deformations under a hydrostatic stress, which is contrary  
490 to experimental observation (this has also been remarked in a recent paper by Vergori et  
491 al. [30]).

## 492 4 Some Particular Material Symmetries

493 The conditions (63) for strict incompressibility and (68) for quasi-incompressibility are  
494 general, and hold regardless of material symmetry. When a material symmetry is given,  
495 conditions (63) and (68) can be employed to find the number of independent components of  
496 the elasticity tensor. As we shall show, the case of isotropy is trivial. For the cases of trans-  
497 verse isotropy and orthotropy, it is convenient to enforce conditions (63) and (68) within  
498 Walpole's formalism [18], which, due to the isomorphism between fourth-order tensors and  
499 the corresponding Walpole's representations, allows for evaluating the double contractions

500 of tensors in conditions (63) and (68) by means of the matrix multiplication of the ma-  
 501 trix parts and regular multiplication of scalars of the scalar parts of the corresponding  
 502 Walpole’s representations of the tensors.

## 503 4.1 Isotropy

504 The linear elasticity tensor of a generic isotropic material is a fourth-order tensor in  
 505  $[TS]_0^4$  with the form

$$\mathbb{L} = 3\kappa \mathbb{K}^\sharp + 2\mu \mathbb{M}^\sharp, \quad (69)$$

506 where  $\kappa$  is the bulk modulus and  $\mu$  is the shear modulus. If strict incompressibility is en-  
 507 forced, conditions (63) impose that the linear elasticity tensor has only one independent  
 508 elastic modulus, the shear modulus  $\mu$ , and representation

$$\mathbb{L}_{\text{strict}} = 2\mu \mathbb{M}^\sharp. \quad (70)$$

509 In contrast, the quasi-incompressibility conditions (68), are always identically verified, and  
 510 therefore the elasticity tensor keeps two independent elastic constants, as in the general  
 511 compressible case, and reads

$$\mathbb{L}_{\text{quasi}} = 3\kappa \mathbb{K}^\sharp + 2\mu \mathbb{M}^\sharp, \quad (71)$$

512 where the bulk modulus  $\kappa$  is much larger than the shear modulus  $\mu$ .

513 **Remark 4.1.** Note that a quite common representation for isotropic elasticity tensors is  
 514 in the form  $\mathbb{L} = 3\lambda \mathbb{K}^\sharp + 2\mu \mathbb{I}^\sharp$ , where  $\lambda$  and  $\mu$  are called *Lamé’s constants*, and  $\mu$  is still  
 515 the shear modulus. This representation is very useful in several circumstances, such as,  
 516 for example, in computations based on the Finite Element Method, where the term  $2\mu \mathbb{I}^\sharp$   
 517 generates the symmetric, positive definite modified stiffness operator relating the nodal  
 518 displacements with the nodal pressures and the external generalised forces (cf., e.g., [31]).  
 519 Nevertheless, we believe that there are cases in which the representation  $\mathbb{L} = 3\kappa \mathbb{K}^\sharp + 2\mu \mathbb{M}^\sharp$   
 520 is more advantageous and physically sound. Indeed, the algebraic computations involving  
 521 the elasticity tensor are easier (and their physical meaning becomes clearer), since  $\mathbb{K}^\sharp$  and  
 522  $\mathbb{M}^\sharp$  form an orthogonal basis [18, 20, 17, 16] (in contrast,  $\mathbb{K}^\sharp$  and  $\mathbb{I}^\sharp$  do not). Moreover, the  
 523 constants  $\kappa$  and  $\mu$ , which must be both strictly positive, have a direct physical meaning.  
 524 For this reason, we prefer the representation terms of in  $\mathbb{K}^\sharp$  and  $\mathbb{M}^\sharp$ .

## 525 4.2 Transverse Isotropy

526 Using Walpole’s formalism (Section 2.3), the linear elasticity tensor  $\mathbb{L}$  of a generic trans-  
 527 versely isotropic material has representation

$$\{\tilde{\mathbb{L}}\} = \left\{ \begin{bmatrix} n & \sqrt{2}l \\ \sqrt{2}l & 2c \end{bmatrix}, 2\mu_t, 2\mu_a \right\}, \quad (72)$$

528 where  $n$  is the elastic modulus in uniaxial strain (compare with the Young’s modulus, which  
 529 is the modulus in uniaxial stress),  $c$  is the plane-strain bulk modulus (in the transverse  
 530 plane),  $l$  is called cross modulus,  $\mu_t$  is the shear modulus in the transverse plane, and  $\mu_a$   
 531 is the shear modulus in any plane  $\mu_t$  containing the symmetry axis.

532 The three strict incompressibility conditions (63) reduce to the two scalar conditions

$$n + 4(c + l) = 0, \quad n - 2c + l = 0, \quad (73)$$

533 where we note that  $\mathbb{K} : \mathbb{L} : \mathbb{M}^T = \mathbb{O}$  and  $\mathbb{M} : \mathbb{L} : \mathbb{K}^T = \mathbb{O}$  both yield  $n - 2c + l = 0$ . These two  
 534 conditions state that only one of  $n$ ,  $c$  and  $l$  is independent. Mathematically, electing any

535 of the three as the independent parameter is indifferent. However, looking at the physical  
 536 meaning of each, we note that the most appropriate choice is

$$\alpha = -l. \quad (74)$$

537 Indeed, both uniaxial strain and plane strain, to which  $n$  and  $c$  refer, respectively, are strain  
 538 states that *cannot* be attained under the constraint of isochoric motion. The parameter  $l$ ,  
 539 instead, can be thought to be related to a triaxial state of strain that is compatible with  
 540 isochoric motion. The cross-modulus  $l$  is the transversely isotropic equivalent of the first  
 541 Lamé's modulus  $\lambda$  of isotropic elasticity, to which it reduces in the limit case, as it can  
 542 be easily verified with Spencer's representation [21]. Note that, in general, similarly to  $\lambda$ ,  
 543  $l$  can be negative, and must indeed be negative to ensure positive semi-definiteness and  
 544 therefore convexity for the case of strict incompressibility, as we shall see in Section 5.  
 545 With this choice, the linear elasticity tensor for strict incompressibility is represented by

$$\{\tilde{\mathbb{L}}_{\text{strict}}\} = \left\{ \left[ \begin{array}{cc} 2\alpha & -\sqrt{2}\alpha \\ -\sqrt{2}\alpha & \alpha \end{array} \right], 2\mu_t, 2\mu_a \right\}, \quad (75)$$

546 with only three (from the original five) independent elastic constants.

547 The quasi-incompressibility conditions (68) yield the single scalar condition

$$n - 2c + l = 0, \quad (76)$$

548 meaning that only two of  $n$ ,  $c$  and  $l$  are independent. Here we choose, as independent  
 549 parameters, the bulk modulus

$$\kappa = \frac{1}{3} \langle \mathbb{K}^\sharp, \mathbb{L} \rangle = \frac{1}{9} g_{ab} g_{cd} L^{abcd} = \frac{1}{9} [n + 4(c + l)], \quad (77)$$

550 which is a linear combination of  $n$ ,  $c$  and  $l$ , obtained by applying Equation (52) to the case  
 551 of transverse isotropy, and

$$\alpha' = \kappa - l. \quad (78)$$

552 With this choice, the linear elasticity tensor for the transversely isotropic quasi-incompressible  
 553 case reads

$$\{\tilde{\mathbb{L}}_{\text{quasi}}\} = \left\{ \left[ \begin{array}{cc} \kappa + 2\alpha' & \sqrt{2}(\kappa - \alpha') \\ \sqrt{2}(\kappa - \alpha') & 2\kappa + \alpha' \end{array} \right], 2\mu_t, 2\mu_a \right\}, \quad (79)$$

554 with four independent elastic constants: one more than for the case of strict incompress-  
 555 ibility. Recalling Walpole's transversely isotropic representation of  $\mathbb{K}^\sharp$  (Equations (24)),  
 556 the elasticity tensor can be written as

$$\{\tilde{\mathbb{L}}_{\text{quasi}}\} = 3\kappa \{\tilde{\mathbb{K}}\} + \{\tilde{\mathbb{L}}'_{\text{strict}}\}, \quad (80)$$

557 where  $\{\tilde{\mathbb{L}}'_{\text{strict}}\}$  has the same form as  $\{\tilde{\mathbb{L}}_{\text{strict}}\}$  of Equation (75), except for  $\alpha$  being replaced  
 558 by  $\alpha'$ . Equation (80) emphasises that the quasi-incompressible case has one additional  
 559 independent elastic constant with respect to the strictly incompressible case.

### 560 4.3 Orthotropy

561 Using Walpole's formalism (Section 2.4), the linear elasticity tensor  $\mathbb{L}$  of a generic or-  
 562 thotropic material has representation

$$\{\tilde{\mathbb{L}}\} = \left\{ \left[ \begin{array}{ccc} L^{1111} & L^{1122} & L^{1133} \\ L^{1122} & L^{2222} & L^{2233} \\ L^{1133} & L^{2233} & L^{3333} \end{array} \right], 2\mu_{23}, 2\mu_{13}, 2\mu_{12} \right\}, \quad (81)$$

563 where the diagonal elements of the symmetric  $3 \times 3$  matrix are the moduli in uniaxial strain  
 564 in the three orthotropic directions, the off-diagonal elements are the cross moduli, and  $\mu_{pq}$   
 565 are the shear moduli in the  $pq$ -planes.

566 Conditions (63) for strict incompressibility reduce to the three independent scalar con-  
 567 ditions

$$L^{1111} + L^{2222} + L^{3333} + 2L^{2233} + 2L^{1133} + 2L^{1122} = 0, \quad (82a)$$

$$2L^{1111} - L^{2222} - L^{3333} - 2L^{2233} + L^{1133} + L^{1122} = 0, \quad (82b)$$

$$-L^{1111} + 2L^{2222} - L^{3333} + L^{2233} - 2L^{1133} + L^{1122} = 0, \quad (82c)$$

568 which imply that only three of the six  $L^{ppqq}$  (no sum on  $p$  and  $q$ ) are independent. Supported  
 569 by arguments analogical to those made for the case of transverse isotropy, we choose, as  
 570 independent parameters, the negatives of the cross moduli, i.e.,

$$\alpha_{pq} = -L^{ppqq}, \quad p \neq q, \text{ no sum on } p \text{ and } q, \quad (83)$$

571 and obtain the representation

$$\{\tilde{L}_{\text{strict}}\} = \left\{ \begin{bmatrix} \alpha_{12} + \alpha_{13} & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{12} & \alpha_{12} + \alpha_{23} & -\alpha_{23} \\ -\alpha_{13} & -\alpha_{23} & \alpha_{13} + \alpha_{23} \end{bmatrix}, 2\mu_{23}, 2\mu_{13}, 2\mu_{12} \right\}, \quad (84)$$

572 with six independent elastic constants (from the original nine).

573 For the case of orthotropy, the quasi-incompressibility conditions (68) yield the two  
 574 scalar conditions

$$2L^{1111} - L^{2222} - L^{3333} - 2L^{2233} + L^{1133} + L^{1122} = 0, \quad (85a)$$

$$-L^{1111} + 2L^{2222} - L^{3333} + L^{2233} - 2L^{1133} + L^{1122} = 0, \quad (85b)$$

575 meaning that only four of the six  $L^{ppqq}$  (no sum on  $p$  and  $q$ ) are independent. If the  
 576 independent parameters are chosen to be the bulk modulus

$$\begin{aligned} \kappa &= \frac{1}{3} \langle \mathbb{K}^\sharp, \mathbb{L} \rangle = \frac{1}{9} g_{ab} g_{cd} L^{abcd} = \\ &= \frac{1}{9} (L^{1111} + L^{2222} + L^{3333} + 2L^{2233} + 2L^{1133} + 2L^{1122}), \end{aligned} \quad (86)$$

577 obtained by applying equation (52) to the case of orthotropy, and

$$\alpha'_{pq} = \kappa - L^{ppqq}, \quad p \neq q, \text{ no sum on } p \text{ and } q, \quad (87)$$

578 the linear elasticity tensor for the orthotropic quasi-incompressible case reads

$$\{\tilde{L}_{\text{quasi}}\} = \left\{ \begin{bmatrix} \kappa + \alpha'_{12} + \alpha'_{13} & \kappa - \alpha'_{12} & \kappa - \alpha'_{13} \\ \kappa - \alpha'_{12} & \kappa + \alpha'_{12} + \alpha'_{23} & \kappa - \alpha'_{23} \\ \kappa - \alpha'_{13} & \kappa - \alpha'_{23} & \kappa + \alpha'_{13} + \alpha'_{23} \end{bmatrix}, 2\mu_{23}, 2\mu_{13}, 2\mu_{12} \right\}, \quad (88)$$

579 with seven independent elastic constants: again, one more than for the case of strict  
 580 incompressibility. Similarly to what has been done for the case of transverse isotropy,  
 581 considering the orthotropic representation of  $\mathbb{K}^\sharp$  (Equations (36)), the elasticity tensor can  
 582 be written

$$\{\tilde{L}_{\text{quasi}}\} = 3\kappa \{\tilde{K}\} + \{\tilde{L}'_{\text{strict}}\}, \quad (89)$$

583 where  $\{\tilde{L}'_{\text{strict}}\}$  has the same form as  $\{\tilde{L}_{\text{strict}}\}$  of Equation (84), except for the parameters  
 584  $\alpha_{pq}$  being replaced by  $\alpha'_{pq}$ .

## 585 5 Positive Definiteness and Invertibility

586 As already remarked at the end of Section 3.2, by comparing Equations (62) and (67),  
 587 we deduce that the strictly incompressible and the quasi-incompressible cases differ from  
 588 each other because of the presence of the bulk modulus  $\kappa$  as an additional parameter in  
 589 the latter case. Here we would like to show that, for this reason, the linear elasticity  
 590 tensor is positive semi-definite for the case of strict incompressibility and positive defi-  
 591 nite for the case of quasi-incompressibility. For the case of quasi-incompressibility, the  
 592 positive definiteness of the elasticity tensor implies its invertibility. For the case of strict  
 593 incompressibility, the positive semi-definiteness of the elasticity tensor, implying its non-  
 594 invertibility, mathematically translates the physical impossibility to have an infinite bulk  
 595 modulus. This can be shown by looking at the examples of isotropy, transverse isotropy,  
 596 and orthotropy reported in Section 4.

597 For the isotropic quasi-incompressible case (but this is identical for the general com-  
 598 pressible case), the inverse of the elasticity tensor  $\mathbb{L}_{\text{quasi}} = 3\kappa \mathbb{K}^\sharp + 2\mu \mathbb{M}^\sharp$  is given by  
 599  $\mathbb{L}_{\text{quasi}}^{-1} = (3\kappa)^{-1} \mathbb{K}^\flat + (2\mu)^{-1} \mathbb{M}^\flat$ , as is immediately verifiable by evaluating  $\mathbb{L}_{\text{quasi}} : \mathbb{L}_{\text{quasi}}^{-1} = \mathbb{I}$  in  
 600 components, or by accounting for the orthogonality and idempotence of  $\mathbb{K}$  and  $\mathbb{M}$  [19, 18, 16].  
 601 Moreover,  $\mathbb{L}_{\text{quasi}}$  is positive definite if, and only if, both  $\kappa$  and  $\mu$  are positive. For the  
 602 isotropic strictly incompressible case, it is evident that  $\mathbb{L}_{\text{strict}} = 2\mu \mathbb{M}^\sharp$  is not invertible, and  
 603 therefore it is only positive semi-definite, provided that  $\mu$  is positive.

604 Exploiting Walpole's formalism [18], the transversely isotropic and orthotropic cases are  
 605 treated in a similar way. In Walpole's representation of the elasticity tensors for transverse  
 606 isotropy (Equations (75) and (79)) and orthotropy (Equations (84) and (88)), the individual  
 607 scalars (shear moduli) must be positive and the matrix must be positive definite to ensure  
 608 positive definiteness of the tensor. The positive definiteness of the matrix parts can be  
 609 checked by evaluating their eigenvalues.

610 For transverse isotropy, the eigenvalues of the  $2 \times 2$  matrix are

$$0, \quad 3\alpha, \quad (90)$$

611 for strict incompressibility (positive semi-definiteness attained for  $\alpha > 0$ , i.e.,  $l < 0$ ), and

$$3\kappa, \quad 3\alpha' = 3(\kappa - l), \quad (91)$$

612 for quasi-incompressibility (positive definiteness attained for  $\kappa > 0$  and  $\kappa > l$ ).

613 For orthotropy, the eigenvalues of the  $3 \times 3$  matrix are

$$0, \quad (\alpha_{23} + \alpha_{13} + \alpha_{12}) \pm \sqrt{(\alpha_{23} + \alpha_{13} + \alpha_{12})^2 - 3(\alpha_{23}\alpha_{13} + \alpha_{13}\alpha_{12} + \alpha_{12}\alpha_{23})}, \quad (92)$$

614 for strict incompressibility (positive semi-definiteness attained for  $(\alpha_{23} + \alpha_{13} + \alpha_{12}) > 0$ ,  
 615 i.e.,  $(L^{2233} + L^{1133} + L^{1122}) < 0$ , as the symmetry of the matrix ensures that the eigenvalues  
 616 are all real, and the term under square root is positive and smaller than  $(\alpha_{23} + \alpha_{13} + \alpha_{12})$ ,  
 617 in absolute value), and

$$3\kappa, \quad (\alpha'_{23} + \alpha'_{13} + \alpha'_{12}) \pm \sqrt{(\alpha'_{23} + \alpha'_{13} + \alpha'_{12})^2 - 3(\alpha'_{23}\alpha'_{13} + \alpha'_{13}\alpha'_{12} + \alpha'_{12}\alpha'_{23})}, \quad (93)$$

618 for quasi-incompressibility (positive definiteness attained for  $\kappa > 0$  and  $(\alpha'_{23} + \alpha'_{13} + \alpha'_{12}) >$   
 619  $0$ , i.e.,  $\kappa > \frac{1}{3}(L^{2233} + L^{1133} + L^{1122})$ ).

620 We conclude noting that, regardless of the material symmetry, if the term  $3\kappa \mathbb{K}^\sharp$ , with  
 621  $\kappa > 0$ , is added to  $\mathbb{L}_{\text{strict}}$  (which is equivalent to referring to the corresponding quasi-  
 622 incompressible material), the resulting fourth-order tensor can be inverted. Then, the  
 623 strictly incompressible case is retrieved by performing the limit for  $\kappa \rightarrow \infty$ .

## 624 6 Discussion

625 In order to retrieve the correct expression of the linear elasticity tensor for incompress-  
 626 ible materials, we followed the path dictated by the non-linear Theory of Elasticity, and  
 627 modelled incompressibility in two ways. In the strict incompressibility approach, one im-  
 628 poses the kinematical constraint of isochoric motion, and treats the hydrostatic pressure  
 629 as the associated Lagrange multiplier. In the quasi-incompressibility approach, one uses  
 630 the bulk modulus as a penalty number to keep volumetric deformations very small. We de-  
 631 rived the algebraic conditions for a fourth-order tensor to represent the elasticity tensor of  
 632 strictly incompressible and quasi-incompressible materials, regardless of the material sym-  
 633 metry. This constitutes a rigorous framework for the determination of the correct form  
 634 of the linear elasticity tensor of incompressible materials, which can be used to enforce  
 635 the physical requirement of compatibility of a non-linear elastic material with its linear  
 636 counterpart [7, 17, 16].

637 By using the elegant formalism introduced by Walpole [18], we studied the cases of  
 638 isotropy, transverse isotropy and orthotropy. We proved that the linear elasticity tensor  
 639 for the case of isotropy, transverse isotropy and orthotropy is characterised by 1, 3, 6 in-  
 640 dependent material parameters, respectively, in the strictly incompressible case (i.e. when  
 641 the kinematically admissible deformations are isochoric), and by 2, 4, 7 independent ma-  
 642 terial parameters, respectively, in the quasi-incompressible case (i.e. when the volumetric-  
 643 deviatoric decoupling of the strain energy function is considered), from the original 2, 5,  
 644 9 parameters, respectively, of compressible linear elasticity. Walpole's formalism makes  
 645 the study of the positive definiteness of the elasticity tensor extremely simple: a tensor  
 646 is positive definite if its Walpole's representation is such that the matrix part is positive  
 647 definite, and all the scalars are positive (note that if the tensor is positive definite then it  
 648 is invertible, and that positive semi-definiteness is treated analogously).

649 An immediate application of the results presented here is for all those elastic potentials  
 650 defined in terms of the linear elasticity tensor. This is the case of Fung-type potentials  
 651 [32, 33, 34], which are monotonic functions of a quadratic form in the Green-Lagrange  
 652 strain  $\mathbf{E}$ , i.e., take the form  $\hat{W}(\mathbf{E}) = a \varphi(\frac{1}{2} \mathbf{E} : \mathbb{Q} : \mathbf{E})$ , where  $a$  is a material constant,  $\mathbb{Q}$   
 653 is a symmetric, positive definite (or positive semi-definite) fourth-order tensor in  $[TB_R]_0^4$ ,  
 654 and  $\varphi$  is a convex, monotonic function (Fung's original potential is exponential, with  
 655  $\varphi = \exp -\text{id}$ , where  $\text{id}$  is the identity in  $\mathbb{R}$ ). It has been shown [35] that, in order to ensure  
 656 convexity of the potential, the fourth-order tensor  $\mathbb{Q}$  of the quadratic form must be related  
 657 to the material linear elasticity  $\mathbb{L}$  by  $\mathbb{Q} = a^{-1} \mathbb{L}$ . Therefore, for a strictly incompressible  
 658 or quasi-incompressible Fung-type potential, since the spatial linear elasticity tensor  $\mathbb{L}$   
 659 must obey the algebraic conditions (63) or (68), respectively, so must the material linear  
 660 elasticity tensor  $\mathbb{L}$  (see Equation (45)), and therefore so must the tensor  $\mathbb{Q}$  of the quadratic  
 661 form (with the appropriate spherical and deviatoric operators: in this case the material  
 662 operators  $\mathbb{K}$  and  $\mathbb{M}$ , which are analogical to the spatial  $\mathbb{K}$  and  $\mathbb{M}$ , and whose expression is  
 663 reported in [16]). An application of incompressible Fung-type potentials can be found in  
 664 the work by Bellini et al. [36].

665 We note that results equivalent to those presented here have been found, for the case  
 666 of transverse isotropy, by deBotton and Ponte-Castañeda [37] based on the earlier - but  
 667 equivalent - version of Walpole's formalism [19]. Based on the spectral decomposition of the  
 668 compliance tensor, Itskov and Aksel [38] introduced a procedure to study the admissible  
 669 values of the elastic constants for the cases of strict and quasi-incompressibility, and found  
 670 a closed expression of the elasticity tensor without explicit use of the eigenvalue problem  
 671 solution. Moreover, our results for the case of quasi-incompressibility coincide with those  
 672 recently reported by Vergori et al. [30], who also proved that, for the case of monoclinic

673 symmetry, the number of independent parameters reduce from 13 to 10. A natural exten-  
674 sion of our work could be to include the monoclinic symmetry to retrieve the results by  
675 Vergori et al. [30] in the quasi-incompressible case, and to study the strictly incompressible  
676 case. Finally, it is an open problem to understand how the results presented in this work  
677 should be generalised to the case of second-gradient continua [39, 40], particularly when  
678 used to describe fibre-reinforced composites or porous media [41, 42, 43, 44] saturated with  
679 incompressible fluids, or in the case of  $N$ -th grade continua [45], or in the case of beams,  
680 plates and shells [46, 47].

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