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A Fractional Approach to Nonlocal Elasticity

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Abstract: If the attenuation function of strain is expressed as a power law, the formalism of fractional calculus may be used to handle Eringen nonlocal elastic model. Aim of the present paper is to provide a mechanical interpretation to this nonlocal fractional elastic model by showing that it is equivalent to a discrete, point-spring model. A one-dimensional geometry is considered; the static, kinematic and constitutive equations are presented and the governing fractional differential equation highlighted. Two numerical procedures to solve the fractional equation are finally implemented and applied to study the strain field in a finite bar under given edge displacements.

Keywords: Nonlocal elasticity, long-range interactions, Riesz derivative, Marchaud derivative.

1. INTRODUCTION

One of the most attractive properties of fractional calculus is that, by varying the order of derivation, it is possible to describe the transition between completely different phenomena. In the last decades, many fractional differential equations have been proposed, solved and used to interpret experimental data that cannot be caught by usual differential models. They include relaxation equations, wave equations and diffusion equations (see e.g. Carpinteri and Mainardi, 1997; Mainardi, 2009).

For what concerns solid mechanics, most of the applications refer to rheological models, where the fractional derivative is taken with respect to the time variable. More recently, two research directions (Carpinteri et al., 2009) have been set that make use of fractional derivative with respect to the space variable. The former one (Carpinteri and Cornetti, 2002) explores the connection between fractal sets and fractional calculus and applies the (local) fractional calculus formalism to address the problem of deformation and damage in solid mechanics. Fractal patterns often arise in heterogeneous materials and developing mathematical models able to catch the fractality of such phenomena is a matter of primary concern. The strength of such approach are the non-integer physical dimensions provided by the fractal geometry and fractional operators. This property has proven to be very effective in the description of the size-scale effects in solid mechanics. For a review, see Carpinteri et al. (2006).

The latter research direction (Di Paola and Zingales, 2008) aims to model nonlocal continua, i.e. solids characterized by nonlocal interactions (Eringen and Edelen, 1972; Aifantis, 1994; Polizzotto, 2001). The novelty is that internal forces are described by fractional derivatives. One of the most remarkable achievements of this approach is that, by exploiting the Marchaud definition of fractional derivative, the fractional operators have a clear mechanical interpretation, i.e. springs connecting non-adjacent points of the body. The related stiffness decays along with the distance among the material points. However, since only the integral part of the Marchaud derivative is retained in the equilibrium equation, the model developed by Di Paola and Zingales (2008) does not coincide with the corresponding Eringen nonlocal fractional elastic model; also the order of fractional derivation is different. On the other hand, aim of the present paper is to provide a mechanical interpretation to Eringen fractional nonlocal model and an efficient algorithm for its solution.

2. FRACTIONAL INTEGRALS AND DERIVATIVES

There are various definitions of fractional differintegral operators, not necessarily equivalent to each other. A complete list of these definitions can be found in the fractional calculus treatises, e.g. Samko et al. (1993). These definitions have different origins. The most frequently used definition of fractional integral of order β ($\beta \in \Re^+$) is due to Riemann-Liouville and is a straightforward generalization to non-integer values of Cauchy formula for repeated integrations:

$$I_{a+}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{f(t)}{(x-t)^{1-\beta}} \mathrm{d}t \tag{1}$$

The Riemann-Liouville fractional derivative of order β is defined as the (integer) derivative of order n ($n \in N$ and $n-1 < \beta < n$) of the fractional integral of order $(n-\beta)$. That is:

$$D_{a+}^{\beta} f(x) = D^n [I_{a+}^{n-\beta} f(x)] \tag{2}$$

However, it is also possible to define the fractional derivative as the fractional integral of order (1−β) of the first derivative. In such a case we obtain the Caputo definition of fractional derivative, ${}_cD_{a+}^{\beta}$:

$${}_cD_{a+}^{\beta} f(x) = I_{a+}^{n-\beta}[D^n f(x)] \tag{3}$$

It is worth observing that the Riemann-Liouville derivative of a constant is not zero, whereas it is null the corresponding Caputo derivative. Since Caputo definition generalizes this well-known property of the derivatives of integer orders, Caputo fractional derivative is usually more practical for applications.

Eqns (1-3) represent the so-called left (or forward) fractional integrals and derivatives. Analogously, it is possible to define the right (or backward) operators as:

$$I_{b-}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b \frac{f(t)}{(t-x)^{1-\beta}} \mathrm{d}t \tag{4}$$

$$D_{b-}^{\beta} f(x) = (-D)^n [I_{a+}^{n-\beta} f(x)] \tag{5}$$

$${}_cD_{b-}^{\beta} f(x) = I_{a+}^{n-\beta}[(-D)^n f(x)] \tag{6}$$

A general result in fractional analysis states that the Caputo fractional derivatives (either forward or backward) of a function $f(x)$ are equal to the Riemann-Liouville derivatives provided that the polynomial of order $n-1$ (evaluated either in $x = a$ or $x = b$) is subtracted from the function itself:

$$D_{a+}^{\beta}\left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k \right] = {}_cD_{a+}^{\beta} f(x) \tag{7}$$

$$D_{b-}^{\beta}\left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!}(x-b)^k \right] = {}_cD_{b-}^{\beta} f(x) \tag{8}$$

By recalling the fractional derivatives of the power functions $(x-a)^k$ and $(b-x)^k$, eqns (7-8) provide:

$$D_{a+}^{\beta} f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(1+k-\beta)}(x-a)^{k-\beta} + {}_cD_{a+}^{\beta} f(x) \tag{9}$$

$$D_{b-}^{\beta} f(x) = \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{\Gamma(1+k-\beta)}(b-x)^{k-\beta} + {}_cD_{b-}^{\beta} f(x) \tag{10}$$

that will be exploited for the numerical implementation in Section 5.

In the case $0 < \beta < 1$, by applying the formula of integration by parts to the Caputo's definition and after some analytical manipulations, it is possible to give an alternative form to the Riemann-Liouville fractional derivative (Samko et al., 1993):

$$D_{a+}^{\beta} f(x) = \frac{f(x)}{\Gamma(1-\beta)(x-a)^{\beta}} + \frac{\beta}{\Gamma(1-\beta)} \int_a^x \frac{f(x)-f(t)}{(x-t)^{1+\beta}} \mathrm{d}t \tag{11}$$

$$D_{b-}^{\beta} f(x) = \frac{f(x)}{\Gamma(1-\beta)(b-x)^{\beta}} + \frac{\beta}{\Gamma(1-\beta)} \int_x^b \frac{f(x)-f(t)}{(t-x)^{1+\beta}} \mathrm{d}t \tag{12}$$

Eqns (11-12) are the so-called Marchaud definitions of fractional derivative. Since, for $0 < \beta < 1$, they coincide with the Riemann-Liouville definitions for a wide class of functions, we will indicate them with the same symbol.

Finally, it is possible to introduce the Riesz fractional integrals and derivatives, defined as the sum of the forward and backward fractional operator up to a multiplicative factor:

$$I_{a,b}^{\beta} f(x) = \frac{1}{2}\left[I_{a+}^{\beta} f(x) + I_{b-}^{\beta} f(x) \right] = \frac{1}{2\Gamma(\beta)} \int_a^b \frac{f(t)}{|x-t|^{1-\beta}} \mathrm{d}t \tag{13}$$

$$D_{a,b}^{\beta} f(x) = \frac{1}{2}\left[D_{a+}^{\beta} f(x) + D_{b-}^{\beta} f(x) \right] \tag{14}$$

Note that the multiplicative constant, here taken simply equal to ½ following, e.g., Agrawal (2007), can vary according to the different fractional calculus treatises (usually it is function of β).

By exploiting the Marchaud definition of fractional derivatives (11-12), for $0 < \beta < 1$, the Riesz fractional derivative (14) can be expressed as:

$$D_{a,b}^{\beta} f(x) = \\ = \frac{1}{2\Gamma(1-\beta)}\left[\frac{f(x)}{(x-a)^{\beta}} + \frac{f(x)}{(b-x)^{\beta}} + \beta \int_a^b \frac{f(x)-f(t)}{|x-t|^{1+\beta}} \mathrm{d}t \right] \tag{15}$$

While the Marchaud definitions (11) and (12) hold true only for $0 < \beta < 1$ (otherwise the integrals at the right-hand side diverge), we were able to prove that eqn (15) is valid in the whole range $0 < \beta < 2$. This non-trivial result (details will be given elsewhere) is of fundamental importance, since it will allow us to provide a mechanical interpretation to the Eringen non-local fractional model in the next section. For fractional operators analogous to (13-14), but defined on infinite domains, see, e.g., Mainardi et al. (2002) and Ortigueira (2008).

3. ERINGEN NONLOCAL FRACTIONAL MODEL

According to Eringen nonlocal elasticity, the stress at a given point depends on the strain in a neighbourhood of that point by means of a convolution integral. This dependence is described by a proper attenuation function g, which decays along with the distance. In the case of a one-dimensional domain (i.e. a bar):

$$\sigma(x) = E\left[\varepsilon(x) + \kappa_{\alpha} \int_a^b \varepsilon(t)\, g(x-t) \mathrm{d}t \right] \tag{16}$$

where σ is the stress, x the longitudinal coordinate, $x = a$ and $x = b$ the bar extreme coordinates, E the Young's modulus, ε the strain defined as the derivative of the longitudinal displacement u and κ_{α} is a material constant. The bar length is l ($l = b-a$). Note that the Eringen model (Eringen and Edelen, 1972) is sometime referred to as strong (or integral) nonlocality, to distinguish it from the weak (or gradient) nonlocal elastic model, where the stress depends on the strain and its derivatives (Aifantis, 1994).

Let us now assume the following form for the attenuation function g:

$$g(\xi) = \frac{1}{2\,\Gamma(1-\alpha)|\xi|^{\alpha}} \tag{17}$$

with $0 < \alpha < 1$.With the choice of eqn (17), the constitutive relationship becomes:

$$\sigma(x) = E\left[\varepsilon(x) + \frac{\kappa_\alpha}{2\Gamma(1-\alpha)}\int_a^b \frac{\varepsilon(t)}{|x-t|^\alpha}\,\mathrm{d}t\right] \tag{18}$$

In eqn (18) we recognize the presence of the Riesz integral (13). Thus, we may rewrite eqn (18) as:

$$\sigma(x) = E\left[\varepsilon(x) + \kappa_\alpha (I_{a,b}^{1-\alpha}\varepsilon)\right] \tag{19}$$

Note that κ_α has anomalous physical dimensions $[\mathrm{L}]^{\alpha-1}$. Since $\varepsilon = \mathrm{d}u / \mathrm{d}x$, it is easy to highlight in eqn (19) the presence of the Caputo fractional derivatives of the displacement. Therefore, by exploiting eqns (3) and (6), the dependence of the stress upon the displacement becomes:

$$\sigma(x) = E\left[\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\kappa_\alpha}{2}({}_{\mathrm{c}}D_{a+}^{\alpha}u - {}_{\mathrm{c}}D_{b-}^{\alpha}u)\right] \tag{20}$$

In order to get the equilibrium equation in terms of the displacement function $u(x)$, we simply need to substitute eqn (20) into the static equation $\mathrm{d}\sigma/\mathrm{d}x + f(x) = 0$, where $f(x)$ is the longitudinal force per unit volume. By means of eqns (7) and (8) and some more analytical manipulations, we get:

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{\kappa_\alpha}{2}\left\{D_{a+}^{1+\alpha}[u(x)-u(a)] + D_{b-}^{1+\alpha}[u(x)-u(b)]\right\} = -\frac{f(x)}{E} \tag{21}$$

Eqn (21) is a fractional differential equation. Note that, while the left fractional derivative coincides always with its integer order counterpart when the order of derivation is an integer number, the right fractional derivative coincides with the corresponding integer derivative only when the order of derivation is even; when the order of derivation is an odd number, it is equal to its opposite. Therefore, the term in curly brackets is equal to $2\,u''(x)$ when $\alpha = 1$, and vanishes when $\alpha = 0$. In order to highlight the presence of the Riesz fractional derivative, eqn (21) can be set in the following form:

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \kappa_\alpha\left\{D_{a,b}^{1+\alpha}[u(x)] + \frac{\alpha}{2\,\Gamma(1-\alpha)}\left[\frac{u(a)}{(x-a)^{1+\alpha}} + \frac{u(b)}{(b-x)^{1+\alpha}}\right]\right\} = -\frac{f(x)}{E} \tag{22}$$

4. EQUIVALENT POINT-SPRING MODEL

A useful interpretation of the governing equation (22) for the nonlocal elastic bar is sought. To this aim, it is convenient to express Riesz fractional derivatives in the Marchaud-like form (Di Paola and Zingales (2008)). Note that this is possible since we proved that eqn (15) holds true also for orders of derivation between 1 and 2, which is exactly the case in eqn (22) (while the order of fractional derivation considered in Di Paola and Zingales (2008) was less than unity). Hence, by letting $\beta = 1 + \alpha$ in eqn (15), we get:

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \kappa_\alpha\frac{\alpha}{2\,\Gamma(1-\alpha)}\left[\frac{u(x)-u(a)}{(x-a)^{1+\alpha}} + \frac{u(x)-u(b)}{(b-x)^{1+\alpha}} + (1+\alpha)\int_a^b \frac{u(x)-u(t)}{|x-t|^{2+\alpha}}\,\mathrm{d}t\right] = -\frac{f(x)}{E} \tag{23}$$

where the gamma function property $\Gamma(1-\alpha) = -\alpha\,\Gamma(-\alpha)$ has been used. In this form it is evident that the first term at the left-hand side rules the local interactions, whereas the terms in the square brackets rule the nonlocal interactions by means of linear elastic springs and can be seen as an extra-force per unit of volume acting at the point of abscissa x. More in detail, the first two terms in the brackets refer to long-range interactions between the inner ($a < x < b$) and outer ($x = a$, $x = b$) points, whereas the integral term takes into account the interaction between two inner generic points. To make the concept even clearer, it is useful to write eqn (23) in discrete form. To this purpose, let us introduce a partition of the interval $[a, b]$ on the x axis made of n ($n \in N$) intervals of length $\Delta x = l/n$. The generic point of the partition has the abscissa x_i, with $i = 1, \ldots, n+1$ and $x_1 = a$, $x_{n+1} = b$; that is, $x_i = a + (i-1)\Delta x$. Hence, for the inner points of the domain ($i = 2,\ldots, n$), the discrete form of eqn (23) reads:

$$\frac{-u_{i+1} + 2u_i - u_{i-1}}{(\Delta x)^2} + \frac{\kappa_\alpha\alpha}{2\,\Gamma(1-\alpha)}\left[\frac{u_i - u_1}{(x_i - x_1)^{1+\alpha}} + \frac{u_i - u_{n+1}}{(x_{n+1} - x_i)^{1+\alpha}}\right] + \frac{\kappa_\alpha\alpha(1+\alpha)}{2\,\Gamma(1-\alpha)}\sum_{\substack{j=1\\ j\neq i}}^{j=n+1}\frac{(u_i - u_j)\Delta x}{|x_i - x_j|^{2+\alpha}} = \frac{f_i}{E} \tag{24}$$

where $u_i \equiv u(x_i)$ and $f_i \equiv f(x_i)$. Multiplying both the sides of eqn (24) by $EA\,\Delta x$, eqn (24) may be rewritten as:

$$k_{i,i+1}^{l}(u_i - u_{i+1}) + k_{i,i-1}^{l}(u_i - u_{i-1}) + k_{i,1}^{vs}(u_i - u_1) + k_{i,n+1}^{vs}(u_i - u_{n+1}) + \sum_{j=1,\, j\neq i}^{j=n+1} k_{i,j}^{vv}(u_i - u_j) = f_i A\Delta x \tag{25}$$

It is evident how the nonlocal fractional model is equivalent to a point-spring model where three kinds of springs appear: the local springs, ruling the local interactions, whose stiffness is k^l; the springs connecting the inner material points with the bar edges, ruling the volume-surface long-range interactions, with stiffness k^{vs}; the springs connecting the inner material points each other, describing the nonlocal interactions between non-adjacent volumes, whose stiffness is k^{vv}. Provided that the indexes are never equal one to the other, the following expressions for the stiffnesses hold ($i = 1,\ldots, n+1$):

$$k_{i,i+1}^{l} = k_{i+1,i}^{l} = EA/\Delta x \tag{26}$$

$$k_{i,1}^{vs} = k_{1,i}^{vs} = EA\kappa_\alpha\frac{\alpha}{2\,\Gamma(1-\alpha)}\frac{\Delta x}{(x_i - x_1)^{1+\alpha}} \tag{27}$$

$$k_{i,n+1}^{vs} = k_{n+1,i}^{vs} = EA\kappa_\alpha\frac{\alpha}{2\,\Gamma(1-\alpha)}\frac{\Delta x}{(x_{n+1} - x_i)^{1+\alpha}} \tag{28}$$

$$k_{i,j}^{vv} = k_{j,i}^{vv} = EA\kappa_\alpha\frac{\alpha(1+\alpha)}{2\,\Gamma(1-\alpha)}\frac{(\Delta x)^2}{|x_i - x_j|^{2+\alpha}} \tag{29}$$

Furthermore, by exploiting the Principle of Virtual Work to derive the proper either kinematic or static boundary conditions, it is possible to show that a fourth set of springs has to be introduced: it is composed by a unique spring connecting the two bar extremes with stiffness:

$$k^{ss}_{1,n+1} = k^{ss}_{n+1,1} = \frac{EA\kappa_\alpha}{2\,\Gamma(1-\alpha)} \frac{1}{(x_{n+1}-x_1)^\alpha} \tag{30}$$

The superscript "ss" for the stiffness (30) is used since the spring connecting the bar edges can be seen as modeling the interactions between material points lying on the surface, which, in the simple one-dimensional model under examination, reduce to the two points $x = a,b$. Note that the presence of such a spring was implicitly embedded in the constitutive equation (19). However, since it provides a constant stress contribution throughout the bar length, its presence was lost by derivation when inserting the constitutive relationship into the differential equilibrium equation, i.e. when passing from eqn (20) to eqn (21).

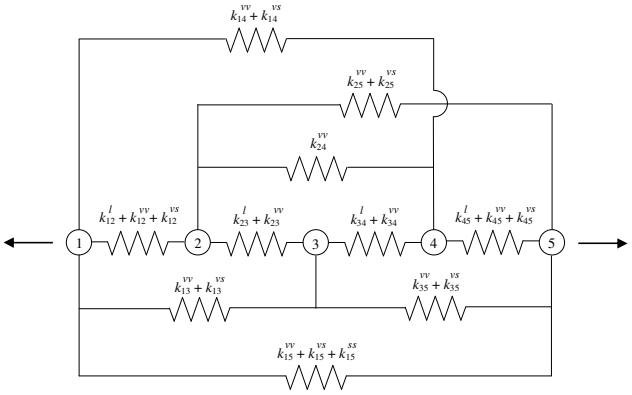


Fig. 1. Pont-spring model equivalent to the nonlocal fractional elastic bar ($n = 4$).

To summarize, the constitutive fractional relationship (19) is equivalent to a point-spring model with four sets of springs, one local (eqn (26)) and three nonlocal (27-30). Note that their stiffnesses all decay with the distance, although the decaying velocity differs from one kind to the other. The equivalent point-spring model is drawn in fig.1. Each internal point is connected to the adjacent points by two local springs, to the bar extremes by two volume-surface nonlocal springs and to all the other material points by the volume-volume nonlocal springs. Finally a surface-surface nonlocal spring connects the bar edges. Turning the attention to the whole bar, the number of the local springs is n, the number of the volume-surface springs is $2n-1$, the number of the volume-volume springs is $n(n+1)/2$.

For what concerns the limit cases, if $\alpha = 0$, the volume-volume and the volume-surface spring interactions ruled by eqns (27-29) vanish, and only the contribution (30) remains (together with the local springs (26)): the nonlocal model corresponds to a classical (local) elastic bar in parallel with a spring of stiffness $EA\kappa_\alpha/2$. The governing equation reverts to the classical case: $u'' = f / E$. On the other hand, in the limit case $\alpha = 1$, since $\Gamma(0) = \infty$, the surface-surface (eqn (30)) and the volume-surface (eqns (27-28)) contributions disappear. For what concerns the interactions between inner material points (eqn (29)), only the interactions between adjacent material points are retained (the Gamma function tends to infinity, but the integral in eqn (23) diverges). Correspondingly, the additive term in eqn (19) has the same form as the classical (local) one, the model representing a bar with a stiffened (by a factor of $(1+\kappa_\alpha)$) Young's modulus, while the governing equation (23) becomes $u'' = f / [E(1+\kappa_\alpha)]$.

5. NUMERICAL ANALYSIS

Based on eqn (15), the equivalence between the fractional model and the point-spring model proved in the previous section provide also a straightforward numerical algorithm to implement the fractional governing equation (22-23). In fact, by using the same partition previously introduced, eqn (23) can be discretized as:

$$\sum_{j=1}^{j=n+1} k_{i,j}\, u_j = F_i \qquad i = 1,\ldots,n+1 \tag{31}$$

where the right hand side F_i is equal to $f_i\, A\, \Delta x$ for the inner points ($i = 2,\ldots, n$) and to the external forces F_a and F_b acting at the bar edges for $i = 1$ and $i = n+1$, respectively; $k_{i,j}$ is the generic element of the stiffness (square) matrix $\mathbf{K}$, which is the sum of four stiffness matrices:

$$\mathbf{K} = \mathbf{K}^{l} + \mathbf{K}^{vv} + \mathbf{K}^{vs} + \mathbf{K}^{ss} \tag{32}$$

whose non-diagonal terms are provided by the opposite of the corresponding stiffnesses (26-30). Furthermore, the diagonal terms $k_{i,i}$ of each matrix is provided by the relationship:

$$k_{i,i} = \sum_{j=1,\, j\neq i}^{j=n+1} k_{i,j}, \qquad i = 1,\ldots,n+1 \tag{33}$$

Note that all the four matrices are symmetrical, with positive elements on the diagonal and negative outside. More in detail, the local matrix $\mathbf{K}^{l}$ is tridiagonal; the nonlocal matrix $\mathbf{K}^{vv}$ ruling the long-range interactions between inner points is fully populated; the nonlocal matrix related to the inner-outer interactions $\mathbf{K}^{vs}$ has only border and diagonal elements different from zero; finally, the nonlocal matrix $\mathbf{K}^{ss}$ describing the interaction between the bar edges is empty except for the four corner elements.

Despite the clear physical-mechanical meaning, however, the discretization (31) is not the most efficient way to solve the fractional differential equation (22). Particularly, it is not able to catch the solution for α approaching unity, when the weight function in the integral in eqn (23) behaves as a Dirac function. Since the order of fractional derivation is comprised between 1 and 2 (i.e. $0 \le \alpha < 1$), we chose to implement the so-called L2 algorithm firstly proposed by Oldham and Spanier (1974) and later applied to discretize the Riesz derivative by Yang et al. (2010). The L2 algorithm is based on the formulae (9) and (10), which now read:

$$D_{a+}^{1+\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{-\alpha\, f(a)}{(x-a)^{1-\alpha}} + \frac{f'(a)}{(x-a)^{\alpha}} + \int_a^x \frac{f''(t)\mathrm{d}t}{(x-t)^{\alpha}} \right] \quad (34)$$

$$D_{b-}^{1+\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{-\alpha\, f(b)}{(b-x)^{1-\alpha}} - \frac{f'(b)}{(b-x)^{\alpha}} + \int_x^b \frac{f''(t)\mathrm{d}t}{(t-x)^{\alpha}} \right] \quad (35)$$

By approximating the second order derivatives by means of the usual finite differences and evaluating analytically the remaining part of the integrals in eqns (34-35), we get the following approximate discrete expressions of the fractional derivatives in the internal points of the interval $[a,b]$, i.e. for $i = 2,\ldots, n$:

$$D_{a+}^{1+\alpha} f(x_i) \approx \frac{(\Delta x)^{-(1+\alpha)}}{\Gamma(2-\alpha)} \times$$
$$\times \left\{ \frac{-\alpha\,(1-\alpha)}{(i-1)^{1+\alpha}} f_1 + \frac{1-\alpha}{(i-1)^{\alpha}} (f_2 - f_1) + \right. \quad (36)$$
$$\left. + \sum_{k=0}^{i-2} (f_{i-k+1} - 2f_{i-k} + f_{i-k-1})[(k+1)^{1-\alpha} - k^{1-\alpha}] \right\}$$

$$D_{b-}^{1+\alpha} f(x_i) \approx \frac{(\Delta x)^{-(1+\alpha)}}{\Gamma(2-\alpha)} \times$$
$$\times \left\{ \frac{-\alpha\,(1-\alpha)}{(n-i+1)^{1+\alpha}} f_{n+1} - \frac{(1-\alpha)}{(n-i+1)^{\alpha}} (f_{n+1} - f_n) + \right. \quad (37)$$
$$\left. + \sum_{k=0}^{n-i} (f_{i+k+1} - 2f_{i+k} + f_{i+k-1})[(k+1)^{1-\alpha} - k^{1-\alpha}] \right\}$$

Developing the sums in eqns (36-37), the Riesz fractional derivative can hence be approximated as:

$$D_{a,b}^{1+\alpha} f(x_i) \approx \frac{(\Delta x)^{-(1+\alpha)}}{2\,\Gamma(2-\alpha)} \sum_{j=1}^{n+1} r_{i,j} f_j \quad (38)$$

where the terms $r_{i,j}$ of the matrix $\mathbf{R}$ are provided in Appendix A. By exploiting eqn (38), we may discretize the fractional differential equation (22) in a form alternative to eqn (24) (i.e. (31)) as:

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} + \frac{\kappa_\alpha}{2\,\Gamma(2-\alpha)} \frac{1}{(\Delta x)^{1+\alpha}} \left\{ \sum_{j=1}^{j=n+1} r_{i,j} u_j + \right.$$
$$\left. + \alpha(1-\alpha) \left[\frac{u_1}{(i-1)^{1+\alpha}} + \frac{u_{n+1}}{(n-i+1)^{1+\alpha}} \right] \right\} = -\frac{f_i}{E} \quad (39)$$

holding for $i = 2,\ldots, n$.

We applied the developed fractional nonlocal model to analyze the strain field in a finite bar with given edge displacement: $u_a = u_0 = 0$ m and $u_b = u_{n+1} = 10^{-3}$ m. The bar length l is 1 m and the material constant κ_α is assumed equal to 1 $\mathrm{m}^{1-\alpha}$. Note that, since no external force is present, the Young's modulus E does not affect the result.

We solved the governing equation (22) in the case $\alpha = 0.5$ both with the physically-based algorithm (31), i.e. the point-spring model, and with the mathematically-based algorithm (39). The strain fields corresponding to the two numerical procedures are plotted in fig. 2a. From a mathematical point of view, the almost perfect coincidence between the two solutions is an important result since it can be seen as an indirect proof that eqn (15) holds true also for order of fractional derivation comprised between 1 and 2, as we claimed in Section 2. On the other hand, from a physical point of view, it is interesting to observe that, with respect to the classical case represented by a uniform strain field, the strain localizes near the bar ends. This effect can be explained observing that the zones close to the borders are less stiff because of lower presence of the long-range interactions. Finally, it is interesting to note that the same behaviour is provided also by gradient elasticity (Vardoulakis et al., 1996).

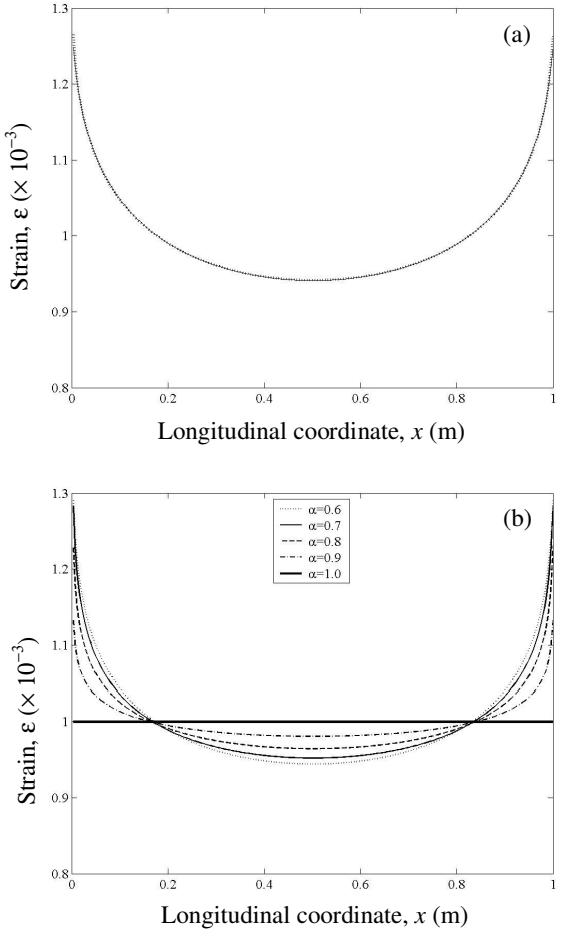


Fig. 2. Strain field for a bar under given edge displacements: (a) comparison between the spring algorithm (continuous line) and the L2 algorithm (dotted line) for $\alpha = 0.5$; (b) strain fields for different α values obtained by the L2 algorithm.

As already observed, algorithm (31) is much less efficient than algorithm (39) for any α, becoming completely unreliable as $\alpha \to 1^-$. Therefore, in fig. 2b, we used algorithm (39) to solve the same geometry considered in fig.2a but for different α values. Note that the area beneath each curve is constant, being equal to the assigned relative displacement between the bar extremes (i.e. 10^{-3} m). As $\alpha \to 1^-$, the solution tend to the homogeneous one since the nonlocal term

in the constitutive equation (19) acquires the same form as the local one. On the other hand, up to some extent, the strain localization increases as α decreases.

6. CONCLUSIONS

As is well-known, in solid mechanics, temporal fractional derivatives can be used to develop improved visco-elastic models. On the other hand, in the present paper we showed that spatial fractional derivative (namely the Riesz fractional derivative) can be used to take into account nonlocal effects in the constitutive material behaviour. It has been shown that, in the one-dimensional case, the nonlocal fractional bar is equivalent to a point-spring model with four sets of springs. It is argued that these springs may describe the long-range interactions between volume and surface elements of the solid, that, as is well-known, play an important role at the smaller scales. A key role in deriving the present model is played by the proof of a suitable formula expressing the Riesz fractional derivative of order comprised between 1 and 2 in a Marchaud-like form.

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Appendix A. APPROXIMATION OF THE RIESZ DERIVATIVE

In this appendix we provide the explicit expressions of the terms $r_{i,j}$ (i = 2,…, n and j = 1,…, n+1) of the square (n+1)×(n+1) matrix **R** (see eqn (38)). The terms on the first and last row depend on the assigned boundary conditions and are not given here. We have:

$$r_{2,1} = 1 + \alpha^2 \tag{A.1}$$

$$r_{2,2} = 2^{1-\alpha} - 4 - \alpha = r_{n,n} \tag{A.2}$$

$$r_{3,1} = -1 + (2 + \alpha + \alpha^2)/\, 2^{1+\alpha} \tag{A.3}$$

$$r_{3,2} = 4 - (3 + \alpha) \,/\, 2^{\alpha} \tag{A.4}$$

The remaining diagonal terms are (i = 3,…, n–1):

$$r_{i,i} = -2\,(3 - 2^{1-\alpha}) \tag{A.5}$$

while the terms on the first and second columns are, respectively (i = 4,…, n):

$$r_{i,1} = -\alpha(1-\alpha)(i-1)^{-(1+\alpha)} - (1-\alpha)(i-1)^{-\alpha} + \\ + (i-1)^{1-\alpha} - (i-2)^{1-\alpha} \tag{A.6}$$

$$r_{i,2} = (1-\alpha)(i-1)^{-\alpha} - 2(i-1)^{1-\alpha} + \\ + 3(i-2)^{1-\alpha} - (i-3)^{1-\alpha} \tag{A.7}$$

The elements on the diagonal close to the main one have the following values (i = 4,…, n):

$$r_{i,i-1} = 4 + 3^{1-\alpha} - 3 \cdot 2^{1-\alpha} \tag{A.8}$$

All the other terms beneath the main diagonal are equal to (i = 5,…, n; j = 3,…, i–2):

$$r_{i,j} = (i-j+2)^{1-\alpha} - 3(i-j+1)^{1-\alpha} + \\ + 3(i-j)^{1-\alpha} - (i-j-1)^{1-\alpha} \tag{A.9}$$

The remaining elements, i.e. those placed above, are immediately obtained by observing that the matrix **R** fulfils a sort of polar symmetry, i.e. (i = 2,…, n; j = i+1,…, n+1):

$$r_{i,j} = r_{n+2-i,n+2-j} \tag{A.10}$$

and the matrix **R** is completely defined.