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# A NOTE ON INTEGER POLYNOMIALS WITH SMALL INTEGRALS

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ABSTRACT. The smart method of Gelfond–Shnirelman–Nair allows to obtain in elementary way a lower bound for the prime counting function  $\pi(x)$  in terms of integrals of suitable integer polynomials. In this paper we studied the properties of the class of integer polynomials relevant for the method.

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## 1. INTRODUCTION

Let  $\pi(x)$  be the number of primes not exceeding  $x$ . The Prime Number Theorem (PNT), independently proved in 1896 by Hadamard and the de la Vallée Poussin, states that

$$\pi(N) \sim \frac{N}{\log N} \quad N \rightarrow +\infty.$$

In 1851, Chebyshev [6] made the first step towards the PNT by proving that, given  $\varepsilon > 0$ ,

$$(c_1 - \varepsilon) \frac{N}{\log N} \leq \pi(N) \leq (c_2 + \varepsilon) \frac{N}{\log N}$$

where  $c_1 = \log(2^{1/2}3^{1/3}5^{1/5}/30^{1/30})$ ,  $c_2 = 6c_1/5$  and  $N$  is sufficiently large. This result was proved using an elementary approaches, i.e. without use of complex analysis and the Riemann zeta function. A survey of elementary methods in the study of the distribution of prime numbers may be found in Diamond [7].

In 1936 Gelfond and Shnirelman, see Gelfond's editorial remarks in the 1944 edition of Chebyshev's Collected Works [6, pag. 287–288], proposed a new elementary and clever method for deriving a lower bound for the prime-counting functions  $\pi(x)$  and  $\psi(x)$ . In 1982 the Gelfond–Shnirelman method was rediscovered and developed by Nair, see [9] and [10]. The method of Gelfond–Shnirelman–Nair allows to obtain in elementary way a lower bound for  $\pi(x)$  in terms of integrals of suitable integer polynomials and runs as follows.

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Let  $d_N$  denote the least common multiple of the integers  $1, 2, \dots, N$  and observe that

$$d_N \leq \prod_{p \leq N} p^{\log N / \log p},$$

where  $p$  belongs to the set of prime numbers. Taking the logarithm of both sides gives

$$\log d_N \leq \log \left( \prod_{p \leq N} p^{\log N / \log p} \right) = \sum_{p \leq N} \log \left( p^{\log N / \log p} \right) = \pi(N) \log N$$

and then

$$(1) \quad \pi(N) \geq \frac{\log d_N}{\log N}.$$

From this we can obtain a lower bound for the prime counting function  $\pi(N)$  from a lower bound for the least common multiple  $d_N$ . An elementary and smart way to proceed is to consider a polynomial with integral coefficients

$$P(x) = \sum_{n=0}^{N-1} a_n x^n$$

and let

$$I(P) = \int_0^1 P(x) dx = \sum_{n=0}^{N-1} \frac{a_n}{n+1}.$$

Since  $I(P)$  is a rational number whose denominator divides  $d_N$ , we see that  $I(P)d_N$  is an integer, and hence if  $I(P) \neq 0$  we have

$$d_N |I(P)| \geq 1$$

and then

$$d_N \geq \frac{1}{|I(P)|}.$$

Form the above and (1) we get

$$(2) \quad \pi(N) \geq \frac{\log(1/|I(P)|)}{\log N}.$$

The easiest way to proceed is to bound the absolute value of the integral  $I(P)$

$$(3) \quad |I(P)| = \left| \int_0^1 P(x) dx \right| \leq \int_0^1 |P(x)| dx$$

and

$$(4) \quad \int_0^1 |P(x)| dx \leq \max_{0 \leq x \leq 1} |P(x)| = \|P\|_{[0,1]},$$

obtaining

$$\pi(N) \geq \frac{\log(1/\|P\|_{[0,1]})}{\log N}.$$

If we could find a sequence of integer polynomials  $p_n$ , of degree  $n$ , with sufficiently small supremum norms such that

$$\lim_{n \rightarrow +\infty} \log \left( \|p_n\|_{[0,1]}^{-1/n} \right) = \lim_{n \rightarrow +\infty} -\frac{1}{n} \log \|p_n\|_{[0,1]} = 1,$$

we can obtain the best possible lower bound consistent with the Prime Number Theorem.

This motivates the study of the integer polynomials  $P_N(x)$  and the quantities  $C_N$  such that

$$\|P_N\|_{[0,1]} = \min_{\substack{P(x) \in \mathbb{Z}[x] \\ \deg(P)=N, \|P\|_{[0,1]} > 0}} \|P\|_{[0,1]}$$

and

$$C_N = -\frac{1}{N} \log \|P_N\|_{[0,1]},$$

the so-called integer Chebyshev problem. Much is known about  $P_N(x)$  and  $C_N$ . It was proved by Snirelman, see [11], that the sequence  $C_N$  converges to a limit  $C$ . Borwein and Erdélyi [5] showed that  $C \in (0.85866, 0.86577)$  and the lower bound was improved by Flammang [8] to 0.85912. The best known result to date, due to Pritsker [12], is that  $C \in (0.85991, 0.86441)$ . See also [1], [2], [3], [4], [5] and [14].

Therefore, following this line, we can get a lower bound in the form

$$\pi(N) \geq C \frac{N}{\log N},$$

only for constant  $C$  less than 0.87, which is quite far from what is expected by the PNT.

In order to avoid the trouble above, in this paper we deal with the problem in a different way. From the definition of  $I(P)$  we have that

$$|I(P)| = \left| \int_0^1 P(x) dx \right| = \left| \sum_{n=0}^{N-1} \frac{a_n}{n+1} \right| = \frac{1}{d_N} \left| \sum_{n=0}^{N-1} \frac{d_N}{n+1} a_n \right|.$$

Since  $d_N/(n+1)$  and  $a_n$  are integers for every  $n = 0, 1, \dots, N-1$ , we have that

$$\left| \sum_{n=0}^{N-1} \frac{d_N}{n+1} a_n \right|$$

is an integer and then the small positive value of  $|I(P)|$  is  $1/d_N$  and it is reached if

$$\sum_{n=0}^{N-1} \frac{d_N}{n+1} a_n = \pm 1.$$

Without loss of generality we can deal with the linear diophantine equation

$$\sum_{n=0}^{N-1} \frac{d_N}{n+1} x_n = 1$$

with integer coefficients  $d_N/(n+1)$ . Observing that the integer coefficients  $d_N, d_N/2, \dots, d_N/N$  are relatively prime, we obtain that for every  $N$  there exists at least one polynomial of degree  $N-1$  such that  $I(P) = 1/d_N$ . Note that the set of the integer polynomials of fixed

degree with integral on  $[0, 1]$  equal to zero is a vector space and then the set of the integer polynomials of fixed degree with integrals on  $[0, 1]$  equal to a constant is an affine space. This leads to define the following affine space of the polynomials with positive and minimal integral on  $[0, 1]$ .

**Definition.** Let  $S_N = \{P(x) \in \mathbb{Z}[x], \deg(P) = N - 1, I(P) = 1/d_N\}$

In this paper we studied the properties of such a class of integer polynomials.

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## 2. SOME PROPERTIES OF THE SET $S_N$

In the set  $S_N$  there are integer polynomials with many of the first coefficients equal to zero, and then with  $x = 0$  as a root of great degree.

**Theorem 1.** For every  $N$ , there exists an integer polynomial

$$P(x) = \sum_{n=K(N)}^{N-1} a_n x^n \in S_N$$

with

$$K(N) \sim \frac{N}{2}.$$

*Proof.* As usual,  $(a_1, a_2, \dots, a_j)$  denotes the greatest common divisor of the integers  $a_1, a_2, \dots, a_j$ . We start to observe that if we have

$$\left( \frac{d_N}{k}, \frac{d_N}{k+1}, \dots, \frac{d_N}{N} \right) = 1,$$

for a fixed natural  $k$ , it follows that

$$\left( \frac{d_N}{i}, \frac{d_N}{i+1}, \dots, \frac{d_N}{N} \right) = 1,$$

for every  $1 \leq i \leq k$  and for the same reason if we have

$$\left( \frac{d_N}{k}, \frac{d_N}{k+1}, \dots, \frac{d_N}{N} \right) > 1,$$

for a fixed natural  $k$ , it follows that

$$\left( \frac{d_N}{i}, \frac{d_N}{i+1}, \dots, \frac{d_N}{N} \right) > 1,$$

for every  $k \leq i \leq N$ . This allows to define  $K(N)$  as the natural number such that

$$(5) \quad \left( \frac{d_N}{K(N)+1}, \frac{d_N}{K(N)+2}, \dots, \frac{d_N}{N} \right) = 1$$

and

$$(6) \quad \left( \frac{d_N}{K(N)+2}, \frac{d_N}{K(N)+3}, \dots, \frac{d_N}{N} \right) > 1.$$

From (5) it follows that the linear diophantine equation

$$\sum_{n=K(N)}^{N-1} \frac{d_N}{n+1} x_n = 1$$

has solutions and this implies that there exists an integer polynomial

$$P(x) = \sum_{n=K(N)}^{N-1} a_n x^n \in S_N.$$

Now we prove that

$$(7) \quad K(N) = \min \{p^m : p \text{ prime}, m \geq 1, p^m > N/2\} - 1$$

Let  $q = p^m$  such that  $N/2 < q = p^m < N$ .  $q \leq N$  implies  $q/d_N$  and then

$$\left( \frac{d_N}{q+1}, \frac{d_N}{q+2}, \dots, \frac{d_N}{N} \right) \geq p,$$

since every natural number between  $q+1$  and  $N$  has strictly less than  $m$  factors  $p$  in his prime decomposition. This prove

$$(8) \quad K(N) \leq \min \{p^m : p \text{ prime}, m \geq 1, p^m > N/2\} - 1.$$

On the other hand, by the definition of  $K(N)$ , we have

$$\left( \frac{d_N}{K(N)+2}, \frac{d_N}{K(N)+3}, \dots, \frac{d_N}{N} \right) > 1$$

which implies that there exists a prime number  $p$  such that

$$p \mid \frac{d_N}{K(N)+2}, p \mid \frac{d_N}{K(N)+3}, \dots, p \mid \frac{d_N}{N}.$$

Let  $m = \max\{i : p^i \mid d_N\}$  and therefore  $p^m \leq N$ . From this follows

$$p^m \nmid (K(N)+2), p^m \nmid (K(N)+3), \dots, p^m \nmid N$$

and then

$$(9) \quad K(N) \geq \min \{p^m : p \text{ prime}, m \geq 1, p^m > N/2\} - 1.$$

From (8) and (9) it follows (7). Now the difference between  $K(N)$  and  $N/2$  can be bound by the maximum of the difference between consecutive elements of the set  $\{p^m \leq N : p \text{ prime}, m \geq 1\}$ , which is less than the maximum of the difference between consecutive primes less than  $N$ . This allow to write

$$K(N) = \frac{N}{2} + O(N^{7/12+\varepsilon}),$$

for every  $\varepsilon > 0$ , which concludes the proof of the theorem.  $\square$

**Corollary 2.** *For every  $N$ , there exists an integer polynomial  $P(x) \in S_N$  with  $x = 1$  as a root of degree  $K(N)$  and*

$$K(N) \sim \frac{N}{2}.$$

*Proof.* The corollary follows immediately from the Theorem 1, observing that the change of variable  $x \rightarrow (1 - x)$  don't change the absolute value of the integral  $I(P)$ .  $\square$

The second result is about the number of roots and the number of changes of sign of the integer polynomials in  $S_N$ .

**Theorem 3.** *For all even  $N$ , there exists an integer polynomial  $P(x) \in S_N$  with  $N - 1$  roots on  $(0, 1)$  and  $N - 1$  changes of sign.*

*Proof.* Let  $N$  even number and  $R(x) = (Nx - 1)(Nx - 2) \cdots (Nx - (N - 1))$ .  $R(x)$  is a polynomial with integer coefficients of degree  $N - 1$ , has  $N - 1$  roots on  $(0, 1)$ ,  $(N - 2)/2$  local maxima,  $(N - 2)/2$  local minima and

$$I(R) = \int_0^1 R(x) dx = 0,$$

since the symmetry of the function. Let  $P(x)$  a fixed polynomial in  $S_N$ ,  $k \in \mathbb{Z}$  and  $Q_k(x) = P(x) + kR(x)$ . For every  $k \in \mathbb{Z}$  we have  $I(Q_k) = I(P) = 1/d_N$  and then  $Q_k(x) \in S_N$ . For every  $N$  there exists a constant  $k$  such that  $Q_k(x)$  has  $N - 1$  roots on  $(0, 1)$  and  $N - 1$  changes of sign.  $\square$

**Corollary 4.** *For every  $N$ , there exists an integer polynomial  $P(x) \in S_N$  with at least  $N - 2$  roots on  $(0, 1)$  and  $N - 2$  changes of sign.*

On the other side we can prove that in the set  $S_N$  there are also integer polynomials with at most one root and one change of sign.

**Theorem 5.** *For every  $N$ , there exists an integer polynomial  $P(x) \in S_N$  with at most one root on  $(0, 1)$  and at most one change of sign on  $(0, 1)$ .*

*Proof.* Let  $P(x)$  a fixed polynomial in  $S_N$ ,  $k \in \mathbb{Z}$  and  $Q_k(x) = P(x) + k(2x - 1)$ . For every  $k \in \mathbb{Z}$  we have  $I(Q_k) = I(P) = 1/d_N$  and then  $Q_k(x) \in S_N$ . Now we observe that  $Q_k(0) = P(0) - k$ ,  $Q_k(1) = P(1) + k$  and  $Q'_k(x) = P'(x) + 2k$  for every  $x \in [0, 1]$ .

For every  $N$  there exists a constant  $k$  such that  $Q_k(0) < 0$ ,  $Q_k(1) > 0$  and  $Q'_k(x) > 0$  for every  $x \in [0, 1]$  and this implies that the polynomial  $Q_k(x)$  has exactly one root and one change of sign on  $(0, 1)$ .  $\square$

### 3. OPEN PROBLEM

In the standard method of Gelfond–Shnirelman–Nair we bound the absolute value of the integral

$$(10) \quad |I(P)| = \left| \int_0^1 P(x) dx \right| \leq \int_0^1 |P(x)| dx$$

and then

$$(11) \quad \int_0^1 |P(x)| dx \leq \max_{0 \leq x \leq 1} |P(x)| = \|P\|_{[0,1]},$$

to obtain

$$\pi(N) \geq \frac{\log(1/\|P\|_{[0,1]})}{\log N}.$$

As observed in the introduction, following this line we can get a lower bound in the form

$$\pi(N) \geq C \frac{N}{\log N},$$

only for constant  $C$  much less than 1. It is not clear if this is only a consequence of the use of supremum norm on the interval  $[0, 1]$  in (11) or if the inequality (10) is also involved.

If the set  $S_N$  contains polynomials of constant sign in  $(0, 1)$  for all  $N$ , or at least for infinite values of  $N$ , the limit of the method would be only due to the inequality (11).

It is simple to verify that for very small values of  $N$  these positive polynomials exist. For  $S_3$ ,  $\deg(P) = 2$  and  $d_3 = 6$ , we have the positive polynomial  $P(x) = x(1 - x)$  and for  $S_4$ ,  $\deg(P) = 3$  and  $d_3 = 12$ , we have the positive polynomial  $P(x) = x^2(1 - x)$ . For  $S_N$  with greater values of  $N$  is not simple to determine what happens, and this leads to the following question.

**Problem:** for every  $N$ , or at least for infinite values of  $N$ , there exists an integer polynomial  $P(x) \in S_N$  such that  $P(x) \geq 0$  ?

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