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# **On the application of the stochastic approach in predicting fatigue reliability using Miner's damage rule**

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**Abstract:**

The present paper investigates the application of the stochastic approach when the commonly adopted Miner's linear damage rule is implemented, both in its traditional form and in its modified form to include the presence of a random stress threshold (random fatigue limit), below which the rate of damage accumulation is reduced.

Main steps are provided to obtain the simulated distribution of the accumulated damage under variable amplitude loading. When the stochastic approach is applied in presence of a random fatigue limit, an additional correlation structure, which takes into account the fatigue limit value, must be introduced in the analysis. If the number of cycles to failure under constant amplitude loading is Weibull (Log-Normal) distributed, then the corresponding accumulated damage is Fréchet (Log-Normal) distributed.

The effects of the correlation structure on reliability prediction under variable amplitude loading are also investigated. To this aim, several experimental datasets are taken from the literature, covering various metallic materials and variable amplitude block sequences.

The results show that the choice of the damage accumulation model is a key factor to value the improvement in the accuracy of reliability predictions introduced by the stochastic approach.

Comparison of the predicted number of cycles to failure with experimental data shows that larger errors are nonconservative, regardless of the adopted correlation structure. When the analysis is limited to reliability levels above 80%, for these large nonconservative errors it is the quantile approach to be closer to actual experimental data, thus limiting the overestimation of component's life. For the experimental datasets considered in the paper, adoption of a stochastic approach would improve the accuracy of Miner's predictions in 10% of cases.

**Keywords:**

Variable amplitude loading; Correlation structure; Fatigue life prediction; Random fatigue limit; Monte Carlo simulations

## Nomenclature

$\{\mathbf{1}_{\mathbf{D}_{n_{TOT}} < 1}\}$  = indicator function of the subvector ( $\{\mathbf{D}_{n_{TOT}}\} < 1$ ) of the vector  $\{\mathbf{D}_{n_{TOT}}\}$   
 $\{\mathbf{1}\}_{n_s \times 1}, \{\mathbf{1}\}_{n_{sim} \times 1}$  = vectors with each element equal to one  
 $a_0, a_1, b_0, \mu_{D_i}, \mu_l, \mu_{lN_i}, \sigma_{D_i}, \sigma_l, \sigma_{lN_i}, \boldsymbol{\theta}$  = parameters to be estimated  
 $\tilde{a}_0, \tilde{a}_1, \tilde{b}_0, \tilde{\mu}_{D_i}, \tilde{\mu}_l, \tilde{\sigma}_{D_i}, \tilde{\sigma}_l, \tilde{\boldsymbol{\theta}}$  = estimates of the parameters  
 $D, D_i, D_{n_{TOT}}, D_{s_l}$  = accumulated damages  
 $[\mathbf{D}], \{\mathbf{D}\}$  = pseudorandom matrix and vector of accumulated damages  
 $\{\mathbf{D}_{n_{TOT}}\}$  = pseudorandom vector of total damage at  $n_{TOT}$   
 $\varepsilon_0, \varepsilon_\infty$  = errors in the prediction of the number of cycles to failure, in a variable amplitude test  
 $f_{lN}(\cdot), f_{lN|S}(\cdot)$  = probability density functions  
 $F(\cdot), F_{lN}(\cdot), F_{lN|S}(\cdot)$  = cumulative distribution functions  
 $\Phi(\cdot)$  = standard Normal cumulative distribution function  
 $L(\cdot)$  = Likelihood function  
 $lL(\cdot)$  = Log-Likelihood function  
 $\lambda$  = positive parameter in correlation coefficients  
 $m$  = total number of blocks  
 $\{\bar{\boldsymbol{\mu}}\}$  = mean vector of a conditional multivariate Normal distribution  
 $n_c$  = number of runouts  
 $n_{c,1}^*, n_{c,2}^*, \dots, n_{c,n_c}^*, n_{f,1}^*, n_{f,2}^*, \dots, n_{f,n_f}^*, n_i, n_{s_l}$  = values of number of cycles  
 $n_f$  = number of failures  
 $N_i$  = number of cycles to failure at  $s_i$   
 $n_{i,tot}, n_{s_l}, n_{TOT}, n_{TOT,\lambda=0}, n_{TOT,\lambda=\infty}, n_{TOT,real}$  = total number of cycles, in a variable amplitude test  
 $n_s$  = number of different stress amplitudes  
 $n_{sim}$  = number of simulations  
 $N_{s_l}$  = number of cycles to failure at  $s_l$ , when Haibach's hypothesis is used  
 $N_{TOT}$  = total number of cycles to failure, in a variable amplitude test  
 $P[\cdot]$  = probability of an event  
 $R_{n_{TOT}}$  = reliability at  $n_{TOT}$   
 $\rho_{D_i,D_j}, \rho_{D_i,D_{s_l}}$  = Spearman's rank correlation coefficients  
 $[\boldsymbol{\rho}]$  = Spearman's rank correlation matrix  
 $s_{c,1}, s_{c,2}, \dots, s_{c,n_c}, s_{f,1}, s_{f,2}, \dots, s_{f,n_f}, s_i$  = values of stress amplitudes  
 $s_l$  = value of fatigue limit  
 $S_l$  = random fatigue limit  
 $\Sigma_{D_i,D_j}, \Sigma_{D_i,D_{s_l}}$  = Pearson's correlation coefficients  
 $[\boldsymbol{\Sigma}]$  = Pearson's correlation matrix of a conditional multivariate Normal distribution  
 $[\mathbf{U}], \{\mathbf{U}\}$  = pseudorandom matrix and vector drawn from a multivariate Uniform distribution  
 $z_\gamma$  =  $\gamma$ -th quantile of the standard Normal distribution  
 $\{\mathbf{Z}\}$  = pseudorandom vector drawn from a Normal distribution with mean  $\{\bar{\boldsymbol{\mu}}\}$  and correlation matrix  $[\boldsymbol{\Sigma}]$   
 $|\cdot|$  = absolute value  
 $[\cdot]$  = matrix  
 $\{\cdot\}$  = vector  
 $\{\cdot\}^T$  = transpose of a vector

## 1. Introduction

It is generally acknowledged that prediction of fatigue reliability is critical for the design and maintainability of many structural components. Due to this criticality, extensive literature is dedicated to the problem. Nevertheless, life prediction and reliability evaluation remain a challenging issue, especially when service loading or variable amplitude loading is considered.

Several damage accumulation models have been proposed in the literature<sup>1</sup>. Though many models have been postulated, none of them has the benefit of widespread acceptance and the Miner's damage rule still remains the most commonly adopted in practice.

Under variable amplitude loading, fatigue lives at different stress amplitudes are commonly assumed to be uncorrelated (e.g., [2]) or, rather, fully correlated (e.g., [3]) Log-Normal or Weibull random variables. More recently, Liu and Mahadevan<sup>4,5</sup> postulated a model that includes actual correlation of fatigue lives across different stress amplitudes, based on the physical evidence that under variable amplitude loading it is the behavior of a given specimen at different stress amplitudes to generate the randomness of the distribution. Therefore, it is reasonable to assume that fatigue properties at different stress amplitudes may be correlated.

In [4], Liu and Mahadevan proposed to treat the fatigue lives as correlated Log-Normal random fields (stochastic approach) which depend on the value of the stress amplitude. Compared to traditional fatigue life prediction methods, the stochastic approach requires one additional variable amplitude loading test. In their works<sup>4,5</sup>, Liu and Mahadevan proposed a nonlinear model to account for damage accumulation under variable amplitude loading. Consistently with the traditional form of Miner's rule, their model does not take into account the existence of a fatigue limit. Based on this nonlinear damage accumulation model, they concluded that the stochastic approach improves accuracy of reliability predictions.

The present paper proposes a general statistical procedure that allows to implement the stochastic approach in a general framework, making use only of Monte Carlo simulations with no need of expansion techniques. The analysis is performed by taking into consideration the possible presence of a random fatigue limit, below which the rate of damage accumulation is reduced, and by assuming that the number of cycles to failure under constant amplitude loading can be Log-Normal or Weibull distributed.

In order to investigate whether the stochastic approach can improve the accuracy of reliability predictions with damage accumulation models different from the one proposed in [4,5], the Miner's linear damage rule is adopted in the present paper and tested over several experimental datasets taken from the literature, including data considered in [4,5].

As a final purpose, the paper attempts to answer the question whether, in spite of the higher complexity, by adopting a stochastic approach, it is possible to improve the accuracy of Miner's predictions, especially in the most critical cases of nonconservative estimates (i.e., overestimation of component's life).

## 2. Damage rule and reliability definition

Consider a fatigue test with specimens subject to a variable amplitude block loading. Let  $m$  be the total number of blocks and the pair  $(s_i, n_i)$  the stress amplitude and the number of cycles of the  $i$ -th block ( $i = 1, \dots, m$ ), respectively.

Different damage accumulation models have been proposed in the literature to predict the total number of cycles to failure in case of variable amplitude block loading. Among them, the linear damage rule postulated by Miner plays a major role.

Miner supposed that the total damage  $D$  sustained by a specimen can be computed as:

$$D = \sum_{i=1}^m \frac{n_i}{N_i}, \quad (1)$$

where  $N_i$  is the number of cycles to failure at  $s_i$ , in a constant amplitude test. Failure occurs when  $D$  reaches unity.

In spite of its simplistic hypotheses (linearity, stress amplitude independence, stress sequence independence and stress interaction independence<sup>1</sup>), the Miner's rule is commonly adopted due to its ease of application and will be used in the following.

Let  $n_{TOT}$  be the total number of cycles in a variable amplitude fatigue test. Let  $n_s$  be the number of different stress amplitudes experienced in the test. Due to the hypotheses of the Miner's rule, all cycles at the same stress level can be grouped together and considered as a single block even though they are applied at different times, so that  $n_{i,tot}$  denotes the total number of cycles at  $s_i$ . In this respect,  $n_s$  can be smaller than the total number of blocks,  $m$ , and the total damage due to the  $n_{TOT}$  cycles,  $D_{n_{TOT}}$ , can be written as:

$$D_{n_{TOT}} = \sum_{i=1}^{n_s} \frac{n_{i,tot}}{N_i} = \sum_{i=1}^{n_s} D_i, \quad (2)$$

where  $D_i = n_{i,tot} N_i^{-1}$  (i.e.,  $D_i$  represents the damage accumulated in  $n_{i,tot}$  cycles at  $s_i$ ).

According to the Miner's rule, if the specimen fails at  $n_{TOT}$ , then  $D_{n_{TOT}} = 1$ , otherwise  $D_{n_{TOT}}$  is smaller than 1. The total number of cycles to failure,  $N_{TOT}$ , is a random variable and its distribution is strictly related to the distribution of  $D_{n_{TOT}}$ . In particular, the probability that  $N_{TOT}$  is smaller or equal than a given value  $n_{TOT}$  (i.e the probability of reaching specimen failure before  $n_{TOT}$  cycles) is equal to the probability that  $D_{n_{TOT}}$  is larger or equal to 1:

$$P[N_{TOT} \leq n_{TOT}] = P[D_{n_{TOT}} \geq 1]. \quad (3)$$

In many engineering applications, it is requested to compute the reliability for a given total number of cycles. Since the reliability at  $n_{TOT}$ ,  $R_{n_{TOT}}$ , is the probability of having no failure (runout) in  $n_{TOT}$  cycles, by considering Equations (2) and (3), the reliability at  $n_{TOT}$  can be computed as:

$$R_{n_{TOT}} = P[N_{TOT} > n_{TOT}] = P\left[\sum_{i=1}^{n_s} D_i < 1\right]. \quad (4)$$

Therefore,  $R_{n_{TOT}}$  can be computed only if the distribution of  $D_{n_{TOT}}$  is known.

### 3. Stochastic approach: introduction

If the following assumptions apply to fatigue data:

- the number of cycles to failure at  $s_i$  follows either a Log-Normal or a Weibull distribution (according to the literature (e.g., [6]), this is a quite common assumption for constant amplitude data);
- a simple Basquin's model with constant standard deviation<sup>7,8</sup> well fits constant amplitude data;
- under variable amplitude loading, the damage at  $s_i$  is correlated with the damage at  $s_j$ <sup>4,5</sup>;

then the random variables  $D_i$  ( $i = 1, \dots, n_s$ ) in Equation (4) are correlated variables and are either Log-Normal or Fréchet distributed<sup>9</sup>, if the number of cycles to failure follows a Log-Normal or a Weibull distribution, respectively.

For sake of clarity, the different steps that must be taken to determine the reliability  $R_{n_{TOT}}$  will be first discussed with reference to the case of the Log-Normal distribution. The case of Weibull distributed number of cycles to failure will be briefly addressed in Section 3.4.

In the Log-Normal case, the logarithm of the number of cycles to failure at  $s_i$  is Normal distributed with mean,  $\mu_{LN_i}$ , and standard deviation,  $\sigma_{LN_i}$ .

According to Basquin's model,  $\mu_{LN_i}$  and  $\sigma_{LN_i}$ , can be expressed as:

$$\begin{cases} \mu_{LN_i} = a_0 + a_1 \ln(s_i) \\ \sigma_{LN_i} = b_0 \end{cases}, (5)$$

where  $a_0$ ,  $a_1$  and  $b_0$  are constant parameters that can be estimated through constant amplitude experimental data.

By applying the well-known formulas for the transformation of a random variable<sup>9</sup>, the distribution of the random variables  $D_i$  is shown to be Log-Normal with parameters:

$$\begin{cases} \mu_{D_i} = \ln(n_{i,tot}) - (a_0 + a_1 \ln(s_i)) \\ \sigma_{D_i} = b_0 \end{cases}. (6)$$

According to Equation (2), the total damage  $D_{n_{TOT}}$  corresponds to the sum of  $n_s$  correlated Log-Normal random variables with parameters given in Equation (6). The correlation between  $D_i$  and  $D_j$  (being  $i, j = 1, \dots, n_s$ ) can be modeled as a function of the stress amplitude distance<sup>4,5</sup>:

$$\rho_{D_i, D_j} = e^{-\lambda |s_i - s_j|}, \quad (7)$$

where  $\lambda$  is a positive parameter and  $|\cdot|$  denotes the absolute value. If  $\lambda = 0$ , then  $\rho_{D_i, D_j} = 1$ , (i.e.,  $D_i$  and  $D_j$  are fully correlated); while, if  $\lambda \rightarrow \infty$ , then  $\rho_{D_i, D_j} = 0$  (i.e.,  $D_i$  and  $D_j$  are uncorrelated). In [4,5], the case of  $\lambda = 0$  is defined as *quantile S-N curve*, while the case of  $\lambda \rightarrow \infty$  is defined as *statistical S-N curve*; when  $0 < \lambda < \infty$ , the S-N curve is defined *stochastic S-N curve*. The above definitions of quantile, statistical and stochastic S-N curve will also be used in this paper.

In Equation (7),  $\rho_{D_i, D_j}$  measures the strength of association between  $D_i$  and  $D_j$  and it is represented by the Spearman's rank correlation coefficient, which is the nonparametric version of the Pearson's correlation coefficient. A monotonic relationship is an important underlying assumption of the Spearman's rank correlation coefficient, which is less restrictive than the linear relationship that has to be met by the Pearson's correlation coefficient. A monotonic function links the two coefficients and it permits to compute one, knowing the other<sup>9</sup>.

To the authors' best knowledge, no exact distribution for the sum of correlated Log-Normal random variables can be found in the literature. In [10], the sum was approximated by a Log-Normal distribution. In this paper, Monte-Carlo simulations, based on a multivariate Copula<sup>9</sup>, are adopted to obtain a quasi-exact distribution. It is worth noting that, in case of a Log-Normal distribution, a multivariate Gaussian Copula allows to obtain simulated damage values which correspond to the values obtainable by randomly drawing from a multivariate Log-Normal distribution. For this reason, the Gaussian Copula is adopted in this paper.

### 3.2. Stochastic approach: S-N curve without fatigue limit

Let us first consider the simplest S-N curve, where an unique straight line describes the entire fatigue domain.

Let  $\tilde{a}_0$ ,  $\tilde{a}_1$  and  $\tilde{b}_0$  be the estimates of the parameters  $a_0$ ,  $a_1$  and  $b_0$  obtained from experimental data acquired through constant amplitude tests. The sequence of steps to be followed in order to obtain the simulated reliability  $R_{n_{TOT}}$  for given values of  $\lambda$ ,  $\tilde{a}_0$ ,  $\tilde{a}_1$  and  $\tilde{b}_0$  are:

1. Compute the  $(n_{sim} \times n_s)$  matrix,  $[\mathbf{U}]$ , containing pseudorandom values drawn from a multivariate Gaussian copula<sup>9</sup> with correlation matrix  $[\boldsymbol{\rho}] = [\rho_{D_i, D_j}]_{n_s \times n_s}$  ( $\rho_{D_i, D_j}$  is given in Equation (7)):

$$[\mathbf{U}] = [\{\mathbf{U}_i\}]_{1 \times n_s},$$

where the  $(n_{sim} \times 1)$  vector  $\{\mathbf{U}_i\}$  denotes the  $i$ -th column of matrix  $[\mathbf{U}]$  (i.e.,  $\{\mathbf{U}_i\} = \{U_{1,i} \ \cdots \ U_{n_{sim},i}\}^T$ , being  $\{\cdot\}^T$  the transpose of vector  $\{\cdot\}$ ).

2. Compute the  $(n_{sim} \times n_s)$  matrix,  $[\mathbf{D}]$ , containing pseudorandom values of fatigue damage with correlation matrix  $[\boldsymbol{\rho}]$ :

$$[\mathbf{D}] = [F^{-1}(\{\mathbf{U}_i\})] = [\{\mathbf{D}_i\}]_{1 \times n_s},$$

where the  $(n_{sim} \times 1)$  vector  $\{\mathbf{D}_i\}$  denotes the  $i$ -th column of matrix  $[\mathbf{D}]$  and  $F^{-1}(\cdot)$  denotes the inverse cumulative distribution function (cdf) of the fatigue damage:

$$F^{-1}(\{\mathbf{U}_i\}) = e^{\tilde{\mu}_{D_i} + \tilde{\sigma}_{D_i} \Phi^{-1}(\{\mathbf{U}_i\})},$$

where  $\Phi(\cdot)$  is the standard Normal cdf and  $\tilde{\mu}_{D_i}$  and  $\tilde{\sigma}_{D_i}$  are the estimates of  $\mu_{D_i}$  and  $\sigma_{D_i}$  obtained by taking into account Equation (6):

$$\begin{cases} \tilde{\mu}_{D_i} = \ln(n_{i,tot}) - (\tilde{a}_0 + \tilde{a}_1 \ln(s_i)) \\ \tilde{\sigma}_{D_i} = \tilde{b}_0 \end{cases}.$$

3. Evaluate the  $(n_{sim} \times 1)$  vector,  $\{\mathbf{D}_{n_{TOT}}\}$ , of pseudorandom values of total damage at  $n_{TOT}$ :

$$\{\mathbf{D}_{n_{TOT}}\} = [\mathbf{D}] \cdot \{\mathbf{1}\}_{n_s \times 1} = \sum_{i=1}^{n_s} \{\mathbf{D}_i\},$$

where  $\{\mathbf{1}\}_{n_s \times 1}$  denotes the  $(n_s \times 1)$  vector with each element equal to one.

4. Finally, evaluate the pseudorandom value,  $R_{n_{TOT}}$ , of reliability at  $n_{TOT}$ :

$$R_{n_{TOT}} = \frac{\{\mathbf{1}\}_{n_{sim} \times 1}^T \{\mathbf{1}_{D_{n_{TOT}} < 1}\}}{n_{sim}},$$

where  $\{\mathbf{1}\}_{n_{sim} \times 1}$  denotes the  $(n_{sim} \times 1)$  vector with each element equal to one and the  $(n_{sim} \times 1)$  vector  $\{\mathbf{1}_{D_{n_{TOT}} < 1}\}$  denotes the indicator function of the sub-vector  $\{\mathbf{D}_{n_{TOT}} < 1\}$  of the vector  $\{\mathbf{D}_{n_{TOT}}\}$ .

### 3.3. Stochastic approach: $S$ - $N$ curve with fatigue limit

It is possible to extend the above methodology to include  $S$ - $N$  curves that show a stress threshold  $s_l$ , called the fatigue limit. In the simplest form, it is assumed that stress amplitudes below the fatigue limit do not lead to failure; consequently specimens are supposed to survive an infinite number of cycles.

When computing the total damage, blocks with stress amplitudes below or equal to the fatigue limit ( $s_i \leq s_l$ ) do not induce any damage and it is therefore necessary to distinguish between blocks with stress amplitudes above the fatigue limit and blocks with stress amplitudes below the fatigue limit. In this respect, a bilinear model can be adopted. With a bilinear model, the parameters of the  $i$ -th damage random variable,  $D_i$ , are equal to:



$$\begin{cases} \mu_{D_i} = \begin{cases} \ln(n_{i,tot}) - (a_{1,0} + a_{1,1} \ln(s_i)) & \text{if } s_i > s_l \\ \ln(n_{i,tot}) - (a_{2,0} + a_{2,1} \ln(s_i)) & \text{if } s_i \leq s_l \end{cases} \\ \sigma_{D_i} = b_0 \end{cases} \quad (8)$$

In real service conditions, constant amplitude fatigue is quite rare, since, in general, components are subject to different stress amplitudes. Under this condition, to allot zero damage to blocks below the fatigue limit was shown to lead to nonconservative estimates of component's life, since fatigue cracks, once nucleated, can propagate even for stress amplitudes below the fatigue limit<sup>11-13</sup>.

In order to reduce nonconservative life estimates, the bilinear model proposed by Haibach<sup>14</sup> may be used. Then, in Equation (8),  $a_{2,1}$  depends on  $a_{1,1}$  and it can be assumed to be equal to  $(2a_{1,1} + 1)$ . To fulfill continuity of  $\mu_{D_i}$  at  $s_i$  equal to  $s_l$ , it is necessary that  $(\mu_{D_i})_1 = (\mu_{D_i})_2$  when  $s_i = s_l$ . As a consequence,  $a_{2,0} = a_{1,0} - (a_{1,1} + 1) \ln(s_l)$  and Equation (8) can be rewritten as:

$$\begin{cases} \mu_{D_i} = \begin{cases} \ln(n_{i,tot}) - a_0 - a_1 \ln(s_i) & \text{if } s_i \geq s_l \\ \ln(n_{i,tot}) - a_0 + (a_1 + 1) \ln(s_l) - (2a_1 + 1) \ln(s_i) & \text{if } s_i \leq s_l \end{cases} \\ \sigma_{D_i} = b_0 \end{cases} \quad (9)$$

Parameters in Equation (9) depend on the value  $s_l$ . The value of the fatigue limit may vary from one specimen to another and therefore, in a statistical framework, the fatigue limit must be considered a random variable. In the literature (see e.g., [6,15]), the random fatigue limit,  $S_l$ , is often assumed to be Normal distributed with mean,  $\mu_l$ , and standard deviation,  $\sigma_l$ . Therefore, any fatigue limit value,  $s_l$ , corresponds to a specific quantile of the Normal distribution and may be expressed as:

$$s_l = \mu_l + z_\gamma \sigma_l, \quad (10)$$

where  $z_\gamma$  denotes the  $\gamma$ -th quantile of the standard Normal distribution.

A fatigue model that is capable to take into account the existence of a random fatigue limit was proposed in [16]. According to this model, the  $S$ - $N$  curve with fatigue limit  $s_l$  is representative of the  $\gamma$ -th quantile. In this respect, the given  $s_l$  value identifies the entire  $\gamma$ -th quantile  $S$ - $N$  curve (Figure 1).

When Haibach's hypothesis is used, the stress amplitude  $s_l$  corresponds to a finite number of cycles to failure,  $N_{s_l}$  (Figure 2). From Equation (5), considering that the logarithm of number of cycles to failure is Normal distributed, it can be written:

$$\ln(N_{s_l}) = a_0 + a_1 \ln(s_l) + z_\gamma b_0. \quad (11)$$

Therefore, for a given specimen (i.e., for a given  $s_l$  value), the point with coordinates  $(N_{s_l}, s_l)$  is completely defined and it is common to the quantile, stochastic and statistical  $S$ - $N$  curves (Figure 2). As a consequence, the point  $(N_{s_l}, s_l)$  is a fundamental point in damage evaluation, regardless of the adopted correlation structure.

Let us define  $D_{s_l}$  the fatigue damage caused by a given number of cycles,  $n_{s_l}$ , at  $s_l$ . Since  $D_{s_l} = n_{s_l}/N_{s_l}$ , the fatigue damage caused by  $n_{s_l}$  cycles at  $s_l$  does not depend on the adopted correlation structure. Therefore, for a given specimen, when  $s_i = s_l$ , the fatigue damage caused by  $n_{i,tot} = n_{s_l}$  cycles at  $s_i$ ,  $D_i$ , must correspond to  $D_{s_l}$ . This condition leads to an additional correlation structure between the generic damage  $D_i$  and the damage  $D_{s_l}$  (dotted arrows in Figure 2). By adopting the same correlation function proposed in Equation (7), the additional correlation structure can be expressed as:

$$\rho_{D_i, D_{s_l}} = e^{-\lambda|s_i - s_l|}, \quad (12)$$

where,  $\rho_{D_i, D_{s_l}}$  measures the strength of association between  $D_i$  and  $D_{s_l}$  and it is represented by the Spearman's rank correlation coefficient.

Having completely defined the correlation structure for a linear  $S-N$  curve with random fatigue limit, it is then possible to evaluate the pseudorandom value of reliability,  $R_{n_{TOT}}$ , similarly to what is illustrated in Section 3.2 for the simpler case of a linear  $S-N$  curve without fatigue limit. Main steps are:

1. Generate a pseudorandom fatigue limit value,  $s_l$ , drawn from a Normal distribution with estimated mean equal to  $\tilde{\mu}_l$  and estimated standard deviation equal to  $\tilde{\sigma}_l$  and compute the standard Normal fatigue limit,  $z_\gamma$ , corresponding to  $s_l$  as follows:

$$z_\gamma = \frac{s_l - \tilde{\mu}_l}{\tilde{\sigma}_l}.$$

2. For any couple  $(D_i, D_j)$  ( $i, j = 1, \dots, n_s$ ), compute the Pearson's correlation coefficient corresponding to the Spearman's rank correlation coefficient<sup>9</sup> given in Equation (7):

$$\Sigma_{D_i, D_j} = 2 \cdot \sin\left(\rho_{D_i, D_j} \frac{\pi}{6}\right) = 2 \cdot \sin\left(e^{-\lambda|s_i - s_j|} \frac{\pi}{6}\right),$$

where  $\Sigma_{D_i, D_j}$  denotes the Pearson's correlation coefficient.

3. For any stress amplitude  $s_i$  ( $i = 1, \dots, n_s$ ), compute the Pearson's correlation coefficient between  $D_i$  and  $D_{s_l}$  corresponding to the Spearman's rank correlation coefficient given in Equation (12):

$$\Sigma_{D_i, D_{s_l}} = 2 \cdot \sin\left(\rho_{D_i, D_{s_l}} \frac{\pi}{6}\right) = 2 \cdot \sin\left(e^{-\lambda|s_i - s_l|} \frac{\pi}{6}\right),$$

where  $\Sigma_{D_i, D_{s_l}}$  denotes the Pearson's correlation coefficient between  $D_i$  and  $D_{s_l}$ .

4. Assemble the mean vector,  $\{\bar{\mu}\}$ , of the conditional multivariate Normal distribution<sup>9</sup> of  $D_i$  ( $i = 1, \dots, n_s$ ) given that  $S_l = s_l$ :

$$\{\bar{\mu}\} = -z_\gamma \begin{Bmatrix} \Sigma_{D_1, D_{s_l}} \\ \vdots \\ \Sigma_{D_{n_s}, D_{s_l}} \end{Bmatrix} = \{\bar{\mu}_i\}_{n_s \times 1},$$

where the minus sign follows from the inverse function  $D_{s_l} = n_{s_l}/N_{s_l}$ .

5. Assemble the Pearson's correlation matrix,  $[\bar{\Sigma}]$ , of the conditional multivariate Normal distribution<sup>9</sup> of  $D_i$  ( $i = 1, \dots, n_s$ ) given that  $S_l = s_l$ :

$$[\bar{\Sigma}] = \begin{bmatrix} 1 & \cdots & \Sigma_{D_1, D_{n_s}} \\ \vdots & \ddots & \vdots \\ \Sigma_{D_{n_s}, D_1} & \cdots & 1 \end{bmatrix} - \begin{bmatrix} \Sigma_{D_1, D_{s_l}} \\ \vdots \\ \Sigma_{D_{n_s}, D_{s_l}} \end{bmatrix} \begin{bmatrix} \Sigma_{D_1, D_{s_l}} & \cdots & \Sigma_{D_{n_s}, D_{s_l}} \end{bmatrix} = [\bar{\Sigma}_{i,j}]_{n_s \times n_s}.$$

6. Compute  $n_s$  pseudorandom values,  $\{\mathbf{Z}\} = \{z_i\}_{n_s \times 1}$ , drawn from a multivariate Normal distribution with mean vector  $\{\bar{\mu}\}$  and correlation matrix  $[\bar{\Sigma}]$ .
7. Compute, from  $\{\mathbf{Z}\}$ ,  $n_s$  pseudorandom values,  $\{\mathbf{U}\}$ , drawn from a standard Uniform distribution:

$$\{\mathbf{U}\} = \{\Phi(\{\mathbf{Z}\})\} = \{u_i\}_{n_s \times 1}.$$

It is worth noting that vector  $\{\mathbf{U}\}$  contains  $n_s$  pseudorandom values drawn from a conditional ( $S_l = s_l$ ) multivariate Gaussian Copula with correlation matrix depending on the rank correlations  $\rho_{D_i, D_j}$  ( $i, j = 1, \dots, n_s$ ) and  $\rho_{D_i, D_{s_l}}$  ( $i = 1, \dots, n_s$ ).

8. Compute  $n_s$  pseudorandom values,  $\{\mathbf{D}\}$ , of correlated fatigue damage:

$$\{\mathbf{D}\} = \{F^{-1}(\{\mathbf{U}\})\} = \{D_i\}_{n_s \times 1},$$

where  $F^{-1}(\cdot)$  denotes the inverse cdf of the fatigue damage. If the fatigue lives  $N_i$  ( $i = 1, \dots, n_s$ ) are supposed to be Log-Normal distributed, then  $F^{-1}(\{\mathbf{U}\})$  is equal to:

$$F^{-1}(\{\mathbf{U}\}) = e^{\tilde{\mu}_{D_i} + \tilde{\sigma}_{D_i} \Phi^{-1}(\{\mathbf{U}\})},$$

where  $\tilde{\mu}_{D_i}$  and  $\tilde{\sigma}_{D_i}$  are the estimates of  $\mu_{D_i}$  and  $\sigma_{D_i}$  obtained by taking into account Haibach's hypothesis (Equation (9)):

$$\begin{cases} \tilde{\mu}_{D_i} = \begin{cases} \ln(n_{i,tot}) - \tilde{a}_0 - \tilde{a}_1 \ln(s_i) & \text{if } s_i > s_l \\ \ln(n_{i,tot}) - \tilde{a}_0 + (\tilde{a}_1 + 1) \ln(s_l) - (2\tilde{a}_1 + 1) \ln(s_i) & \text{if } s_i \leq s_l. \end{cases} \\ \tilde{\sigma}_{D_i} = \tilde{b}_0 \end{cases}$$

9. Evaluate the pseudorandom value,  $D_{n_{TOT}}$ , of total damage at  $n_{TOT}$ :

$$D_{n_{TOT}} = \{\mathbf{D}\}^T \cdot \{\mathbf{1}\}_{n_s \times 1} = \sum_{i=1}^{n_s} D_i.$$

10. Compute  $n_{sim}$  pseudorandom total damage values (i.e.,  $D_{n_{TOT},j}$  with  $j = 1, \dots, n_{sim}$ ), by repeating  $n_{sim}$  times points 1 to 9, and assemble the  $(n_{sim} \times 1)$  vector of pseudorandom total damage values:

$$\{\mathbf{D}_{n_{TOT}}\} = \{D_{n_{TOT},j}\}_{n_{sim} \times 1}.$$

11. Finally, evaluate the pseudorandom value,  $R_{n_{TOT}}$ , of reliability at  $n_{TOT}$ :

$$R_{n_{TOT}} = \frac{\{\mathbf{1}\}_{n_{sim} \times 1}^T \cdot \{\mathbf{1}_{D_{n_{TOT}} < 1}\}}{n_{sim}}.$$

### 3.4. Extension to Weibull distribution

The illustrated procedure may be adopted also in the case of Weibull distributed number of cycles to failure. By applying the well-known formulas for the transformation of a random variable<sup>9</sup>, the damage variable  $D_i$  is shown to be Fréchet distributed with parameters equal to:

$$\begin{cases} \mu_{D_i} = e^{\ln(n_{i,tot}) - (a_0 + a_1 \ln(s_i))} \\ \sigma_{D_i} = 1/b_0 \end{cases}.$$

For brevity, only the more general case of  $S-N$  curve with fatigue limit will be discussed. The simpler case of  $S-N$  curve without fatigue limit can indeed be seen as a sub-case, where  $\mu_l \rightarrow 0$  and  $\sigma_l \rightarrow 0$ .

Referring to Section 3.3, of all steps 1)-11), it is only in step 8) that the Fréchet distribution needs to be taken into account. In particular, step 8) must be rewritten as follows:

8. Compute  $n_s$  pseudorandom values,  $\{\mathbf{D}\}$ , of correlated fatigue damage:

$$\{\mathbf{D}\} = \{F^{-1}(\{\mathbf{U}\})\} = \{D_i\}_{n_s \times 1},$$

where  $F^{-1}(\cdot)$  denotes the inverse cdf of the fatigue damage. If the fatigue lives  $N_i$  ( $i = 1, \dots, n_s$ ) are supposed to be Weibull distributed, then  $F^{-1}(\{\mathbf{U}\})$  is equal to:

$$F^{-1}(\{\mathbf{U}\}) = \tilde{\mu}_{D_i}(-\ln(\{\mathbf{U}\}))^{-1/\tilde{\sigma}_{D_i}},$$

where  $\tilde{\mu}_{D_i}$  and  $\tilde{\sigma}_{D_i}$  are the estimates of  $\mu_{D_i}$  and  $\sigma_{D_i}$  obtained by taking into account Haibach's hypothesis:

$$\begin{cases} \tilde{\mu}_{D_i} = \begin{cases} e^{\ln(n_{i,tot}) - \tilde{a}_0 - \tilde{a}_1 \ln(s_i)} & \text{if } s_i > s_l \\ e^{\ln(n_{i,tot}) - \tilde{a}_0 + (\tilde{a}_1 + 1) \ln(s_l) - (2\tilde{a}_1 + 1) \ln(s_i)} & \text{if } s_i \leq s_l. \end{cases} \\ \tilde{\sigma}_{D_i} = 1/\tilde{b}_0 \end{cases}$$

#### 4. Influence of the correlation structure: literature examples

To investigate the influence of the correlation structure in predicting fatigue reliability, prediction results are compared with experimental data. The experimental data includes the datasets considered by Liu and Mahadevan<sup>4,5</sup> as well as additional literature data. The complete experimental data, covering a wide range of metallic materials under different types of variable amplitude loadings, is summarized in Table 1.

##### 4.1. Influence of the correlation structure: constant amplitude loading data

As an initial step, constant amplitude data are used to obtain estimates of the parameters  $a_0, a_1, b_0, \mu_l, \sigma_l$  through the Maximum Likelihood (ML) Principle. Parametric estimation based on the ML Principle is a common practice, since it allows for censoring of experimental data and it provides estimators with good asymptotic properties (consistency, unbiasedness, efficiency and normality<sup>6</sup>).

In case of  $n_f$  failure data,  $n_{f,1}^*, n_{f,2}^*, \dots, n_{f,n_f}^*$ , at stress amplitudes  $s_{f,1}, s_{f,2}, \dots, s_{f,n_f}$  and  $n_c$  right-censored (runout) data,  $n_{c,1}^*, n_{c,2}^*, \dots, n_{c,n_c}^*$ , at stress amplitudes  $s_{c,1}, s_{c,2}, \dots, s_{c,n_c}$ , the Log-Likelihood function<sup>16</sup>,  $LL(\boldsymbol{\theta})$ , takes the form:

$$\begin{aligned} LL(\boldsymbol{\theta}) = \ln[L(\boldsymbol{\theta})] &= \ln \left[ \prod_{i=1}^{n_f} f_{LN}(n_{f,i}^*; s_{f,i}, \boldsymbol{\theta}) \prod_{j=1}^{n_c} (1 - F_{LN}(n_{c,j}^*; s_{c,j}, \boldsymbol{\theta})) \right] = \\ &= \sum_{i=1}^{n_f} \ln(f_{LN}(n_{f,i}^*; s_{f,i}, \boldsymbol{\theta})) + \sum_{j=1}^{n_c} \ln((1 - F_{LN}(n_{c,j}^*; s_{c,j}, \boldsymbol{\theta}))), \end{aligned} \quad (13)$$

where  $L(\boldsymbol{\theta})$  is the Likelihood function,  $\boldsymbol{\theta} = (a_0, a_1, b_0, \mu_l, \sigma_l)$ , while  $f_{LN}(n_{f,i}^*; s_{f,i}, \boldsymbol{\theta})$  and  $F_{LN}(n_{c,j}^*; s_{c,j}, \boldsymbol{\theta})$  denote the probability density function (pdf) and the cdf of the logarithm of the number of cycle to failure, respectively. According to the ML Principle, the ML estimate  $\tilde{\boldsymbol{\theta}}$  of  $\boldsymbol{\theta}$  is the set of parameter values that maximizes  $LL(\boldsymbol{\theta})$  in Equation (13).

For the more general case of  $S$ - $N$  curve with fatigue limit, the pdf and the cdf of the logarithm of the number of cycle to failure are given by<sup>15,16,23</sup>:

$$\begin{cases} f_{LN}(n_{f,i}^*; s_{f,i}, \boldsymbol{\theta}) = \frac{1}{b_0} f_{LN|S} \left( \frac{\ln(n_{f,i}^*) - (a_0 + a_1 \ln(s_{f,i}))}{b_0} \right) \Phi \left( \frac{s_{f,i} - \mu_l}{\sigma_l} \right) \\ F_{LN}(n_{c,j}^*; s_{c,j}, \boldsymbol{\theta}) = F_{LN|S} \left( \frac{\ln(n_{c,j}^*) - (a_0 + a_1 \ln(s_{c,j}))}{b_0} \right) \Phi \left( \frac{s_{c,j} - \mu_l}{\sigma_l} \right) \end{cases}, \quad (14)$$

where  $f_{LN|S}(\cdot)$  and  $F_{LN|S}(\cdot)$  denote the standardized pdf and cdf of the Normal distribution, in the Log-Normal case, or the standardized pdf and cdf of the Gumbel distribution, in the Weibull case<sup>9</sup>.

Table 2 reports the experimental stress amplitudes, the adopted distribution for the number of cycles to failure and the parameter estimates. For each experimental dataset, Weibull and Log-Normal distributions are considered. Parameter estimates reported in the Table refer to the distribution that best fits experimental data (bold character in the table).

#### 4.2. Influence of the correlation structure: variable amplitude loading data

The procedure illustrated in Section 3 is valid for any value of  $\lambda$ . Thus, it is valid for a quantile  $S-N$  curve, a statistical  $S-N$  curve, and for any stochastic  $S-N$  curve based on the exponential correlation function in Equation (7). The reliability function which originates from the stochastic approach will always lie between the two extreme cases of  $\lambda = 0$  (quantile  $S-N$  curve) and  $\lambda \rightarrow \infty$  (statistical  $S-N$  curve). Therefore, it is worthwhile analyzing how the two simulated reliability functions for the two extreme cases compare with the empirical reliability function.

The empirical reliability function may be obtained through application of the Kaplan-Meier estimation method<sup>6</sup>, which is commonly adopted for dealing with lifetime data since it allows for including runouts. The empirical reliability function, which originates from the application of the method, is a stepped function; however, when the number of experimental data is large enough, the function tends to be continuous.

For illustrative purpose, the empirical and simulated curves are shown in Figure 3 for one experimental dataset<sup>17</sup>. The horizontal distance between the empirical and each simulated reliability function is linked to the error that is committed in the estimation of the number of cycles to failure,  $\varepsilon_0$  for the quantile approach and  $\varepsilon_\infty$  for the statistical approach:

$$\begin{cases} \varepsilon_0 = \frac{n_{TOT,\lambda=0} - n_{TOT,real}}{n_{TOT,real}} 100 \\ \varepsilon_\infty = \frac{n_{TOT,\lambda=\infty} - n_{TOT,real}}{n_{TOT,real}} 100 \end{cases}$$

where  $n_{TOT,real}$  denotes the empirical number of cycles to failure and  $n_{TOT,\lambda=0}$  and  $n_{TOT,\lambda=\infty}$  are the estimates of the number of cycles to failure for the two extreme cases (Figure 3).

Both  $\varepsilon_0$  and  $\varepsilon_\infty$  are computed for each reliability level of the stepped empirical reliability function.

Figure 4 synthesizes error computation for the complete experimental dataset. As it can be observed, the best fit line (black line) obtained through the Least Square Method is very close to the bisector (grey line). It is also worth noting that the majority of data points concentrate around the origin; however, the larger errors are on the positive side, meaning a nonconservative estimate of the number of cycles to failure, regardless of the adopted correlation structure.

From an engineering point of view, high reliability levels are usually required. In this respect, Figure 5 plots error computation for reliability levels above 80%. As it can be observed, in this case the best fit line (black line) and the bisector (grey line) differ significantly, confirming what reported in [4]. It is also worth noting that the distance between the two lines increases with the error value and, in particular, that, for large error values,  $\varepsilon_0$  is smaller than  $\varepsilon_\infty$ . In particular, of all data points in the first quadrant (non-conservative region) 75% are below the bisector and the remaining 25% are very close to the bisector. Therefore, overestimation of component's life is less critical when the quantile approach is adopted.

As already pointed out, in Figure 4 the majority of data points are concentrated around the origin, confirming that, in spite of its simplistic hypotheses, the Miner's rule is, with reason, the most commonly adopted damage rule. A further question arises: can the correlation structure (stochastic approach) improve the accuracy of Miner's predictions? This would occur in the case where the experimental data points lie between the two extreme cases. In other words, when the two errors,  $\varepsilon_0$  and  $\varepsilon_\infty$ , have opposite sign.

Of all the experimental datasets considered in this paper, only in 10% of cases did the two errors have opposite sign. About the same percentage is found when considering the high reliability levels of Figure 5. Therefore, it can be concluded that only in 10% of the analyzed cases, the stochastic approach would improve the accuracy of Miner's predictions. In the remaining 90%, the higher complexity introduced in the analysis by the correlation structure does not seem to provide significant advantages.

## 5. Conclusions

The paper investigated the application of the stochastic approach when the commonly adopted Miner's linear damage rule is applied in its traditional form (without fatigue limit) or in its modified form to include the presence of a random fatigue limit, below which the rate of damage accumulation is reduced. The steps that must be followed for the application of the stochastic approach showed that:

- if the number of cycles to failure under constant amplitude loading is Weibull (Log-Normal) distributed, the corresponding accumulated damage is Fréchet (Log-Normal) distributed;
- when the stochastic approach is applied in presence of a random fatigue limit, an additional correlation structure, which takes into account the fatigue limit value, must be introduced in the analysis.

The effects of the correlation structure on reliability prediction under variable amplitude loading were also investigated. Comparison of the predicted number of cycles to failure with experimental data showed that:

- larger errors are positive (i.e., nonconservative), regardless of the adopted correlation structure;
- no significant statistical difference exists between the two extreme cases of quantile and statistical  $S-N$  curve when all reliability levels are considered;
- when the analysis is limited to reliability levels above 80%, for the larger errors in the nonconservative region, the quantile approach is closer to the experimental data (i.e., overestimation of component's life is less critical);
- for the experimental datasets considered in the paper, only in 10% of cases would adoption of a stochastic approach improve the accuracy of Miner's predictions;
- in summary, the choice of the damage accumulation rule is a key factor to value the improvement in the accuracy of reliability predictions introduced by the stochastic approach: indeed, the differing results obtained in the present paper and in [4,5] are due to the different damage accumulation models adopted for the analysis.

It is worth recalling that the distribution of accumulated damage in the variable amplitude case cannot be described with an analytical statistical distribution but must be found by simulations. The steps described in the paper remain valid when damage accumulation models different from the Miner's rule are adopted. In this respect, the described procedure could be the basis for the application of the nonlinear damage model proposed in [4,5] that has been proven to provide more accurate reliability predictions. The advantage of the described procedure would be in the more general framework of Log-Normal and Weibull distributed number of cycles to failure (Log-Normal and Fréchet distributed accumulated damage) and in the possibility to take into account the presence of a fatigue limit, below which the rate of damage accumulation is reduced. The numerical efficiency is guaranteed since the described procedure utilizes only Monte Carlo simulations with no need of the expansion techniques adopted in [4,5].

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## Figure captions

Figure 1: Random fatigue limit model and quantile  $S$ - $N$  curves.

Figure 2: Illustrative plot of the correlation structure in presence of a random fatigue limit. Dash-dotted arrow shows the correlation structure of Equation (7). Dotted arrows highlight correlation structure with point  $(N_{s_l}, s_l)$ .

Figure 3: Illustrative example of empirical and simulated reliability functions. Experimental data taken from [17].

Figure 4: Error computation plot for the complete experimental dataset (571 data points). Nonconservative estimates correspond to overestimation of component's life.

Figure 5: Error computation plot for reliability levels above 80% (97 data points). Highlighted data points are cases for which the quantile approach reduces overestimation of component's life.

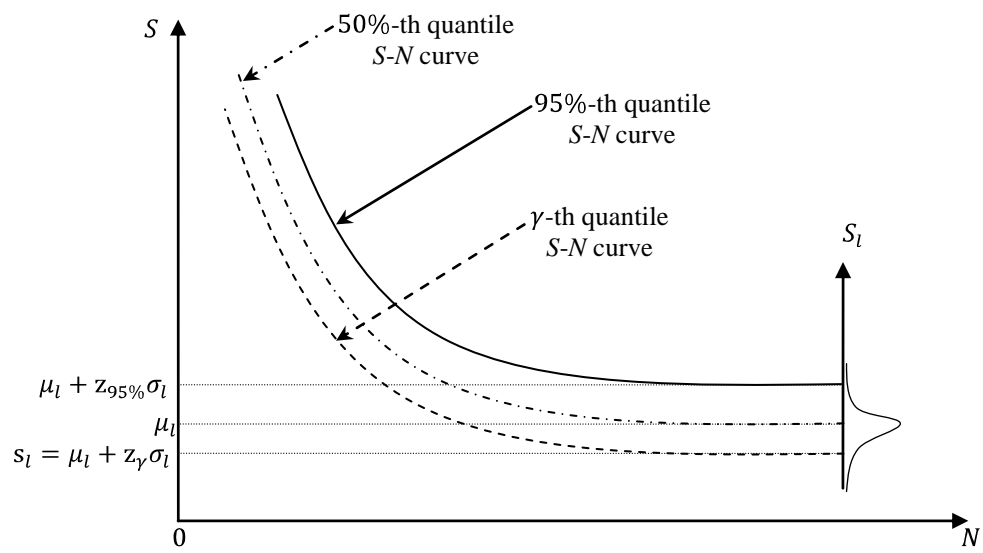


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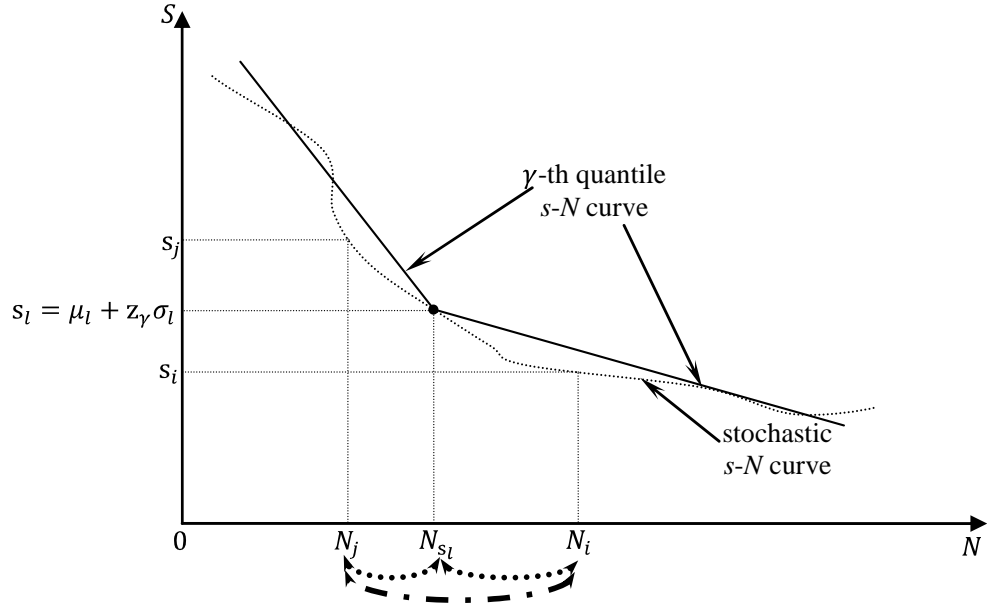


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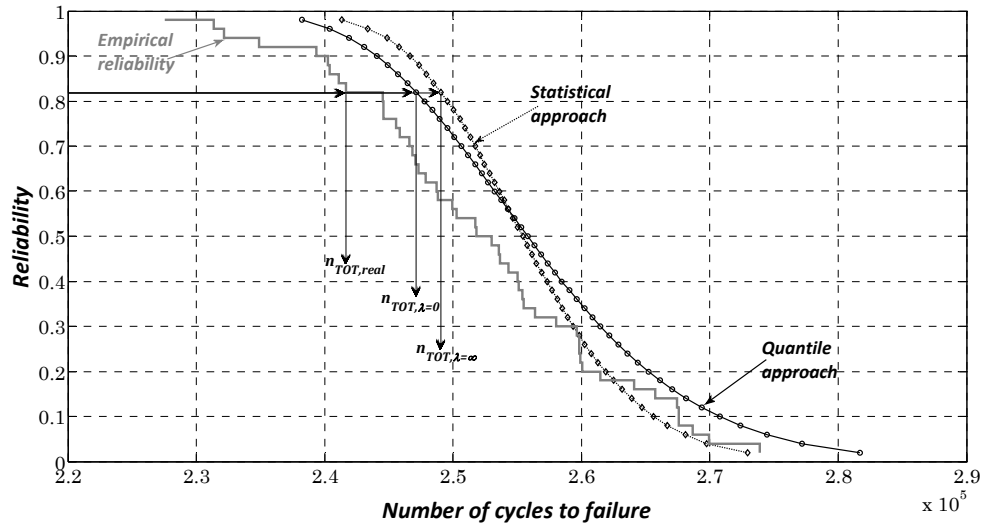


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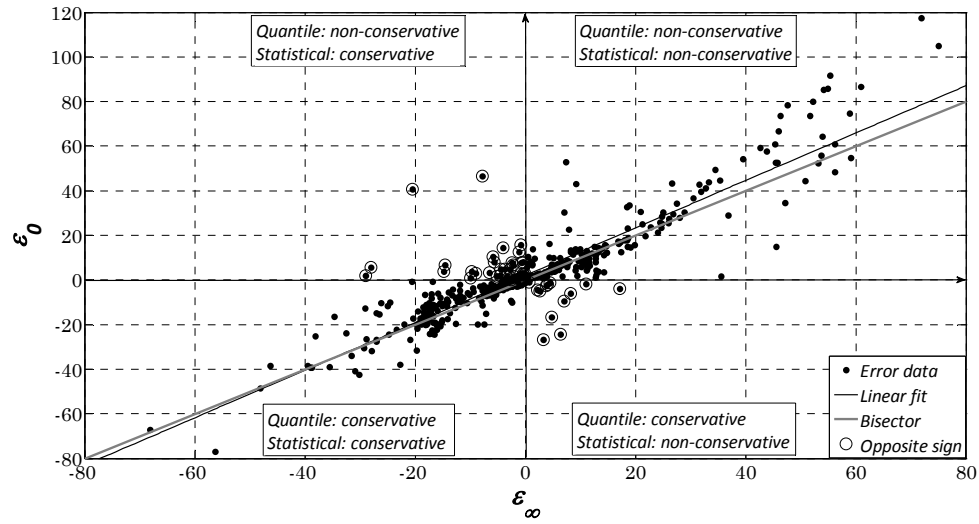


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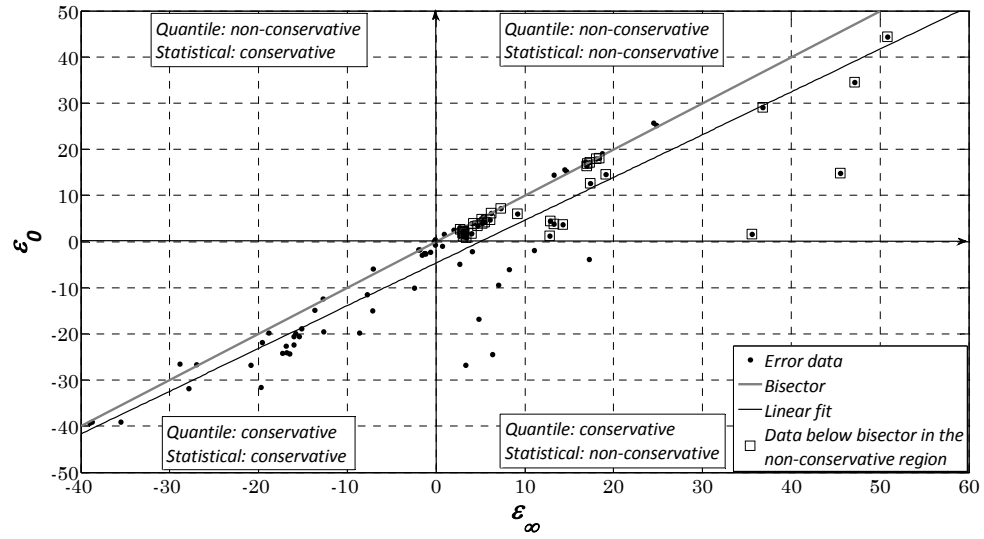


Figure 5: Error computation plot for reliability levels above 80% (97 data points). Highlighted data points are cases for which the quantile approach reduces overestimation of component's life.

Table 1: Experimental data taken from the literature

Material	Reference	Fatigue Limit	Number of specimens	
			Constant Amplitude	Variable Amplitude
Nickel-silver	Tanaka et al. <sup>17</sup>	No	200	50
16Mn steel	Xie <sup>18</sup>	No	15	10
45 steel-1	Xie <sup>18</sup>	No	15-18	12-15
45 steel-2	Yan et al. <sup>19</sup>	No	10	6
45 steel-3	Yan et al. <sup>20</sup>	Yes	10	9
45 steel-4	Zheng and Wei <sup>21</sup>	Yes	10	9
Aluminum alloy 319-T7	Mayer et al. <sup>22</sup>	Yes	6	5-7

Table 2: Constant amplitude tests and parameter estimates

Material	Stress amplitudes [MPa]	Parameter estimates ( $\tilde{a}_0, \tilde{a}_1, \tilde{b}_0, \tilde{\mu}_l, \tilde{\sigma}_l$ )	Log-Likelihood values
Nickel-silver	478,666	(51.54,-6.25,0.19,0,0)	$ll = \begin{cases} 99.93 & \text{Log-Normal} \\ 93.49 & \text{Weibull} \end{cases}$
16Mn steel	344,373,394	(94.34,-13.85,0.20,0,0)	$ll = \begin{cases} 9.52 & \text{Log-Normal} \\ 5.66 & \text{Weibull} \end{cases}$
45 steel-1	309,331,366	(104.62,-15.94,0.38,0,0)	$ll = \begin{cases} -18.61 & \text{Log-Normal} \\ -17.73 & \text{Weibull} \end{cases}$
45 steel-2	520,590,630,650,750	(58.08,-7.21,0.35,409.73,19.43)	$ll = \begin{cases} -17.93 & \text{Log-Normal} \\ -24.32 & \text{Weibull} \end{cases}$
45 steel-3			
45 steel-4	400,450,475,500,525	(56.16,-7.01,0.68,332.86,27.43)	$ll = \begin{cases} -51.90 & \text{Log-Normal} \\ -52.80 & \text{Weibull} \end{cases}$
Aluminum alloy 319-T7	38,44,50,56,67,87,108	(42.24,-6.62,0.57,47.07,4.20)	$ll = \begin{cases} -33.65 & \text{Log-Normal} \\ -36.99 & \text{Weibull} \end{cases}$