Wiener algebras of Fourier integral operators

THE WIENER PROPERTY FOR A CLASS OF FOURIER INTEGRAL OPERATORS

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Abstract. We construct a one-parameter family of algebras $FIO(\Xi, s)$, $s \geq 0$, consisting of Fourier integral operators. We derive boundedness results, composition rules, and the spectral invariance of the operators in $FIO(\Xi, s)$. The operator algebra is defined by the decay properties of an associated Gabor matrix around the graph of the canonical transformation.

1. Introduction

Wiener's lemma, in its original version [38], [39], is a classical statement about absolutely convergent series. In a more general setting, Wiener's lemma represents now one of the driving forces in the development of Banach algebra theory.

In this paper we will consider algebras of Fourier integral operators (FIOs) and their properties. Let us first fix our attention on pseudodifferential operators, which we may express in the Kohn-Nirenberg form

$$\sigma(x,D)f(x) = \int e^{2\pi i x\eta}\sigma(x,\eta)\hat{f}(\eta)\,d\eta\,. \tag{1}$$

The best known result about Wiener's property for pseudodifferential operators is maybe that in [2], see also the subsequent contributions of [5, 37]. It concerns symbols σ in the Hörmander's class $S^0_{0,0}(\mathbb{R}^{2d})$, i.e., smooth functions on $\mathbb{R}^{2d}$ such that, for every multi-index α and every $z \in \mathbb{R}^{2d}$,

$$|\partial_z^\alpha \sigma(z)| \leq C_\alpha. \tag{2}$$

The corresponding pseudodifferential operators form a subalgebra of $\mathcal{L}(L^2(\mathbb{R}^d))$, usually denoted by $L^0_{0,0}$. The standard symbolic calculus concerning the principal part of symbols of products does not hold, nevertheless Wiener's lemma is still valid. Namely, if $\sigma(x,D)$ is invertible in $\mathcal{L}(L^2(\mathbb{R}^d))$, then its inverse is again a pseudodifferential operator with symbol in $S^0_{0,0}(\mathbb{R}^{2d})$, hence belonging to $L^0_{0,0}$. In the absence of a symbolic calculus, such a version of Wiener's lemma seems to be the minimal property required of any reasonable algebra of pseudodifferential operators. A subalgebra $\mathcal{A}$ of operators in $\mathcal{L}(L^2(\mathbb{R}^d))$ that satisfies Wiener's lemma and is thus closed under inversion, is usually called *spectrally invariant* or inverse-closed or sometimes also a Wiener algebra. See [21] for a survey of the theory of spectral invariance.

2000 *Mathematics Subject Classification.* 35S30, 47G30, 42C15.

Key words and phrases. Fourier Integral operators, modulation spaces, short-time Fourier transform, Gabor frames, Wiener algebra.

K. G. was supported in part by the project P22746-N13 of the Austrian Science Fund (FWF).

From the point of view of time-frequency analysis and signal processing, which we are going to adopt in this paper, Wiener's lemma provides an important justification of the engineering practice to model $\sigma(x,D)^{-1}$ as an almost diagonal matrix (this is a peculiar property of pseudodifferential operators, see Theorem 1.2 in the sequel).

Actually, in the applications to signal processing, the symbol $\sigma(x,\eta)$ is not always smooth, and it is convenient to use some generalized version of $S^0_{0,0}(\mathbb{R}^{2d})$ [33]. Let us recall some results in this connection. To give a unified presentation, we use the modulation spaces $M^{p,q}_m$ introduced by Feichtinger, cf. [16, 18]. See Section 2.1 for the definition. We are particularly interested in the so-called Sjöstrand class

$$S_w = M^{\infty,1}(\mathbb{R}^{2d}) \tag{3}$$

and the related scale of spaces

$$S^s_w = M^{\infty,\infty}_{1\otimes v_s}(\mathbb{R}^{2d}), \quad v_s(z) = \langle z\rangle^s = (1+|z|^2)^{s/2},\ z\in\mathbb{R}^{2d}, \tag{4}$$

with the parameter $s\in[0,\infty)$. Defining $S^\infty_w = \bigcap_{s\geq 0} S^s_w$, we recover the Hörmander class $S^\infty_w = S^0_{0,0}(\mathbb{R}^{2d})$, whereas for $s\to 2d$ the symbols in S^s_w have a smaller regularity, until in the maximal space S_w even differentiability is lost.

Theorem 1.1 ([31, 23])**.** *The pseudodifferential operators with a symbol in S_w form a Wiener subalgebra of $\mathcal{L}(L^2(\mathbb{R}^d))$, the so-called Sjöstrand algebra. The symbol classes S^s_w with $s>2d$ provide a scale of Wiener subalgebras of the Sjöstrand algebra, and their intersection coincides with $L^0_{0,0}$.*

In this paper we construct and investigate Wiener subalgebras consisting of Fourier integral operators and generalize Sjöstrand's theory in [31, 32] to FIOs. We will consider FIOs of type I, that is

$$Tf(x) = T_{I,\Phi,\sigma}f(x) = \int_{\mathbb{R}^d} e^{2\pi i\Phi(x,\eta)}\sigma(x,\eta)\widehat{f}(\eta)\,d\eta\,, \tag{5}$$

where we first assume $\sigma\in S^0_{0,0}(\mathbb{R}^{2d})$. For the real-valued phase Φ we assume that $\partial^\alpha\Phi(z)\in S^0_{0,0}(\mathbb{R}^{2d})$ for $|\alpha|\geq 2$ and that a standard non-degeneracy condition is satisfied, cf. Section 2. When $\Phi(x,\eta)=x\eta$ we recapture the pseudodifferential operators in the Kohn-Nirenberg form. The L^2-adjoint of a FIO of type I is a FIO of type II

$$Tf(x) = T_{II,\Phi,\tau}f(x) = \int_{\mathbb{R}^{2d}} e^{-2\pi i[\Phi(y,\eta)-x\eta]}\tau(y,\eta)f(y)\,dy\,d\eta. \tag{6}$$

For the L^2-boundedness of such FIOs of types I and II see for example [1].

It is worth to observe that in contrast to the standard setting of Hörmander [27], we argue globally on $\mathbb{R}^d$, our basic examples being the propagators for Schrödinger-type equations. The second remark is that operators of the reduced form (5) or (6) do not form an algebra, quite in line with the calculus of [27]. The composition of FIOs requires heavier machinery and is addressed, for example, in [24] in the case when symbol and phase belong to the more restrictive Shubin class of [30].

As minimal objective of the present paper we want to present a cheap definition of a subalgebra of $\mathcal{L}(L^2(\mathbb{R}^d))$ containing FIOs of type I, II, and hence $L^0_{0,0}$, and prove the Wiener property for this class.

As a more ambitious objective we will extend our analysis to the case when σ in (5) belongs to the symbol class S^s_w, and we will define a corresponding scale of Wiener algebras of FIOs (we will not treat the full Sjöstrand algebra in this paper). The new algebras of FIOs will be constructed by means of Gabor frames and the decay properties of the corresponding Gabor matrix outside the graph of a symplectic map χ. For FIOs of type I such a decay was already pointed out in [12, 13] with applications to boundedness properties and numerical analysis in [14].

For the formulation of the results we now introduce the basic notions of time-frequency analysis and refer to Section 2 for details. The most suitable representation for our purpose is the short-time Fourier transform, where the localization on the time-frequency plane $\mathbb{R}^{2d}$ occurs on the unit scale both in time and in frequency. For a point $z = (x, \eta) \in \mathbb{R}^{2d}$ and a function f on $\mathbb{R}^d$, we denote the time-frequency shifts (or phase-space shifts) by

$$\pi(z) = M_\eta T_x f(t) = e^{2\pi i t\eta} f(t - x), \quad \text{where} \quad t\eta = t \cdot \eta = \sum_{i=1}^{d} t_i \eta_i \, .$$

The short-time Fourier transform (STFT) of a function/distribution f on $\mathbb{R}^d$ with respect to a Schwartz window function $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ is defined by

$$V_g f(x, \eta) = \langle f, M_\eta T_x g\rangle = \langle f, \pi(z)g\rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t - x)} e^{-2\pi i t\eta} \, dt, \tag{7}$$

for $z = (x, \eta) \in \mathbb{R}^{2d}$. We can now define the generalized Sjöstrand class S^s_w in (4) as the space of distributions $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ such that

$$|\langle \sigma, \pi(z, \zeta)g\rangle| \le C\langle\zeta\rangle^{-s}, \quad \forall z, \zeta \in \mathbb{R}^{2d} \tag{8}$$

for some constant $C > 0$, whereas σ is in the Sjöstrand class S_w if

$$\int_{\mathbb{R}^d} \sup_{z \in \mathbb{R}^d} |\langle \sigma, \pi(z, \zeta)g\rangle| \, d\zeta < \infty.$$

For the discrete description of function spaces and operators we use Gabor frames. Let $\Lambda = A\mathbb{Z}^{2d}$ with $A \in GL(2d, \mathbb{R})$ be a lattice of the time-frequency plane. The set of time-frequency shifts $\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g : \ \lambda \in \Lambda\}$ for a non-zero $g \in L^2(\mathbb{R}^d)$ is called a Gabor system. The set $\mathcal{G}(g, \Lambda)$ is a Gabor frame, if there exist constants $A, B > 0$ such that

$$A\|f\|_2^2 \le \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \le B\|f\|_2^2 \qquad \forall f \in L^2(\mathbb{R}^d). \tag{9}$$

Gabor frames allow us to discretize any continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ into an infinite matrix that captures the properties of the original operator.

For the case of pseudodifferential operators the Gabor discretization provides an equivalent characterization of the Sjöstrand algebra in Theorem 1.1.

Theorem 1.2 ([19, 23]). *Assume that $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$ with $g \in \mathcal{S}(\mathbb{R}^d)$ and fix $s > 2d$. Then the following statements are equivalent for a distribution $\sigma \in \mathcal{S}'(\mathbb{R}^d)$:*

(i) $\sigma \in S_w^s$.
(ii) *There exists* $C>0$ *such that*

$$|\langle \sigma(x,D)\pi(\lambda)g, \pi(\mu)g\rangle| \leq C\langle \mu-\lambda\rangle^{-s}, \qquad \forall \lambda,\mu\in\Lambda. \tag{10}$$

Hence, the assumption $\sigma \in S_w^\infty = S_{0,0}^0(\mathbb{R}^d)$ *is equivalent* (10) *being satisfied for all* $s \geq 0$.

Moreover $\sigma \in S_w$ *if and only if there exists a sequence* $h \in \ell^1(\Lambda)$, *such that* $|\langle \sigma(x,D)\pi(\lambda)g, \pi(\mu)g\rangle| \leq h(\lambda-\mu)$.

We refer to the matrix of an operator T with respect to a Gabor frame as the *Gabor matrix* of T. The above theorem gives a precise meaning to the statement that the Gabor matrix of a pseudodifferential operators is almost diagonal, or that pseudodifferential operators are almost diagonalized by Gabor frames.

We now describe our results about FIOs in Section 3.

Roughly speaking, Fourier integral operators can be defined as follows (Definition 3.2). Consider a bi-Lipschitz canonical transformation $\chi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ (see Definition 2.1) and $s > 2d$. Let $\mathcal{G}(g,\Lambda)$ be a Gabor frame for $L^2(\mathbb{R}^d)$ with $g \in \mathcal{S}(\mathbb{R}^d)$. We say that a continuous linear operator $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is in the class $FIO(\chi, s)$ if its Gabor matrix satisfies the decay condition

$$|\langle T\pi(\lambda)g, \pi(\mu)g\rangle| \leq C\langle \mu-\chi(\lambda)\rangle^{-s}, \qquad \forall \lambda,\mu\in\Lambda. \tag{11}$$

If $\chi = \mathrm{Id}$, the identity operator, then the corresponding Fourier integral operators are simply pseudodifferential operators.

The decomposition of a FIO with respect to a Gabor frame provides a technique to settle the following issues.

(i) *Boundedness of* T *on* $L^2(\mathbb{R}^d)$ (Theorem 3.4):
If $s > 2d$ and $T \in FIO(\chi, s)$, then T can be extended to a bounded operator on $L^2(\mathbb{R}^d)$.
(ii) *The algebra property* (Theorem 3.6): For $i = 1, 2$, $s > 2d$,

$$T^{(i)} \in FIO(\chi_i, s_i) \quad \Rightarrow \quad T^{(1)}T^{(2)} \in FIO(\chi_1 \circ \chi_2, s)\,.$$

(iii) *Wiener property* (Theorem 3.7): If $s > 2d$, $T \in FIO(\chi, s)$ and T is invertible on $L^2(\mathbb{R}^d)$, then $T^{-1} \in FIO(\chi^{-1}, s)$.

These three properties can be summarized neatly by saying that the union $\bigcup_\chi FIO(\chi, s)$ is a Wiener subalgebra of $\mathcal{L}(L^2(\mathbb{R}^d))$ consisting of FIOs.

In Section 4 we return to concrete FIOs of type I and II. Denoting by χ the symplectic transformation related to a phase Φ, we prove the expected extension of Theorem 1.1 to FIOs. Namely: A FIO T of type I as in (5) belongs to $FIO(\chi, s)$ for some $s > 2d$, if and only if its symbol σ belongs to S_w^s (Theorem 4.3). We further prove that the inverse in $\mathcal{L}(L^2(\mathbb{R}^d))$ of an operator of type I is an operator of type II, with symbol belonging to the same class S_w^s (Theorem 4.6). As an example, in Section 5 we treat a Wiener algebra of generalized metaplectic operators.

Although it is impossible to do justice to the vast literature on Fourier integral operators, let us mention some of the contributions that are most related to our ideas.

From the formal point of view, our approach is very similar to that in [3, 4] and [34], where $FIO(\chi, \infty)$ was treated. Instead of Gabor frames, in [3, 4] partitions of unity of the Weyl-Hörmander calculus are used, whereas in [34] the Bargmann transform is the main tool. The boundedness and composition of FIOs are treated in [6, 7, 9, 10, 15, 24].

The time-frequency analysis of pseudodifferential operators was propagated in [20, 22, 23, 28, 36]. Many aspects of Wiener's lemma and spectral invariance of operators are surveyed in [21].

Notation. We write $xy = x \cdot y$ for the scalar product on $\mathbb{R}^d$ and $|t|^2 = t \cdot t$ for $t, x, y \in \mathbb{R}^d$.

The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$. We use the brackets $\langle f, g\rangle$ to denote the extension to $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ of the inner product $\langle f, g\rangle = \int f(t)\overline{g(t)}dt$ on $L^2(\mathbb{R}^d)$. The Fourier transform is normalized to be $\hat{f}(\eta) = \mathcal{F}f(\eta) = \int f(t)e^{-2\pi i t\eta}dt$.

For $1 \leq p \leq \infty$ and a weight m, the space $\ell^p_m(\Lambda)$ is the Banach space of sequences $a = \{a_\lambda\}_{\lambda\in\Lambda}$ on a lattice Λ, such that

$$\|a\|_{\ell^p_m} := \left(\sum_{\lambda\in\Lambda} |a_\lambda|^p m(\lambda)^p\right)^{1/p} < \infty$$

(with obvious changes when $p = \infty$).

Throughout the paper, we shall use the notation $A \lesssim B$ to express the inequality $A \leq cB$ for a suitable constant $c > 0$, and $A \asymp B$ for the equivalence $c^{-1}B \leq A \leq cB$.

2. Preliminaries

2.1. Phase functions and canonical transformations.

Definition 2.1. *A real phase function Φ on $\mathbb{R}^{2d}$ is called* tame*, if the following three properties are satisfied:*

A1. $\Phi \in \mathcal{C}^\infty(\mathbb{R}^{2d})$*;*
A2. For $z = (x, \eta)$*,*

$$|\partial^\alpha_z \Phi(z)| \leq C_\alpha, \quad |\alpha| \geq 2; \tag{12}$$

A3. There exists $\delta > 0$ *such that*

$$|\det \partial^2_{x,\eta}\Phi(x, \eta)| \geq \delta. \tag{13}$$

If we set

$$\begin{cases} y = \nabla_\eta \Phi(x, \eta) \\ \xi = \nabla_x \Phi(x, \eta), \end{cases} \tag{14}$$

we can solve with respect to (x, ξ) by the global inverse function theorem (see e.g. [26]) and obtain a mapping χ defined by $(x, \xi) = \chi(y, \eta)$. The canonical transformation χ enjoys the following properties:

B1. $\chi:\mathbb{R}^{2d}\to\mathbb{R}^{2d}$ is smooth, invertible, and preserves the symplectic form in $\mathbb{R}^{2d}$, i.e., $dx\wedge d\xi = dy\wedge d\eta$; χ is a *symplectomorphism*.
B2. For $z=(y,\eta)$,

$$|\partial_z^\alpha \chi(z)|\leq C_\alpha, \quad |\alpha|\geq 1; \tag{15}$$

B3. There exists $\delta>0$ such that, for $(x,\xi)=\chi(y,\eta)$,

$$|\det \frac{\partial x}{\partial y}(y,\eta)|\geq \delta. \tag{16}$$

Conversely, to every transformation χ satisfying *B1*, *B2*, *B3* corresponds a tame phase Φ, uniquely determined up to a constant. This can be easily proved by (16), the global inverse function theorem [26] and using the pattern of [29, Theorem 4.3.2.] (written for the local case).

From now on we shall define by Φ_χ the phase function (up to constants) corresponding to the canonical transformation χ.

Observe that *B1* and *B2* imply that χ and χ^{-1} are globally Lipschitz. This property implies that

$$\langle w-\chi(z)\rangle \asymp \langle \chi^{-1}(w)-z\rangle \qquad w,z\in\mathbb{R}^{2d},$$

which we will use frequently. Moreover, if χ and $\tilde{\chi}$ are two transformation satisfying *B1* and *B2*, the same is true for $\chi\circ\tilde{\chi}$, whereas the additional property *B3* is not necessarily preserved, even if χ and $\tilde{\chi}$ are linear. This reflects the lack of the algebra property of the corresponding FIOs of type I; see Section 5 below.

2.2. **Time-frequency concepts.** We recall the basic concepts of time-frequency analysis and refer the reader to [18] for the full details. Consider a distribution $f\in\mathcal{S}'(\mathbb{R}^d)$ and a Schwartz function $g\in\mathcal{S}(\mathbb{R}^d)\setminus\{0\}$ (the so-called *window*). The short-time Fourier transform of f with respect to g was defined in (7) by $V_gf(z)=\langle f,\pi(z)g\rangle$. The short-time Fourier transform is well-defined whenever the bracket $\langle\cdot,\cdot\rangle$ makes sense for dual pairs of function or distribution spaces, in particular for $f\in\mathcal{S}'(\mathbb{R}^d)$ and $g\in\mathcal{S}(\mathbb{R}^d)$ or $f,g\in L^2(\mathbb{R}^d)$. We recall the covariance formula for the short-time Fourier transform that will be used in the sequel (Section 5):

$$V_g(M_\xi T_y f)(x,\eta)=e^{-2\pi i(\eta-\xi)y}(V_gf)(x-y,\eta-\xi), \qquad x,y,\omega,\xi\in\mathbb{R}^d. \tag{17}$$

The symbol spaces are provided by the modulation spaces. These were introduced by Feichtinger in the 80's (see the original paper [16]) and now are well-known in the framework of time-frequency analysis. For the fine-tuning of decay properties in the definition of modulation spaces we use weight functions of polynomial growth. For $s\geq 0$ we set $v(x,\eta)=v_s(x,\eta)=\langle(x,\eta)\rangle^s=(1+|x|^2+|\eta|^2)^{s/2}$ and denote by $\mathcal{M}_v(\mathbb{R}^{2d})$ the space of v-moderate weights on $\mathbb{R}^{2d}$; these are measurable functions $m>0$ satisfying $m(z+\zeta)\leq Cv(z)m(\zeta)$ for every $z,\zeta\in\mathbb{R}^d$. In particular, $v_s(z)^{-1}=\langle z\rangle^{-s}$ is v_s-moderate. The corresponding inequality $\langle z+w\rangle^{-s}\leq\langle z\rangle^{-s}\langle w\rangle^s$ is also called Peetre's inequality.

Let g be a non-zero Schwartz function. For $1\leq p,q\leq\infty$ and $m\in\mathcal{M}_v(\mathbb{R}^{2d})$ the modulation space $M_m^{p,q}(\mathbb{R}^d)$ is the space of distributions $f\in\mathcal{S}'(\mathbb{R}^d)$ such that their

STFTs belong to the space $L^{p,q}_m(\mathbb{R}^{2d})$ with norm

$$\|f\|_{M^{p,q}_m(\mathbb{R}^d)} := \|V_g f\|_{L^{p,q}_m(\mathbb{R}^{2d})} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_g f(x,\eta)|^p m(x,\eta)^p dx\right)^{\frac{q}{p}} d\eta\right)^{\frac{1}{q}}.$$

This definition does not depend on the choice of the window $g \in \mathcal{S}(\mathbb{R}^d), g \neq 0$, and different windows yield equivalent norms on $M^{p,q}_m$ [18, Thm. 11.3.7]. Moreover, the space of admissible windows can be enlarged to $M^1_v(\mathbb{R}^d)$. The symbol spaces we shall be mainly concerned with are $S^s_w = M^{\infty,\infty}_{1\otimes v_s}(\mathbb{R}^{2d})$ with the norm

$$\|\sigma\|_{S^s_w} = \sup_{z\in\mathbb{R}^{2d}}\sup_{\zeta\in\mathbb{R}^{2d}} |V_\Psi \sigma(z,\zeta)|\,\langle\zeta\rangle^s,$$

where $\Psi \in \mathcal{S}(\mathbb{R}^{2d}) \setminus \{0\}$.

The Hörmander symbol class $S^0_{0,0}(\mathbb{R}^{2d})$ can be characterized by means of modulation spaces as follows, see for example [23]:

$$S^0_{0,0} = \bigcap_{s\geq 0} S^s_w.$$

2.2.1. *Gabor frames.* Fix a function $g \in L^2(\mathbb{R}^d)$ and a lattice $\Lambda = A\mathbb{Z}^d$, for $A \in GL(2d,\mathbb{R})$. The Gabor system $\mathcal{G}(g,\Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$ is a Gabor frame if there exist constants $A, B > 0$ such that (9) is satisfied. We define the coefficient operator C_g, which maps functions to sequences as follows:

$$(C_g f)_\lambda := \langle f, \pi(\lambda)g\rangle, \quad \lambda \in \Lambda, \tag{18}$$

the synthesis operator

$$D_g c := \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)g, \quad c = \{c_\lambda\}_{\lambda\in\Lambda} \tag{19}$$

and the Gabor frame operator

$$S_g f := D_g C_g f = \sum_{\lambda\in\Lambda}\langle f, \pi(\lambda)g\rangle \pi(\lambda)g. \tag{20}$$

Equivalently, the set $\mathcal{G}(g,\Lambda)$ is called a Gabor frame for the Hilbert space $L^2(\mathbb{R}^d)$, if S_g is a bounded and invertible operator on $L^2(\mathbb{R}^d)$. If $\mathcal{G}(g,\Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$, then the so-called *dual window* $\gamma = S_g^{-1}g$ is well-defined and the set $\mathcal{G}(\gamma,\Lambda)$ is a frame (the so-called canonical dual frame of $\mathcal{G}(g,\Lambda)$). Every $f \in L^2(\mathbb{R}^d)$ possesses the frame expansion

$$f = \sum_{\lambda\in\Lambda}\langle f, \pi(\lambda)g\rangle\pi(\lambda)\gamma = \sum_{\lambda\in\Lambda}\langle f, \pi(\lambda)\gamma\rangle\pi(\lambda)g \tag{21}$$

with unconditional convergence in $L^2(\mathbb{R}^d)$, and norm equivalence

$$\|f\|_{L^2} \asymp \|C_g f\|_{\ell^2} \asymp \|C_\gamma f\|_{\ell^2}.$$

Gabor frames give the following characterization of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and of the modulation spaces $M^p_m(\mathbb{R}^d)$, $1 \le p \le \infty$. If $g \in \mathcal{S}(\mathbb{R}^d)$ and $\mathcal{G}(g, \Lambda)$ is a frame, then

$$f \in \mathcal{S}(\mathbb{R}^d) \Leftrightarrow \sup_{\lambda \in \Lambda} \langle \lambda \rangle^N |\langle f, \pi(\lambda) g \rangle| < \infty \quad \forall N \in \mathbb{N}, \tag{22}$$

$$f \in M^p_m(\mathbb{R}^d) \Leftrightarrow \Big(\sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda) g \rangle|^p m(\lambda)^p \Big)^{1/p} < \infty, \tag{23}$$

These results are contained in [18, Ch. 13]. In particular, if $\gamma = g$, then the frame is called *Parseval* frame and the expansion (21) reduces to

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \pi(\lambda) g. \tag{24}$$

We may take the existence of Parseval frames with $g \in \mathcal{S}(\mathbb{R}^d)$ for granted. From now on we work with Parseval frames, so that we will not have to deal with the dual window γ. Let us underline that the properties of FIOs written for Parseval frames work exactly the same with general Gabor frames with dual windows γ different from g.

3. A Wiener Algebra of Fourier Integral Operators

We first present an equivalence between continuous decay conditions and the decay of the discrete Gabor matrix for a linear operator $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$.

Theorem 3.1. *Let T be a continuous linear operator $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ and χ a canonical transformation which satisfies B1 and B2 of Definition 2.1. Let $\mathcal{G}(g, \Lambda)$ be a Parseval frame with $g \in \mathcal{S}(\mathbb{R}^d)$ and $s \ge 0$. Then the following properties are equivalent.*

(i) *There exists $C > 0$ such that*

$$|\langle T\pi(z) g, \pi(w) g \rangle| \le C \langle w - \chi(z) \rangle^{-s}, \qquad \forall z, w \in \mathbb{R}^{2d}. \tag{25}$$

(ii) *There exists $C > 0$ such that*

$$|\langle T\pi(\lambda) g, \pi(\mu) g \rangle| \le C \langle \mu - \chi(\lambda) \rangle^{-s}, \qquad \forall \lambda, \mu \in \Lambda. \tag{26}$$

Proof. The implication (i) $\Longrightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (i). The argument is borrowed from the proof of a similar result for pseudodifferential operators in [19, Theorem 3.2]. Let C be a relatively compact fundamental domain of the lattice Λ. Given $z, w \in \mathbb{R}^{2d}$, we can write $w = \lambda + u$, $z = \mu + u'$ for unique $\lambda, \mu \in \Lambda$, $u, u' \in C$, and

$$|\langle T\pi(z) g, \pi(w) g \rangle| = |\langle T\pi(\mu)\pi(u') g, \pi(\lambda)\pi(u) g \rangle|.$$

Now we expand $\pi(u) g = \sum_{\nu \in \Lambda} \langle \pi(u) g, \pi(\nu) g \rangle \pi(\nu) g$, and likewise $\pi(u') g$. Since $V_g g \in \mathcal{S}(\mathbb{R}^{2d})$, the coefficients in this expansion satisfy

$$\sup_{u \in C} |\langle \pi(u) g, \pi(\nu) g \rangle| = \sup_{u \in C} |V_g g(\nu - u)| \lesssim \sup_{u \in C} \langle \nu - u \rangle^{-N} \lesssim (\sup_{u \in C} \langle u \rangle^N) \langle \nu \rangle^{-N} \lesssim \langle \nu \rangle^{-N}$$

for every N. Using (26), we now obtain that

$$|\langle T\pi(z)g, \pi(w)g\rangle| \leq \sum_{\nu,\nu'\in\Lambda} |\langle T\pi(\mu+\nu')g, \pi(\lambda+\nu)g\rangle||\langle\pi(u')g, \pi(\nu')g\rangle||\langle\pi(u)g, \pi(\nu)g\rangle|$$
$$\lesssim \sum_{\nu,\nu'\in\Lambda} \langle\lambda+\nu-\chi(\mu+\nu')\rangle^{-s}\langle\nu'\rangle^{-N}\langle\nu\rangle^{-N}.$$

Since $v_s(z)^{-1} = \langle z\rangle^{-s}$ is v_s-moderate, we majorize the main term of the sum as

$$\begin{aligned}\langle\lambda+\nu-\chi(\mu+\nu')\rangle^{-s} &= \langle\lambda-\chi(\mu)+\nu-\chi(\mu+\nu')+\chi(\mu)\rangle^{-s}\\ &\leq \langle\lambda-\chi(\mu)\rangle^{-s}\langle\chi(\mu+\nu')-\chi(\mu)-\nu\rangle^{s} \lesssim \langle\lambda-\chi(\mu)\rangle^{-s}\langle\nu'\rangle^{s}\langle\nu\rangle^{s},\end{aligned}$$

since $\chi(\mu+\nu')-\chi(\mu) = \mathcal{O}(\nu')$ by the Lipschitz property of χ.

Hence,

$$|\langle T\pi(z)g, \pi(w)g\rangle| \lesssim \langle\lambda-\chi(\mu)\rangle^{-s}\sum_{\nu,\nu'\in\Lambda}\langle\nu'\rangle^{s-N}\langle\nu\rangle^{s-N} \lesssim \langle\lambda-\chi(\mu)\rangle^{-s},$$

for $N \in \mathbb{N}$ large enough.

Finally, with $\lambda = w-u$, $\mu = z-u'$, we apply the above estimate again and obtain

$$\begin{aligned}\langle\lambda-\chi(\mu)\rangle^{-s} &= \langle w-u-\chi(z-u')-\chi(z)+\chi(z)\rangle^{-s}\\ &\leq \langle w-\chi(z)\rangle^{-s}\langle u+\chi(z-u')-\chi(z)\rangle^{s}\\ &\lesssim \langle w-\chi(z)\rangle^{-s}\langle u\rangle^{s}\langle u'\rangle^{s} \lesssim \langle w-\chi(z)\rangle^{-s},\end{aligned}$$

since $\sup_{u\in C}\langle u\rangle^s < \infty$. Thus we have proved that $|\langle T\pi(z)g, \pi(w)g\rangle| \lesssim \langle w-\chi(z)\rangle^{-s}$. □

Inspired by the characterization of Theorem 3.1, we now define a class of FIOs associated to a canonical transformation χ. In view of the equivalence (25) and (26), we focus on the decay of the *discrete* Gabor matrix.

Definition 3.2. *Let χ be a transformation satisfying B1 and B2, and $s \geq 0$. Fix $g \in \mathcal{S}(\mathbb{R}^d)\setminus\{0\}$ and let $\mathcal{G}(g,\Lambda)$ be a Parseval frame for $L^2(\mathbb{R}^d)$. We say that a continuous linear operator $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is in the class $FIO(\chi, s)$, if its Gabor matrix satisfies the decay condition*

$$|\langle T\pi(\lambda)g, \pi(\mu)g\rangle| \leq C\langle\mu-\chi(\lambda)\rangle^{-s}, \qquad \forall\lambda,\mu\in\Lambda. \tag{27}$$

The class $FIO(\Xi, s) = \bigcup_\chi FIO(\chi, s)$ is the union of these classes where χ runs over the set of all transformations satisfying $B1, B2$.

Note that we do not require the assumption *B3*.

We first observe that this definition does not depend on the choice of the Gabor frame.

Lemma 3.3. *The definition of $FIO(\chi, s)$ is independent of the Gabor frame $\mathcal{G}(g,\Lambda)$.*

Proof. Let $\mathcal{G}(\varphi,\Lambda')$ be a Gabor frame with a window $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and a possibly different lattice Λ'. As in the proof of Theorem 3.1 we expand $\pi(\lambda)\varphi = \sum_{\nu\in\Lambda}\langle\pi(\lambda)\varphi, \pi(\nu)g\rangle\pi(\nu)g$ with convergence in $\mathcal{S}(\mathbb{R}^d)$ and likewise $\pi(\mu)\varphi$, where $\lambda,\mu \in \Lambda'$. Consequently $T\pi(\lambda)\varphi =$

$\sum_{\nu\in\Lambda}\langle\pi(\lambda)\varphi,\pi(\nu)g\rangle T\pi(\nu)g$ converges weak* in $\mathcal{S}'(\mathbb{R}^d)$ and the following identity is well-defined:

$$\begin{aligned}\langle T\pi(\lambda)\varphi,\pi(\mu)\varphi\rangle &= \sum_{\nu\in\Lambda}\langle\pi(\lambda)\varphi,\pi(\nu)g\rangle\langle T\pi(\nu)g,\pi(\mu)\varphi\rangle\\ &= \sum_{\nu\in\Lambda}\sum_{\nu'\in\Lambda}\langle\pi(\lambda)\varphi,\pi(\nu)g\rangle\langle T\pi(\nu)g,\pi(\nu')g\rangle\langle\pi(\mu)\varphi,\pi(\nu')g\rangle\end{aligned}$$

Since $\varphi, g\in\mathcal{S}(\mathbb{R}^d)$, the characterization (22) and the covariance property (17) imply that $|\langle\pi(\lambda)\varphi,\pi(\nu)g\rangle| = |\langle\varphi,\pi(\nu-\lambda)g\rangle| \lesssim \langle\nu-\lambda\rangle^{-N}$ for $\nu\in\Lambda,\lambda\in\Lambda'$ and every $N\geq 0$. After substituting these estimates and choosing N large enough, we obtain the majorization

$$\begin{aligned}|\langle T\pi(\lambda)\varphi,\pi(\mu)\varphi\rangle| &\lesssim \sum_{\nu\in\Lambda}\sum_{\nu'\in\Lambda}\langle\nu-\lambda\rangle^{-N}\langle\nu'-\chi(\nu)\rangle^{-s}\langle\nu'-\mu\rangle^{-N}\\ &\lesssim \sum_{\nu\in\Lambda}\langle\nu-\lambda\rangle^{-N}\langle\mu-\chi(\nu)\rangle^{-s} \asymp \sum_{\nu\in\Lambda}\langle\nu-\lambda\rangle^{-N}\langle\chi^{-1}(\mu)-\nu\rangle^{-s}\\ &\lesssim \langle\chi^{-1}(\mu)-\lambda\rangle^{-s}\asymp\langle\mu-\chi(\lambda)\rangle^{-s}.\end{aligned}$$

□

As in [23], the definition of classes of operators by their Gabor matrices facilitates the investigation of their basic properties. In line with Sjöstrand's original program we next derive the boundedness, the composition rules, and properties of the inverse operator in the classes $FIO(\chi,s)$.

For $s>2d$ the class $FIO(\chi,s)$ possesses many desired properties.

Theorem 3.4. *Let $s>2d$ and $T\in FIO(\chi,s)$. Then T extends to a bounded operator on $M^p(\mathbb{R}^d)$, $1\leq p\leq\infty$, and in particular on $L^2(\mathbb{R}^d)=M^2(\mathbb{R}^d)$.*

Proof. Let $\mathcal{G}(g,\Lambda)$ be a Parseval frame with $g\in\mathcal{S}(\mathbb{R}^d)$. Since the frame operator $S_g=D_gC_g$ is the identity operator, we can write T as $T=D_gC_gTD_gC_g$, where D_g and C_g are the synthesis and coefficient operators of (18) and (19). Since $\mathcal{G}(g,\Lambda)$ is a frame and $g\in\mathcal{S}(\mathbb{R}^d)$, C_g is bounded from $M^p(\mathbb{R}^d)$ to $\ell^p(\Lambda)$ and $D_g=C_g^*$ is bounded from $\ell^p(\Lambda)$ to $M^p(\mathbb{R}^d)$. (For $p=2$ this is contained in the definition of a frame, for $p\neq 2$ this is slightly less obvious and stated in [18, Ch. 12.2].)

The operator C_gTD_g maps sequences to sequences, and its matrix K is precisely the Gabor matrix of T, namely, $K_{\mu,\lambda}=\langle T\pi(\lambda)g,\pi(\mu)g\rangle$. Since by assumption $|\langle T\pi(\lambda)g,\pi(\mu)g\rangle|\lesssim\langle\mu-\chi(\lambda)\rangle^{-s}$ and $s>2d$, Schur's test implies that the matrix K representing C_gTD_g is bounded on $\ell^p(\Lambda)$. Consequently T is bounded on $M^p(\mathbb{R}^d)$ for $1\leq p\leq\infty$. □

Remark 3.5. *As it is clear from the preceeding proof, the boundedness of the operator $T\in FIO(\chi,s)$, $s>2d$, fails in general on the modulation spaces $M^{p,q}(\mathbb{R}^{2d})$, with $p\neq q$. Indeed, in this case the change of variables $u=\chi(z)$ is not allowed. A concrete counter-example is the FIO of type I in* (35) *below.*

Next, we show that the class $FIO(\Xi,s)$ for $s>2d$ is an algebra.

Theorem 3.6. *If $T^{(i)} \in FIO(\chi_i, s_i)$ with $s_i > 2d$, $i = 1, 2$; then the composition $T^{(1)}T^{(2)}$ is in $FIO(\chi_1 \circ \chi_2, s)$ with $s = \min(s_1, s_2)$. Consequently the class $FIO(\Xi, s) = \bigcup_\chi FIO(\chi, s)$ is an algebra with respect to the composition of operators.*

Proof. We write the product $T^{(1)}T^{(2)}$ as

$$T^{(1)}T^{(2)} = D_g C_g T^{(1)} T^{(2)} D_g C_g = D_g (C_g T^{(1)} D_g)(C_g T^{(2)} D_g) C_g \,.$$

Then $C_g T^{(1)} T^{(2)} D_g$ is the Gabor matrix of $T^{(1)}T^{(2)}$ with entries $K_{\mu,\lambda} = \langle T^{(1)}T^{(2)}\pi(\lambda)g, \pi(\mu)g\rangle$ and $C_g T^{(i)} D_g$, $i = 1, 2$, is the Gabor matrix of $T^{(i)}$ with entries $K^{(i)}_{\mu,\lambda} = \langle T^{(i)}\pi(\lambda)g, \pi(\mu)g\rangle$. Thus the composition of operators corresponds to the multiplication of their Gabor matrices. Using the decay estimates for $K^{(i)}_{\mu,\lambda}$ and $s = \min(s_1, s_2)$, we estimate the size of the Gabor matrix of $T^{(1)}T^{(2)}$ as follows:

$$\begin{aligned} |K_{\mu,\lambda}| &= \sum_{\nu\in\Lambda} K^{(1)}_{\mu,\nu} K^{(2)}_{\nu,\lambda} \lesssim \sum_{\nu\in\Lambda} \langle \mu - \chi_1(\nu)\rangle^{-s_1} \langle \nu - \chi_2(\lambda)\rangle^{-s_2} \\ &\lesssim \sum_{\nu\in\Lambda} \langle \chi_1^{-1}(\mu) - \nu\rangle^{-s} \langle \nu - \chi_2(\lambda)\rangle^{-s} \,. \end{aligned}$$

Since $v_s^{-1} = \langle \nu\rangle^{-s} \in \ell^1(\Lambda)$ is subconvolutive for $s > 2d$, i.e., $v_s^{-1} * v_s^{-1} \le C v_s^{-1}$ [18, Lemma 11.1.1 (d)], the last expression is dominated by $\langle \chi_1^{-1}(\mu) - \chi_2(\lambda)\rangle^{-s} \asymp \langle \mu - \chi_1(\chi_2(\lambda))\rangle^{-s}$.

By Proposition 3.4, $T^{(1)}$ and $T^{(2)}$ extend to bounded operators on $L^2(\mathbb{R}^d)$, so that the product $T^{(1)}T^{(2)}$ is well-defined and bounded on $L^2(\mathbb{R}^d)$. □

Finally, we consider the invertibility in the class $FIO(\chi, s)$ and show that $FIO(\Xi, s)$ is inverse-closed in $\mathcal{B}(L^2(\mathbb{R}^d))$ for $s > 2d$.

Theorem 3.7. *Let $T \in FIO(\chi, s)$ with $s > 2d$. If T is invertible on $L^2(\mathbb{R}^d)$, then $T^{-1} \in FIO(\chi^{-1}, s)$. Consequently, the algebra $FIO(\Xi, s)$ is inverse-closed in $\mathcal{L}(L^2(\mathbb{R}^d))$.*

Proof. We first show that the adjoint operator T^* belongs to the class $FIO(\chi^{-1}, s)$. Indeed, since χ is bi-Lipschitz, we have

$$\begin{aligned} |\langle T^*\pi(\lambda)g, \pi(\mu)g\rangle| &= |\langle \pi(\lambda)g, T(\pi(\mu)g)\rangle| = |\langle T(\pi(\mu)g, \pi(\lambda)g)\rangle| \\ &\lesssim \langle \lambda - \chi(\mu)\rangle^{-s} \asymp \langle\, \chi^{-1}(\lambda) - \mu\rangle^{-s}. \end{aligned}$$

Hence, by Theorem 3.6, the operator $P := T^*T$ is in $FIO(\mathrm{Id}, s)$ and satisfies the estimate $|\langle P\pi(\lambda)g, \pi(\mu)g\rangle| \lesssim \langle \lambda - \mu\rangle^{-s}, \forall \lambda, \mu \in \Lambda$.

We now exploit the characterization for pseudodifferential operators contained in Theorem 1.2 and deduce that P is a pseudodifferential operator with a symbol in S^s_w. Since T and therefore T^* are invertible on $L^2(\mathbb{R}^d)$, P is also invertible on $L^2(\mathbb{R}^d)$. Now we apply Theorem 1.1 and conclude that the inverse P^{-1} is again a pseudodifferential operator with a symbol in S^s_w. Hence P^{-1} is in $FIO(\mathrm{Id}, s)$. Finally, using the algebra property of Theorem 3.6 once more, we obtain that $T^{-1} = P^{-1}T^*$ is in $FIO(\chi^{-1}, s)$ and thus satisfies the estimate $|\langle T^{-1}\pi(\lambda)g, \pi(\mu)g\rangle| \lesssim \langle \chi^{-1}(\lambda) - \mu\rangle^{-s}, \forall \lambda, \mu \in \Lambda$. □

Combining Theorems 3.4, 3.6, and 3.7, we see that $FIO(\Xi, s)$ with $s > 2d$ is a Wiener subalgebra of $\mathcal{L}(L^2(\mathbb{R}^d))$ consisting of FIOs.

4. FIOs of type I

The abstract class of Fourier integral operators $FIO(\chi, s)$ was defined by decay properties of the Gabor matrix. Our next step is to relate the Gabor matrix of an operator to the phase and symbol of concrete FIOs. This step is more technical and resumes our investigations in [12, 13, 14].

By the Schwartz' Kernel Theorem every continuous linear operator $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ can be written as a FIO of type I with a given phase $\Phi(x, \eta)$ for some symbol $\sigma(x, \eta)$ in $\mathcal{S}'(\mathbb{R}^{2d})$. Hence, if T is a continuous linear operator $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ and χ satisfies *B1*, *B2*, *and B3*, then $T = T_{I,\Phi_\chi,\sigma}$ is a FIO of type I with symbol σ and phase Φ_χ.

As a first step we formulate the following result.

Proposition 4.1. *Let $T = T_{I,\Phi,\sigma}$ be a FIO of type I with symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ and a phase Φ satisfying A1 and A2. If the Gabor matrix of T satisfies*

$$|\langle T\pi(x,\eta)g, \pi(x',\eta')g\rangle| \le C\langle \nabla_x\Phi(x',\eta) - \eta', \nabla_\eta\Phi(x',\eta) - x\rangle^{-s} \qquad x, x', \eta, \eta' \in \mathbb{R}^d, \tag{28}$$

for some $C > 0$ and $s \ge 0$, then σ is in the generalized Sjöstrand class $S^s_w(\mathbb{R}^{2d})$.

In particular, if $s > 2d$, then we have $\sigma \in S_w(\mathbb{R}^{2d})$.

Proof. The proof uses techniques from [12]. To set up notation, let $\Phi_{2,z}$ be the remainder in the second order Taylor expansion of the phase Φ, i.e.,

$$\Phi_{2,z}(w) = 2\sum_{|\alpha|=2}\int_0^1 (1-t)\partial^\alpha\Phi(z+tw)dt\frac{w^\alpha}{\alpha!} \qquad z, w \in \mathbb{R}^{2d}, \tag{29}$$

and set

$$\Psi_z(w) = e^{2\pi i\Phi_{2,z}(w)}\overline{g} \otimes \widehat{g}(w). \tag{30}$$

We recall the fundamental relation between the Gabor matrix of a FIO and the STFT of its symbol from [11, Prop. 3.2] and [13, Section 6]: for $g \in \mathcal{S}(\mathbb{R}^d)$ we have

$$|\langle T\pi(x,\eta)g, \pi(x',\eta')g\rangle| = |V_{\Psi_{(x',\eta)}}\sigma((x',\eta), (\eta' - \nabla_x\Phi(x',\eta), x - \nabla_\eta\Phi(x',\eta)))|\,.$$

Writing $u = (x', \eta)$, $v = (\eta', x)$, (28) translates into

$$|V_{\Psi_u}\sigma(u, v - \nabla\Phi(u))| \le C\langle v - \nabla\Phi(u)\rangle^{-s},$$

and then into the estimate

$$\sup_{(u,w)\in\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \langle w\rangle^s |V_{\Psi_u}\sigma(u,w)| < \infty. \tag{31}$$

Now, setting $G = \overline{g} \otimes \widehat{g} \in \mathcal{S}(\mathbb{R}^{2d})$, we can write

$$\begin{aligned} V_{G^2}\sigma(u,v) &= \int e^{-2\pi i t v}\sigma(t)\overline{G^2(t-u)}\,dt \\ &= \int e^{-2\pi i t v}\sigma(t)e^{-2\pi i \Phi_{2,u}(t-u)}\overline{G(t-u)}e^{2\pi i \Phi_{2,u}(t-u)}\overline{G(t-u)}\,dt \\ &= \int e^{-2\pi i t v}\sigma(t)\overline{\Psi_u(t-u)}e^{2\pi i \Phi_{2,u}(t-u)}\overline{G(t-u)}\,dt \\ &= \mathcal{F}(\sigma T_u\overline{\Psi_u}) *_v \mathcal{F}(T_u(e^{2\pi i\Phi_{2,u}}\overline{G}))(v) \\ &= V_{\Psi_u}\sigma(u,\cdot) * \mathcal{F}(T_u(e^{2\pi i\Phi_{2,u}}\overline{G}))(v). \end{aligned} \tag{32}$$

Using (32) and the weighted Young inequality $L^\infty_s(\mathbb{R}^{2d}) * L^1_s(\mathbb{R}^{2d}) \hookrightarrow L^\infty_s(\mathbb{R}^{2d})$ we get

$$\|\sigma\|_{S^s_w} \asymp \sup_u \|V_{G^2}\sigma(u,\cdot)\|_{L^\infty_s} \lesssim \sup_u \|V_{\Psi_u}\sigma(u,\cdot)\|_{L^\infty_s} \sup_u \|\mathcal{F}\big(e^{2\pi i\Phi_{2,u}}\overline{G}\big)\|_{L^1_s}.$$

The first factor in the right-hand side is finite by (31). The second one is finite because the set $\{e^{2\pi i\Phi_{2,u}}\overline{G} : u \in \mathbb{R}^{2d}\}$ is bounded in $\mathcal{S}(\mathbb{R}^{2d})$, and the embedding $\mathcal{S} \hookrightarrow \mathcal{F}L^1_s$ is continuous. This gives $\sigma \in S^s_w$.

The last statement follows from the inclusion relations for modulation spaces in [16, Proposition 6.5], namely, $M^{\infty,\infty}_{1\otimes v_s}(\mathbb{R}^{2d}) \hookrightarrow M^{\infty,1}(\mathbb{R}^{2d})$ if and only if $s > 2d$. ∎

The next lemma clarifies further the relation between the phase Φ and the canonical transformation χ .

Lemma 4.2. *Consider a phase function Φ satisfying* A1, A2, *and* A3. *Then*

$$|\nabla_x\Phi(x',\eta)-\eta'|+|\nabla_\eta\Phi(x',\eta)-x| \asymp |\chi_1(x,\eta)-x'|+|\chi_2(x,\eta)-\eta'| \qquad \forall x,x',\eta,\eta' \in \mathbb{R}^d. \tag{33}$$

Proof. The estimate $\gtrsim$ was already proved in [13, Lemma 3.1]).

For the converse estimate observe that $x = \nabla_\eta\Phi(\chi_1(x,\eta),\eta)$ by definition of χ_1, hence

$$|\nabla_\eta\Phi(x',\eta)-x| = |\nabla_\eta\Phi(x',\eta) - \nabla_\eta\Phi(\chi_1(x,\eta),\eta)| \le C|x'-\chi_1(x,\eta)|, \tag{34}$$

because of assumption (A2) on Φ.

Since $\nabla_x\Phi(x',\eta) = \chi_2(\nabla_\eta\Phi(x',\eta),\eta))$, the first term on the left-hand side of (33) can be estimated as

$$|\nabla_x\Phi(x',\eta)-\eta'| \le |\chi_2(\nabla_\eta\Phi(x',\eta),\eta) - \chi_2(x,\eta)| + |\chi_2(x,\eta)-\eta'|\,.$$

Finally the Lipschitz continuity of χ and (34) imply that

$$|\nabla_x\Phi(x',\eta)-\eta'| \le |\eta'-\chi_2(x,\eta)| + C|x'-\chi_1(x,\eta)|.$$

∎

The following theorem identifies the abstract class $FIO(\chi,s)$ with a class of concrete Fourier integral operators and is perhaps the main result of this paper.

Theorem 4.3. *Fix $\mathcal{G}(g,\Lambda)$ be a Parseval frame with $g \in \mathcal{S}(\mathbb{R}^d)$ and let $s \ge 0$.*

Let T be a continuous linear operator $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ and χ a canonical transformation which satisfies B1, B2 and B3. Then the following properties are equivalent.

(i) $T = T_{I,\Phi_\chi,\sigma}$ *is a FIO of type I for some* $\sigma \in S^s_w$.
(ii) $F \in FIO(\chi, s)$.

Proof. The implication (i) $\Rightarrow$ (ii) was proved in [12, Theorem 3.3].

The implication (ii) $\Longrightarrow$ (i) follow immediately from Proposition 4.1 and Lemma 4.2, since $\langle (x,\eta') - \nabla\Phi(x,\eta)\rangle^{-s} \lesssim \langle (x',\eta') - \chi(x,\eta)\rangle^{-s}$. □

Corollary 4.4. *Under the same assumptions as in Theorem 4.3 the following statements are equivalent:*
(i) $T = T_{I,\Phi_\chi,\sigma}$ *is a FIO of type I for some* $\sigma \in S^0_{0,0}$.
(ii) $T \in FIO(\chi,\infty) = \bigcap_{s\geq 0} FIO(\chi, s)$.

Remark 4.5. *The corollary should be juxtaposed to Tataru's characterization of* $FIO(\chi,\infty)$ *in* [34, Theorem 4]. *Tataru assumes only conditions B1 and B2, but not B3 on the canonical transformation* χ *and therefore obtains a larger class of FIOs satisfying the decay condition* (25). *The new insight of Corollary 4.4 is that under the additional assumption B3* every *FIO admits the classical representation* (5). *As a byproduct we see that the integral operator in (14) of* [34] *possesses the classical representation, provided that* χ *also satisfies B3. This observation might be useful for the solution of the Cauchy problem in* [34, Section 5]*, where, for small time,* χ *is a small perturbation of the identity transformation and therefore certainly satisfies B3.*

Thanks to Theorem 3.4, the FIOs of type I with symbol in S^s_w, with $s > 2d$, are bounded on $M^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$ (with the obvious modification for $p = \infty$). Consider now the multiplication operator

$$Tf(x) = e^{\pi i|x|^2} f(x). \tag{35}$$

Then T is a FIO of type I having phase $\Phi(x,\eta) = |x|^2/2 + x\eta$ and symbol $\sigma(x,\eta) = 1$, for every $(x,\eta) \in \mathbb{R}^{2d}$. Observe that $\sigma = 1 \in S^0_{0,0}$, hence $\sigma \in S^s_w$, for every $s \geq 0$. However, the multiplication operator T is not bounded on $M^{p,q}$, when $p \neq q$, as proved in [13, Proposition 7.1].

We now look at the Wiener property of the class of FIOs of type I with symbol in S^s_w, with $s > 2d$. As we will see in Section 5, this class is not closed under composition, therefore the Wiener property must necessarily involve FIOs of type II (see (6)).

Theorem 4.6. *Let* T *be a FIO of type I with a tame phase* Φ *and a symbol* $\sigma \in S^s_w$, *with* $s > 2d$. *If* T *is invertible on* $L^2(\mathbb{R}^d)$, *then* T^{-1} *is a FIO of type II with same phase* Φ *and a symbol* $\tau \in S^s_w$.

Proof. Let χ be the canonical transformation associated to Φ. Then by Theorem 4.3 T belongs to $FIO(\chi, s)$. As in the proof of Theorem 3.7 we consider $P = T^*T$ and write

$$T^{-1} = P^{-1}T^* = (T(P^{-1})^*)^* = (TP^{-1})^*.$$

We have already shown that P is in $FIO(\mathrm{Id}, s)$ and a pseudodifferential operator with a symbol in S^s_w and that also $P^{-1} \in FIO(\mathrm{Id}, s)$ by the spectral invariance of pseudodifferential operators of Theorem 1.1. Now Theorem 3.6 implies that $TP^{-1} \in FIO(\chi, s)$ and Theorem 4.3 implies that TP^{-1} is a FIO of type I with tame phase Φ and a symbol $\rho \in S^s_w$. Since $T^{-1} = (TP^{-1})^*$, T^{-1} is a FIO of type II with phase Φ

and the symbol $\tau(x,\eta) = \overline{\rho(\eta,x)}$. An easy computation as in [11, Lemma 2.11]) shows that $\tau \in S^s_w$. $\square$

Although the proof of Theorem 4.6 is short, it combines the main insights of Sections 3 and 4 and uses the spectral invariance of pseudodifferential operators with symbols in S^s_w (Theorem 1.1 from [23]).

5. Generalized Metaplectic Operators

As an example, we will consider the class of $FIO(\chi,s)$ whose phase is a linear transformation $\chi(z) = \mathcal{A}z$ for some invertible matrix $\mathcal{A} \in GL(2d,\mathbb{R})$. Since χ must preserve the symplectic form (assumption B2), $\mathcal{A}$ must be a symplectic matrix, $\mathcal{A} \in Sp(d,\mathbb{R})$. Recall that the symplectic group is defined by

$$Sp(d,\mathbb{R}) = \left\{\mathcal{A} \in GL(2d,\mathbb{R}) :\, {}^t\mathcal{A}J\mathcal{A} = J\right\},$$

where

$$J = \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix}.$$

Definition 5.1. *Let $\mathcal{A} \in Sp(d,\mathbb{R})$ and $s \geq 0$. Fix a Parseval frame $\mathcal{G}(g,\Lambda)$ with $g \in \mathcal{S}(\mathbb{R}^d)$. We say that a continuous linear operator $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is a* generalized metaplectic operator*, in short, $T \in FIO(\mathcal{A},s)$, if its Gabor matrix satisfies the decay condition*

$$|\langle T\pi(\lambda)g, \pi(\mu)g\rangle| \leq C\langle \mu - \mathcal{A}\lambda\rangle^{-s}, \qquad \forall \lambda,\mu \in \Lambda.$$

The union $\bigcup_{\mathcal{A}\in Sp(d,\mathbb{R})} FIO(\mathcal{A},s)$ is called the class of generalized metaplectic operators and denoted by $FIO(Sp,s)$.

Since $Sp(d,\mathbb{R})$ is a group, Theorems 3.6 and 3.7 imply the following statement.

Theorem 5.2. *For $s > 2d$, $FIO(Sp,s)$ is a Wiener subalgebra of $FIO(\Xi,s)$.*

The main examples in $FIO(Sp,s)$ are the operators of the metaplectic representation of $Sp(d,\mathbb{R})$. Given $\mathcal{A} \in Sp(d,\mathbb{R})$, the metaplectic operator $\mu(\mathcal{A})$ is defined by the intertwining relation

$$\pi(\mathcal{A}z) = c_{\mathcal{A}}\,\mu(\mathcal{A})\pi(z)\mu(\mathcal{A})^{-1} \quad \forall z \in \mathbb{R}^d\,, \tag{36}$$

where $c_{\mathcal{A}} \in \mathbb{C}, |c_{\mathcal{A}}| = 1$ is a phase factor. The existence of the metaplectic operators is a consequence of the Stone-von Neumann theorem and the irreducibility of the (projective) representation of $\mathbb{R}^{2d}$ by the time-frequency shifts $\pi(z), z \in \mathbb{R}^{2d}$. The phase $c_{\mathcal{A}}$ can be chosen in such a way that μ lifts to a unitary representation of the double cover of the symplectic group (which we will assume in the sequel). For the group theoretical background and the construction of the metaplectic representation we refer to [17].

Let $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(d,\mathbb{R})$ with $d \times d$ blocks A,B,C,D. Then condition *B3* of Definition 2.1 is equivalent to $\det A \neq 0$. In this case, $\mu(\mathcal{A})$ is explicitly given by the FIO of type I

$$\mu(\mathcal{A})(x) = (\det A)^{-1/2}\int_{\mathbb{R}^d} e^{2\pi i\Phi(x,\eta)}\hat{f}(\eta)\,d\eta \tag{37}$$

with the phase Φ given by

$$\Phi(x,\eta)=\frac{1}{2}xCA^{-1}x+\eta A^{-1}x-\frac{1}{2}\eta A^{-1}B\eta\,. \tag{38}$$

(see [17, Theorem 4.51])

Even without the explicit form of $\mu(\mathcal{A})$, Definition 5.1 yields some interesting information about the metaplectic representation.

Proposition 5.3. *If $\mathcal{A}\in Sp(d,\mathbb{R})$, then $\mu(\mathcal{A})\in\bigcap_{s\geq 0}FIO(\mathcal{A},s)$.*

Proof. According to Theorem 3.1 it is enough to prove that $\mu(\mathcal{A})$ satisfies the continuous decay condition (25). Using the definition of $\mu(\mathcal{A})$ in (36) and the covariance property of the STFT (17), we write the Gabor matrix of $\mu(\mathcal{A})$ as

$$\begin{aligned}|\langle\mu(\mathcal{A})\pi(z)g,\pi(w)g\rangle| &= |\langle\pi(\mathcal{A}z)\mu(\mathcal{A})g,\pi(w)g\rangle|\\ &= |V_g(\pi(\mathcal{A}z)\mu(\mathcal{A})g)(w)|\\ &= |V_g\big(\mu(\mathcal{A})g\big)\big(w-\mathcal{A}z\big)|\,. \end{aligned}\tag{39}$$

Since both $g\in\mathcal{S}(\mathbb{R}^d)$ and $\mu(\mathcal{A})g\in\mathcal{S}(\mathbb{R}^d)$, we also have $V_g(\mu(\mathcal{A})g)\in\mathcal{S}(\mathbb{R}^{2d})$, e.g., by [18, Ch. 11.2.5]. This gives

$$|\langle\mu(\mathcal{A})\pi(z)g,\pi(w)g\rangle|\leq C\langle w-\mathcal{A}z\rangle^{-s}\,, \tag{40}$$

for every $s\geq 0$, as desired. □

The following theorem shows that every generalized metaplectic operator is a product of a metaplectic operator and a classical pseudodifferential operator.

Theorem 5.4. *Let $\mathcal{A}\in Sp(d,\mathbb{R})$ and $T\in FIO(\mathcal{A},s)$ with $s>2d$. Then there exist symbols $\sigma_1,\sigma_2\in S^s_w$ with the corresponding pseudodifferential operators $\sigma_1(x,D)$ and $\sigma_2(x,D)$, such that*

$$T=\sigma_1(x,D)\mu(\mathcal{A})\quad \textit{and}\quad T=\mu(\mathcal{A})\sigma_2(x,D). \tag{41}$$

Proof. We prove the factorization $T=\sigma_1(x,D)\mu(\mathcal{A})$, the other factorization is obtained analogously.

Since $\mu(\mathcal{A})^{-1}=\mu(\mathcal{A}^{-1})$ is in $FIO(\mathcal{A}^{-1},s)$ by Proposition 5.3, the algebra property of Theorem 3.6 implies that $T\mu(\mathcal{A}^{-1})\in FIO(Id,s)$. The fundamental characterization of pseudodifferential operators of Theorem 1.2 implies that existence of a symbol $\sigma_1\in S^s_w$, such that $T\mu(\mathcal{A})^{-1}=\sigma_1(x,D)$, which is what we wanted to show. □

Finally, we check the counterpart of FIO I and II for generalized metaplectic operators.

Let $\mathcal{A}=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in Sp(d,\mathbb{R})$ with $\det A\neq 0$. As proved in Theorem 4.3, every generalized metaplectic operator $T\in FIO(\mathcal{A},s)$ with $s>2d$ is a FIO of type I

$$Tf(x)=(\det\mathcal{A})^{-1/2}\int e^{2\pi i\Phi(x,\eta)}\sigma(x,\eta)\hat{f}(\eta)\,d\eta, \tag{42}$$

with a symbol $\sigma\in S^s_w$ and phase $\Phi(x,\eta)=\frac{1}{2}xCA^{-1}x+\eta A^{-1}x-\frac{1}{2}\eta A^{-1}B\eta$.

We obtain examples of FIOs of type II by taking adjoints. If $T\in FIO(\mathcal{A},s)$, then T^* is a FIO of type II.

As is well-known, the generalized metaplectic operators of type I defined in (42) do not enjoy the algebra property. Consider, for instance, the operators $T_1 = \mu(\mathcal{A}_1)$ and $T_2 = \mu(\mathcal{A}_2)$, with

$$\mathcal{A}_1 = \begin{pmatrix} I_d & I_d \\ 0 & I_d \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} I_d & I_d \\ -I_d & 0 \end{pmatrix}.$$

Then both T_1 and T_2 are FIOs of type I but their product

$$T_1 T_2 = \mu(\mathcal{A}_1)\mu(\mathcal{A}_2) = \mu(\mathcal{A}_1 \mathcal{A}_2) = \mu(-J) = \mathcal{F}$$

cannot be a FIO of type I. Indeed, the Fourier transform $\mathcal{F} = \mu(-J)$ is an example of a metaplectic operator that is a FIO of neither type I nor of type II. Note that in this case assumption *B3* is not satisfied for $\mathcal{J}$.

As in Remark 4.5 we see again that there is a crucial difference between FIOs satisfying all axioms *B1, B2,* and *B3* or only *B1* and *B2*.

Acknowledgment

We would like to thank J. M. Bony for several discussions on the subject of this paper.

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