Comparisons of Concordance in Additive Models

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Abstract

In this note we compare bivariate additive models with respect to their Pearson correlation coefficients, Kendall's τ concordance coefficients, and Blomqvist β medial correlation coefficients. The conditions that enable the comparisons involve variability stochastic orders such as the dispersive and the peakedness orders. Specifically we show that we can compare the Kendall's τ concordance coefficients of Cheriyan and Ramabhadran's bivariate gamma distributions, in spite of the fact that it is hard (and not necessary) to compute them.

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1 Introduction

Let Z_1 and Z_2 be two random variables that a probabilist uses to approximately describe some real world situation. It is often desired that Z_1 and Z_2, on one hand not be independent, and on the other hand not be totally dependent. A common way of doing this sort of modelling is to introduce three independent random variables, X_1, X_2, and Y, and then model Z_1 and Z_2 by

$$Z_1 = g(X_1, Y) \quad \text{and} \quad Z_2 = g(X_2, Y), \tag{1.1}$$

where g is some bivariate function. In the setup (1.1), X_1 and X_2 indicate the "individuality" that is associated with Z_1 and Z_2, whereas Y indicates the factors that give rise to the partial dependence between Z_1 and Z_2.

In the setup (1.1), it is sometimes of importance to figure out the influence of Y on the strength of positive dependence between Z_1 and Z_2. That is, suppose that the dependence between the two random variables in (1.1) can be chosen to be modelled using Y yielding (Z_1, Z_2) as in (1.1), or that it can be chosen to be modelled using $\widetilde{Y}$ yielding $(\widetilde{Z}_1, \widetilde{Z}_2)$ as follows

$$\widetilde{Z}_1 = g(X_1, \widetilde{Y}) \quad \text{and} \quad \widetilde{Z}_2 = g(X_2, \widetilde{Y}).$$

The question that arises then is what conditions on Y and $\widetilde{Y}$ imply that (Z_1, Z_2) is "less positively dependent" than $(\widetilde{Z}_1, \widetilde{Z}_2)$.

For example, Li and Pellerey (2011) considered, among other things, the comparison of

$$(Z_1, Z_2) = (\min\{X_1, Y\}, \min\{X_2, Y\})$$

and

$$(\widetilde{Z}_1, \widetilde{Z}_2) = (\min\{X_1, \widetilde{Y}\}, \min\{X_2, \widetilde{Y}\}).$$

They showed that if Y is larger than $\widetilde{Y}$ in the ordinary stochastic order, then $(\widetilde{Z}_1, \widetilde{Z}_2)$ is more positively dependent than (Z_1, Z_2) in the sense of the concordance order, that is, the copula of $(\widetilde{Z}_1, \widetilde{Z}_2)$ is greater than, or equal to, the copula of (Z_1, Z_2) over the whole unit square; see, for example, Nelsen (2006). Fang and Li (2011) considered the comparison of

$$(Z_1, Z_2) = (\max\{X_1, Y\}, \max\{X_2, Y\})$$

and

$$(\widetilde{Z}_1, \widetilde{Z}_2) = (\max\{X_1, \widetilde{Y}\}, \max\{X_2, \widetilde{Y}\}).$$

They showed that if Y is smaller than $\widetilde{Y}$ in the ordinary stochastic order, then $(\widetilde{Z}_1, \widetilde{Z}_2)$ is more positively dependent than (Z_1, Z_2) in the same sense that was described above.

The purpose of this note is to compare

$$(Z_1, Z_2) = (X_1 + Y, X_2 + Y) \tag{1.2}$$

and

$$(\widetilde{Z}_1, \widetilde{Z}_2) = (X_1 + \widetilde{Y}, X_2 + \widetilde{Y}) \tag{1.3}$$

in a sense of positive dependence. More explicitly, we find conditions on Y and $\widetilde{Y}$ that yield a stronger positive dependence between $\widetilde{Z}_1$ and $\widetilde{Z}_2$, than between Z_1 and Z_2.

Random vectors of the form (1.2) have been used in the literature to model a variety of applications. Here is a sample of such usages:

- *Reliability theory*: The random vector in (1.2) can represent a replacement model similar to a model in Marshall and Shaked (1982, page 263). Specifically, in a reliability system that performs two tasks, Y is the lifetime of the original device that performs both tasks, and upon its failure, it is replaced by two devices with lifetimes X_1 and X_2, each of which performs only one of the tasks. Then (Z_1, Z_2) is the vector of the time periods of the performance of the two tasks.

- *Risk analysis*: Bauerle and Muller (1998, page 66) studied models of pairs of n-dimensional risky portfolios. In the case when $n = 1$, their model for (dependent) risks, that belong to a certain group, is $(g(X_1, Y), g(X_2, Y))$, for some bivariate function g, where X_1 and X_2 are the individual risk factors, and Y is the group-specific risk factor. Specifically, when $g(x, y) = x + y$, the model of Bauerle and Muller (1998) reduces to (1.2).

- *Combat target detection*: Youngren (1991) considered modelling the detection of an enemy unit that has some target elements such as a tank or a truck. When the unit has two elements, Youngren (1991, page 574) modelled the times to the detection of the elements by (1.2), where the random quantity Y captures the contribution of the common environmental factors on the time for detection of both elements, and the random quantity X_i captures the contribution of the other factors to the time of detection of element i, $i = 1, 2$.

Remark 1.1. At the first glance it may not be clear what we may assume about Y and $\widetilde{Y}$ in (1.2) and (1.3) in order for $\widetilde{Z}_1$ and $\widetilde{Z}_2$ to be "more positively dependent" than Z_1 and Z_2. However, upon some reflection we may guess that if $\widetilde{Y}$ is "more variable" (or "more dispersed") than Y, then we may expect $\widetilde{Z}_1$ and $\widetilde{Z}_2$ to be "more positively dependent" than Z_1 and Z_2. The intuitive reason behind this is that the role of Y is to introduce the dependence between Z_1 and Z_2, and it does that by *adding* the same random quantity to both X_1 and X_2. Thus, the "more variable" Y is, the more it "forces" the sums $X_1 + Y$ and $X_2 + Y$ to vary, but to do it "together" and hence "be like each other", and as a result the "more dependent" Z_1 and Z_2 should be. Note that in the extreme case when Y is degenerate (that is, Y is "as small in variability as possible"), then Z_1 and Z_2 are independent. ◀

Verifying the intuition that is described in Remark 1.1, some of the results in this note are of the following form: If Y is smaller than $\widetilde{Y}$ in some variability sense, then (Z_1, Z_2) of (1.2) is smaller than $(\widetilde{Z}_1, \widetilde{Z}_2)$ of (1.3) with respect to some positive dependence sense.

Technically, we found the models in (1.2) and (1.3) to be quite complex for the purpose of comparing the copulas that are associated with (Z_1, Z_2) and with $(\widetilde{Z}_1, \widetilde{Z}_2)$. Thus our present study is more humble in the sense that we compare the Pearson product-moment correlation coefficients, the Kendall's τ concordance coefficients, and the Blomqvist's β medial correlation coefficients of (Z_1, Z_2) and $(\widetilde{Z}_1, \widetilde{Z}_2)$ in (1.2) and (1.3).

It is worthwhile to mention that a comparison of the strength of dependence in the sense of SI (stochastic increasingness) of models that are similar to the ones in (1.2) and (1.3), but still quite different than these, is given in Proposition 3.1 of Khaledi and Kochar (2005).

In the next section we obtain results that compare (Z_1, Z_2) and $(\widetilde{Z}_1, \widetilde{Z}_2)$ with respect to their Pearson product-moment correlation coefficients. These results are quite straightforward, but their importance is that they make up our first formalization of the intuition that is described in Remark 1.1. Our second formalization of that intuition is given in Section 3, where we develop comparisons of (Z_1, Z_2) and $(\widetilde{Z}_1, \widetilde{Z}_2)$ with respect to their Kendall's τ concordance coefficients. A third formulation of the above intuition is described in Section 4, where we compare (Z_1, Z_2) and $(\widetilde{Z}_1, \widetilde{Z}_2)$ with respect to their Blomqvist's β medial correlation coefficients. Further comments, including some remarks about the relationship between the dispersive variability order and our conditions for the comparisons of the various concordant coefficients, are given in Section 5.

In the sequel, 'increasing' stands for 'nondecreasing' and 'decreasing' stands for 'nonincreasing'. For every random variable X, and an event A, we denote by $[X\big|A]$ a random variable that is distributed according to the conditional distribution of X given A.

2 Comparisons via Pearson correlation coefficients

Let X_1, X_2, and Y be independent random variables. In this section we assume that the second moments of these three random variables are finite, and we denote their variances by $\sigma^2_{X_1}$, $\sigma^2_{X_2}$, and σ^2_Y. Define (Z_1, Z_2) by (1.2). A straightforward computation yields the Pearson product-moment correlation coefficient of Z_1 and Z_2 as

$$\mathrm{Corr}(Z_1, Z_2) = \frac{\sigma^2_Y}{\sqrt{\sigma^2_{X_1} + \sigma^2_Y} \cdot \sqrt{\sigma^2_{X_2} + \sigma^2_Y}}. \tag{2.1}$$

It is easy to verify that the partial derivative of (2.1) with respect to σ^2_Y is nonnegative, no matter what $\sigma^2_{X_1}$ and $\sigma^2_{X_2}$ are. That is, for fixed $\sigma^2_{X_1}$ and $\sigma^2_{X_2}$, the correlation coefficient $\mathrm{Corr}(Z_1, Z_2)$ is increasing in σ^2_Y. This observation yields the following result. In this result we assume not only that X_1, X_2, and Y have finite second moments, but also that $\widetilde{Y}$ has this property.

Proposition 2.1. *Let (Z_1, Z_2) and $(\widetilde{Z}_1, \widetilde{Z}_2)$ be as defined in* (1.2) *and* (1.3). *If*

$$\mathrm{Var}(Y) \le \mathrm{Var}(\widetilde{Y}) \tag{2.2}$$

then

$$\mathrm{Corr}(Z_1, Z_2) \le \mathrm{Corr}(\widetilde{Z}_1, \widetilde{Z}_2).$$

Note that if Y is smaller than $\widetilde{Y}$ with respect to any common variability order (some of these will be defined or mentioned in the sequel), then (2.2) holds. In this sense Proposition 2.1 is quite strong because the assumption (2.2) can be considered to be quite a weak variability order.

Noting that (2.1) is a decreasing function of $\sigma^2_{X_1}$ and of $\sigma^2_{X_2}$, Proposition 2.1 can be strengthened as follows.

Theorem 2.2. *Let X_1, X_2, Y, $\widetilde{X}_1$, $\widetilde{X}_2$, and $\widetilde{Y}$ be independent random variables. Define*

$$(Z_1, Z_2) = (X_1 + Y, X_2 + Y)$$

and

$$(\widetilde{Z}_1, \widetilde{Z}_2) = (\widetilde{X}_1 + \widetilde{Y}, \widetilde{X}_2 + \widetilde{Y}).$$

If

$$\mathrm{Var}(X_1) \geq \mathrm{Var}(\widetilde{X}_1),\ \mathrm{Var}(X_2) \geq \mathrm{Var}(\widetilde{X}_2),\ \textit{and}\ \mathrm{Var}(Y) \leq \mathrm{Var}(\widetilde{Y}), \tag{2.3}$$

then

$$\mathrm{Corr}(Z_1, Z_2) \leq \mathrm{Corr}(\widetilde{Z}_1, \widetilde{Z}_2).$$

3 Comparisons via Kendall's concordance coefficients

First we recall the definition of the Kendall's τ concordance coefficient; more details about this coefficient can be found, for instance, in Drouet Mari and Kotz (2001). Let (X, Y) be a random vector. In order to define the Kendall's $\tau(X, Y)$, consider also a copy (X', Y') that is independent of (X, Y); that is, $(X', Y') =_{\mathrm{st}} (X, Y)$. Then $\tau(X, Y)$ is defined as follows:

$$\tau(X, Y) = 2P\{(X - X')(Y - Y') \geq 0\} - 1.$$

We will use below the ordinary stochastic order that is defined as follows: Let X and Y be two random variables with respective distribution functions F and G, and respective survival functions $\overline{F} \equiv 1 - F$ and $\overline{G} \equiv 1 - G$. If $\overline{F}(x) \leq \overline{G}(x)$ for all $x \in \mathbb{R}$ then X is said to be smaller than Y in the ordinary stochastic order, and we denote is as $X \leq_{\mathrm{st}} Y$. A useful property of the ordinary stochastic order is the following: If $X \leq_{\mathrm{st}} Y$ then

$$E[\phi(X)] \leq E[\phi(Y)] \tag{3.1}$$

for all increasing functions for which the above expectations are well defined; see Müller and Stoyan (2002) or Shaked and Shanthikumar (2007) for further properties of the ordinary stochastic order.

Proceeding to the comparison of random vectors, we start with a special case of the setup of (1.2) and (1.3) where X_1 and X_2 are identically distributed.

Proposition 3.1. *Let X_1, X_2, Y and $\widetilde{Y}$ be independent random variables such that X_1 and X_2 are identically distributed. Let (Z_1, Z_2) and $(\widetilde{Z}_1, \widetilde{Z}_2)$ be as defined in* (1.2) *and* (1.3)*. Furthermore, let Y' be an independent copy of the Y in* (1.2)*, and let $\widetilde{Y}'$ be an independent copy of the $\widetilde{Y}$ in* (1.3)*. If*

$$|Y - Y'| \leq_{\mathrm{st}} |\widetilde{Y} - \widetilde{Y}'| \tag{3.2}$$

then

$$\tau(Z_1, Z_2) \leq \tau(\widetilde{Z}_1, \widetilde{Z}_2).$$

Proof. First let us derive a useful expression for

$$\tau(Z_1, Z_2) = \tau(X_1 + Y, X_2 + Y).$$

Let (X_1', X_2', Y') be a copy of (X_1, X_2, Y) [that is, $(X_1', X_2', Y') =_{\mathrm{st}} (X_1, X_2, Y)$] that is independent of (X_1, X_2, Y). Denote $W_1 = X_1 - X_1'$, $W_2 = X_2 - X_2'$, and $W_3 = Y - Y'$, and let F_i be the distribution of W_i, $i = 1, 2, 3$. Note that W_1, W_2, and W_3 are independent, and that F_i is a distribution that is symmetric about 0, $i = 1, 2, 3$. We have

$$\begin{aligned}\tau(X_1 + Y, X_2 + Y) &= 2P\big\{\big[(X_1 + Y) - (X_1' + Y')\big] \cdot \big[(X_2 + Y) - (X_2' + Y')\big] \geq 0\big\} - 1 \\ &= 2P\big\{(W_1 + W_3)(W_2 + W_3) \geq 0\big\} - 1.\end{aligned}$$

Now,

$$\begin{aligned}P\big\{(W_1 + W_3)(W_2 + W_3) \geq 0\big\} &= \int_{-\infty}^{\infty} P\big\{(W_1 + w)(W_2 + w) \geq 0\big\} dF_3(w) \\ &= \int_{-\infty}^{\infty} \big[P\{W_1 > -w\}P\{W_2 > -w\} + P\{W_1 \leq -w\}P\{W_2 \leq -w\}\big] dF_3(w) \\ &= \int_{-\infty}^{0} \Big[\overline{F}_1(-w)\overline{F}_2(-w) + F_1(-w)F_2(-w)\Big] dF_3(w) \\ &\quad + \int_{0}^{\infty} \Big[\overline{F}_1(-w)\overline{F}_2(-w) + F_1(-w)F_2(-w)\Big] dF_3(w).\end{aligned}$$

By changing $-w$ to w in the first integral, and using the symmetry property of F_3, F_1, and F_2, we get that

$$P\big\{(W_1 + W_3)(W_2 + W_3) \geq 0\big\} = 2\int_{0}^{\infty} \big[F_1(w)F_2(w) + \overline{F}_1(w)\overline{F}_2(w)\big] dF_3(w). \tag{3.3}$$

From the symmetry of $W_3 = Y - Y'$ we see that $|Y - Y'| =_{\mathrm{st}} [W_3 | W_3 > 0]$. Denote by $F_{[W_3|W_3>0]}$ the distribution of $[W_3 | W_3 > 0]$, and note that $F_{[W_3|W_3>0]}(w) = 2F_3(w) - 1$ for $w > 0$. Thus we have

$$\begin{aligned}\tau(Z_1, Z_2) &= 4\int_{0}^{\infty} \big[F_1(w)F_2(w) + \overline{F}_1(w)\overline{F}_2(w)\big] dF_3(w) - 1 \\ &= 4\int_{0}^{\infty} \big[F_1^2(w) + \overline{F}_1^2(w)\big] dF_3(w) - 1 \\ &= 2\int_{0}^{\infty} \big[F_1^2(w) + \overline{F}_1^2(w)\big] dF_{[W_3|W_3>0]}(w) - 1,\end{aligned} \tag{3.4}$$

where the second equality follows from the assumption that X_1 and X_2 are identically distributed. Letting $h(w) = F_1^2(w) + \overline{F}_1^2(w)$, $w > 0$, we can write

$$\tau(Z_1, Z_2) = 2E[h(W_3)\big|W_3 > 0] - 1. \tag{3.5}$$

Similarly, consider an independent copy $\widetilde{Y}'$ of $\widetilde{Y}$, that is independent of X_1, X_1', X_2, and X_2', and denote $\widetilde{W}_3 = \widetilde{Y} - \widetilde{Y}'$. Again, note that $|\widetilde{Y} - \widetilde{Y}'| =_{\text{st}} \left[\widetilde{W}_3|\widetilde{W}_3 > 0\right]$. As above we can write

$$\tau(\widetilde{Z}_1, \widetilde{Z}_2) = 2E\big[h(\widetilde{W}_3)\big|\widetilde{W}_3 > 0\big] - 1. \tag{3.6}$$

Next, express h as $h(w) = F_1^2(w)+\overline{F}_1^2(w) = g(F_1(w))$ for $w > 0$, where $g(p) = 1-2p+2p^2$. It is easy to see that $g(p)$ is increasing in $p \in [1/2, 1]$. Now, for $w > 0$, we have that $F_1(w) \geq \frac{1}{2}$, and F_1 is increasing. Hence h is increasing on $[0, \infty)$.

From the equalities $|Y - Y'| =_{\text{st}} [W_3|W_3 > 0]$, $|\widetilde{Y} - \widetilde{Y}'| =_{\text{st}} \left[\widetilde{W}_3|\widetilde{W}_3 > 0\right]$, and (3.2), it follows that $[W_3|W_3 > 0] \leq_{\text{st}} \left[\widetilde{W}_3|\widetilde{W}_3 > 0\right]$. Thus the increasingness of h, the expressions (3.5) and (3.6), and the inequality (3.1), yield $\tau(Z_1, Z_2) \leq \tau(\widetilde{Z}_1, \widetilde{Z}_2)$. □

In the next result we compare the random vector from (1.2), that is,

$$(Z_1, Z_2) = (X_1 + Y, X_2 + Y), \tag{3.7}$$

with

$$(\widehat{Z}_1, \widehat{Z}_2) = (\widehat{X}_1 + Y, \widehat{X}_2 + Y), \tag{3.8}$$

where X_1, X_2, $\widehat{X}_1$, $\widehat{X}_2$, and Y are all independent, and $X_1 =_{\text{st}} X_2$ and $\widehat{X}_1 =_{\text{st}} \widehat{X}_2$.

Proposition 3.2. *Let X_1, X_2, $\widehat{X}_1$, $\widehat{X}_2$, and Y be independent random variables such that $X_1 =_{\text{st}} X_2$ and $\widehat{X}_1 =_{\text{st}} \widehat{X}_2$, and let (Z_1, Z_2) and $(\widetilde{Z}_1, \widetilde{Z}_2)$ be as defined in (3.7) and (3.8). Also, let X_1' and $\widehat{X}_1'$ be independent copies of X_1 and $\widehat{X}_1$, respectively. If*

$$|\widehat{X}_1 - \widehat{X}_1'| \geq_{\text{st}} |X_1 - X_1'|, \tag{3.9}$$

then

$$\tau(\widehat{Z}_1, \widehat{Z}_2) \leq \tau(Z_1, Z_2).$$

Proof. Let F_1 be the distribution function of $X_1 - X_1'$, and let F_3 be the distribution function of $Y - Y'$, where Y' is an independent copy of Y. Then, from (3.4) we have

$$\begin{aligned}\tau(Z_1, Z_2) &= 4\int_0^\infty \big[F_1^2(w) + \overline{F}_1^2(w)\big]dF_3(w) - 1 \\ &= 4\int_0^\infty \big[1 - 2F_1(w)(1 - F_1(w))\big]dF_3(w) - 1. \end{aligned}\tag{3.10}$$

Similarly, if we denote the distribution function of $\widehat{X}_1 - \widehat{X}_1'$ by $\widehat{F}_1$, then

$$\tau(\widehat{Z}_1, \widehat{Z}_2) = 4 \int_0^\infty \big[1 - 2\widehat{F}_1(w)(1 - \widehat{F}_1(w))\big] dF_3(w) - 1. \tag{3.11}$$

From the symmetry of F_1 and $\widehat{F}_1$ it follows that $F_1(w) \geq 1/2$ and that $\widehat{F}_1(w) \geq 1/2$ for $w \geq 0$. From (3.9) it follows that $F_1(w) \geq \widehat{F}_1(w) \geq 1/2$ for all $w \geq 0$. Since the function $p(1-p)$ is decreasing in $p \in [1/2, 1]$, it follows that the integrands in (3.10) and (3.11) satisfy

$$\big[1 - 2F_1(w)(1 - F_1(w))\big] \geq \big[1 - 2\widehat{F}_1(w)(1 - \widehat{F}_1(w))\big], \quad \text{for all } w \geq 0.$$

This, together with (3.10) and (3.11), yields $\tau(\widehat{Z}_1, \widehat{Z}_2) \leq \tau(Z_1, Z_2)$. □

Combination of Propositions 3.1 and 3.2 gives the following main result of this section.

Theorem 3.3. *Let $\widehat{X}_1$, $\widehat{X}_2$, X_1, X_2, Y, and $\widetilde{Y}$ be independent random variables such that $\widehat{X}_1 =_{\text{st}} \widehat{X}_2$ and $X_1 =_{\text{st}} X_2$. Define*

$$(\widehat{Z}_1, \widehat{Z}_2) = (\widehat{X}_1 + Y, \widehat{X}_2 + Y)$$

and

$$(\widetilde{Z}_1, \widetilde{Z}_2) = (X_1 + \widetilde{Y}, X_2 + \widetilde{Y}).$$

Let $\widehat{X}_1'$, X_1', Y', and $\widetilde{Y}'$ be independent copies of $\widehat{X}_1$, X_1, Y, and $\widetilde{Y}$, respectively. If

$$|\widehat{X}_1 - \widehat{X}_1'| \geq_{\text{st}} |X_1 - X_1'| \qquad \textit{and} \qquad |Y - Y'| \leq_{\text{st}} |\widetilde{Y} - \widetilde{Y}'| \tag{3.12}$$

then

$$\tau(\widehat{Z}_1, \widehat{Z}_2) \leq \tau(\widetilde{Z}_1, \widetilde{Z}_2).$$

Remark 3.4. Note that the conditions in (2.3) are weaker than than the conditions in (3.12) (see the argument that follows (5.1) in Section 5 below). However, the conditions in (2.3) require the finiteness of the second moments. Thus, when the second moments do not exist, Theorem 3.3 becomes useful because it does not require the finiteness of second moments.◄

4 Comparisons via Blomqvist's medial correlation coefficients

In this section we consider only random variables that have symmetric distributions about their medians. For such random variables X and Y, with medians m_X and m_Y, respectively, the Blomqvist's β is defined as follows:

$$\begin{aligned}\beta(X, Y) &= P\{(X - m_X)(Y - m_Y) > 0\} - P\{(X - m_X)(Y - m_Y) < 0\} \\ &= 2P\{(X - m_X)(Y - m_Y) > 0\} - 1;\end{aligned} \tag{4.1}$$

see Blomqvist (1950).

Proceeding to the comparison of random vectors, we start with a special case of the setup of (1.2) and (1.3) where X_1 and X_2 are identically distributed.

Proposition 4.1. *Let X_1, X_2, Y and $\widetilde{Y}$ be independent random variables such that X_1 and X_2 are identically distributed. Assume that all these random variables have distributions that are symmetric about their respective medians. Let (Z_1, Z_2) and $(\widetilde{Z}_1, \widetilde{Z}_2)$ be as defined in* (1.2) *and* (1.3)*. If*

$$|Y - m_Y| \leq_{\text{st}} |\widetilde{Y} - m_{\widetilde{Y}}| \tag{4.2}$$

then

$$\beta(Z_1, Z_2) \leq \beta(\widetilde{Z}_1, \widetilde{Z}_2).$$

Proof. From (4.1) we can write

$$\beta(X_1 + Y, X_2 + Y) = 2P\{(X_1 + Y - m_{13})(X_2 + Y - m_{23}) > 0\} - 1,$$

where m_{13} and m_{23} are the medians of X_1+Y and X_2+Y, respectively. Denote the (common) median of X_1 and X_2 by m_X. Consider $W_1 = X_1 - m_X$, $W_2 = X_2 - m_X$, and $W_3 = Y - m_Y$. We have

$$\begin{aligned}
&P\{(X_1 + Y - m_{13})(X_2 + Y - m_{23}) > 0\} \\
&\qquad = P\{(W_1 + W_3 + m_X + m_Y - m_{13})(W_2 + W_3 + m_X + m_Y - m_{23}) > 0\}.
\end{aligned}$$

Note that W_i, $i = 1, 2, 3$ are independent, have distributions that are symmetric about zero, and

$$m_1 + m_Y - m_{13} = m_2 + m_Y - m_{23} = 0.$$

Thus we get that

$$P\{(X_1 + Y - m_{13})(X_2 + Y - m_{23}) > 0\} = P\{(W_1 + W_3)(W_2 + W_3) > 0\}.$$

Denote the distribution function of W_i by F_i, $i = 1, 2, 3$, and note that, with this notation, the expression (3.3) holds for $P\{(W_1 + W_3)(W_2 + W_3) > 0\}$. Following the argument in the proof of Proposition 3.1, that leads from (3.3) to (3.5), we see that

$$\beta(Z_1, Z_2) = 2E[h(W_3)\big|W_3 > 0] - 1, \tag{4.3}$$

where $h(w) = F_1^2(w) + \overline{F}_1^2(w)$, $w > 0$. Similarly, if we let $\widetilde{W}_3 = \widetilde{Y} - m_{\widetilde{Y}}$, then

$$\beta(\widetilde{Z}_1, \widetilde{Z}_2) = 2E\big[h(\widetilde{W}_3)\big|\widetilde{W}_3 > 0\big] - 1. \tag{4.4}$$

From the equalities $|Y - m_Y| =_{\text{st}} [W_3|W_3 > 0]$, $|\widetilde{Y} - m_{\widetilde{Y}}| =_{\text{st}} \Big[\widetilde{W}_3|\widetilde{W}_3 > 0\Big]$, and (4.2), it follows that $[W_3|W_3 > 0] \leq_{\text{st}} \Big[\widetilde{W}_3|\widetilde{W}_3 > 0\Big]$. Thus the increasingness of h (argued in the proof of Proposition 3.1), the expressions (4.3) and (4.4), and the inequality (3.1), yield $\beta(Z_1, Z_2) \leq \beta(\widetilde{Z}_1, \widetilde{Z}_2)$. □

In the next result we compare the random vectors (Z_1, Z_2) and $(\widehat{Z}_1, \widehat{Z}_2)$ of the form given in (3.7) and (3.8).

Proposition 4.2. *Let X_1, X_2, $\widehat{X}_1$, $\widehat{X}_2$, and Y be independent random variables such that $X_1 =_{\mathrm{st}} X_2$ and $\widehat{X}_1 =_{\mathrm{st}} \widehat{X}_2$. Assume that all these random variables have distributions that are symmetric about their respective medians. Let (Z_1, Z_2) and $(\widetilde{Z}_1, \widetilde{Z}_2)$ be as defined in* (3.7) *and* (3.8). *If*

$$|\widehat{X}_1 - m_{\widehat{X}_1}| \geq_{\mathrm{st}} |X_1 - m_{X_1}|, \tag{4.5}$$

then

$$\beta(\widehat{Z}_1, \widehat{Z}_2) \leq \beta(Z_1, Z_2).$$

Proof. As in the proof of Proposition 4.1, let F_1 be the distribution function of $W_1 = X_1 - m_{X_1}$, and let F_3 be the distribution function of $W_3 = Y - m_Y$. Again, as in the proof of Proposition 4.1, with this notation of W_1 and W_3, the expression (3.3) holds for $P\{(W_1 + W_3)(W_2 + W_3) > 0\}$. As a result, the expression (3.4) holds for $\beta(Z_1, Z_2)$. Thus, also the expression (3.10) holds for $\beta(Z_1, Z_2)$; that is,

$$\beta(Z_1, Z_2) = 4 \int_0^\infty \big[1 - 2F_1(w)(1 - F_1(w))\big] dF_3(w) - 1. \tag{4.6}$$

Similarly, if we denote the distribution function of $\widehat{X}_1 - m_{\widehat{X}_1}$ by $\widehat{F}_1$, then

$$\beta(\widehat{Z}_1, \widehat{Z}_2) = 4 \int_0^\infty \big[1 - 2\widehat{F}_1(w)(1 - \widehat{F}_1(w))\big] dF_3(w) - 1. \tag{4.7}$$

Following the argument at the end of the proof of Proposition 3.2, it is seen that (4.6) and (4.7) yield $\beta(\widehat{Z}_1, \widehat{Z}_2) \leq \beta(Z_1, Z_2)$. □

Combination of Propositions 4.1 and 4.2 gives the following main result of this section.

Theorem 4.3. *Let $\widehat{X}_1$, $\widehat{X}_2$, X_1, X_2, Y, and $\widetilde{Y}$ be independent random variables such that $\widehat{X}_1 =_{\mathrm{st}} \widehat{X}_2$ and $X_1 =_{\mathrm{st}} X_2$. Assume that all these random variables have distributions that are symmetric about their respective medians. Define*

$$(\widehat{Z}_1, \widehat{Z}_2) = (\widehat{X}_1 + Y, \widehat{X}_2 + Y)$$

and

$$(\widetilde{Z}_1, \widetilde{Z}_2) = (X_1 + \widetilde{Y}, X_2 + \widetilde{Y}).$$

Let $\widehat{X}_1'$, X_1', Y', and $\widetilde{Y}'$ be independent copies of $\widehat{X}_1$, X_1, Y, and $\widetilde{Y}$, respectively. If

$$|\widehat{X}_1 - m_{\widehat{X}_1}| \geq_{\mathrm{st}} |X_1 - m_{X_1}| \qquad \text{and} \qquad |Y - m_Y| \leq_{\mathrm{st}} |\widetilde{Y} - m_{\widetilde{Y}}| \tag{4.8}$$

then

$$\beta(\widehat{Z}_1, \widehat{Z}_2) \leq \beta(\widetilde{Z}_1, \widetilde{Z}_2).$$

5 Further comments

Recall the dispersive stochastic order that is defined as follows. Let X and Y be random variables with the corresponding distribution functions F and G. Denote by F^{-1} and G^{-1} the respective right continuous inverses. If $F^{-1}(\beta) - F^{-1}(\alpha) \le G^{-1}(\beta) - G^{-1}(\alpha)$ for all $0 < \alpha < \beta < 1$ then X is said to be smaller than Y in the dispersive order, and we denote it as $X \le_{\text{disp}} Y$. See Müller and Stoyan (2002) or Shaked and Shanthikumar (2007) for thorough studies of the dispersive order.

The following result, which is Theorem 3.B.42 in Shaked and Shanthikumar (2007), states a useful relationship between the dispersive and the ordinary stochastic orders.

Theorem 5.1. *Let X and X' be two independent and identically distributed random variables and let Y and Y' be two other independent and identically distributed random variables. Then*

$$X \le_{\text{disp}} Y \Longrightarrow |X - X'| \le_{\text{st}} |Y - Y'|.$$

Recall also the peakedness order that is defined as follows. Let X and Y be random variables with distribution functions that are symmetric about their medians m_X and m_Y. If $|X - m_X| \le_{\text{st}} |Y - m_Y|$ then X is said to be smaller than Y in the peakedness order, and we denote it as $X \le_{\text{peak}} Y$; see Section 3.D in Shaked and Shanthikumar (2007). That is,

$$X \le_{\text{peak}} Y \Longleftrightarrow |X - m_X| \le_{\text{st}} |Y - m_Y|.$$

Thus, (4.2), (4.5), and (4.8) are all conditions of ordering random variables in the peakedness stochastic order.

The orders $\le_{\text{disp}}$ and $\le_{\text{peak}}$ are variability orders in the intuitive sense that if $X \le_{\text{disp}} Y$ or $X \le_{\text{peak}} Y$, then we expect Y to be "more variable" than X. Even the condition

$$|X - X'| \le_{\text{st}} |Y - Y'|, \tag{5.1}$$

where X, X', Y, and Y' are as in Theorem 5.1, can stand for some kind of variability order. For example, note that if $|X - X'| \le_{\text{st}} |Y - Y'|$, then $\text{Var}(X) = \frac{1}{2}E(X - X')^2 \le \frac{1}{2}E(Y - Y')^2 = \text{Var}(Y)$.

Now note that the inequalities (2.2), (3.2), (3.9), (4.2), and (4.5), or more generally, (2.3), (3.12), and (4.8), all compare random variables in some sense of variability or dispersion. In fact, (3.2) and (4.2) compare the variability of the *common parts* (the Ys) of the compared random vectors, whereas (3.9) and (4.5) compare the variability of the *non-common parts* (the Xs) of the compared random vectors. The results show that if the common part (the Y) of one vector is more variable than the common part of the second vector, and/or the non-common parts (the Xs) of one vector are less variable than the non-common parts of the second vector, then the first vector is "less positively dependent" than the second vector. These informal explanations of the results of this paper go along with the intuition. We were able to notice these intuitive explanations only after we proved the inequalities in Sections 2–4.

Note that if $|Y - Y'| \le_{\text{st}} |\widetilde{Y} - \widetilde{Y}'|$ (this is (3.2)), where Y' be an independent copy of Y and $\widetilde{Y}'$ is an independent copy of $\widetilde{Y}$, then, as argued above, $\text{Var}(Y) \le \text{Var}(\widetilde{Y})$ (which is (2.2)).

That is, the condition of Proposition 3.1 is stronger than the condition of Proposition 2.1; a similar observation was given in Remark 3.4. As we can see from Theorem 5.1, a condition that is even stronger than (3.2) is $Y \leq_{\mathrm{disp}} \widetilde{Y}$. This is useful because plenty of examples of random variables that satisfy $Y \leq_{\mathrm{disp}} \widetilde{Y}$ can be found in the literature (for instance, see Example 5.2 below). Also, plenty of examples of random variables that satisfy $Y \leq_{\mathrm{peak}} \widetilde{Y}$ can be found in the literature.

Example 5.2. Let X_1, X_2, Y, and $\widetilde{Y}$ be independent gamma random variables with shape parameters α_X, α_X, α_Y, and $\alpha_{\widetilde{Y}}$, respectively, and with the same scale parameter. Without losing any generality we take the common scale parameter to be 1. Then both

$$(Z_1, Z_2) = (X_1 + Y, X_2 + Y) \qquad \text{and} \qquad (\widetilde{Z}_1, \widetilde{Z}_2) = (X_1 + \widetilde{Y}, X_2 + \widetilde{Y})$$

have Cheriyan and Ramabhadran's bivariate gamma distributions; see Kotz, Balakrishnan, and Johnson (2000, page 432). From Theorem 1 of Saunders and Moran (1978) we see that if

$$\alpha_{\widetilde{Y}} > \alpha_Y > 0 \tag{5.2}$$

then $Y \leq_{\mathrm{disp}} \widetilde{Y}$. It follows, from Theorem 5.1 and Proposition 3.1, that if (5.2) holds then

$$\tau(Z_1, Z_2) \leq \tau(\widetilde{Z}_1, \widetilde{Z}_2). \tag{5.3}$$

Let, furthermore, $\widehat{X}_1$ and $\widehat{X}_2$ be independent gamma random variables with a common shape parameter $\alpha_{\widehat{X}}$, and scale parameter 1. Then also

$$(\widehat{Z}_1, \widehat{Z}_2) = (\widehat{X}_1 + Y, \widehat{X}_2 + Y)$$

has a Cheriyan and Ramabhadran's bivariate gamma distribution. If

$$\alpha_{\widehat{X}} > \alpha_X > 0 \tag{5.4}$$

then, as argued above, $X \leq_{\mathrm{disp}} \widehat{X}$. Now it follows from Theorem 5.1 and Proposition 3.2, that if (5.4) holds then

$$\tau(\widehat{Z}_1, \widehat{Z}_2) \leq \tau(Z_1, Z_2). \tag{5.5}$$

Note that in this example, using our theoretical result, we obtain the inequalities (5.3) and (5.5) without having to explicitly compute the quantities $\tau(Z_1, Z_2)$, $\tau(\widetilde{Z}_1, \widetilde{Z}_2)$, and $\tau(\widehat{Z}_1, \widehat{Z}_2)$ for the Cheriyan and Ramabhadran's bivariate gamma random vectors. In fact, the explicit expressions for $\tau(Z_1, Z_2)$, $\tau(\widetilde{Z}_1, \widetilde{Z}_2)$, and $\tau(\widehat{Z}_1, \widehat{Z}_2)$ are far from trivial, and as a result, directly obtaining (5.3) and (5.5) is not easy, if at all possible. ♢

Looking at (1.2) and (1.3) one may get the impression that if $\widetilde{Y} \geq_{\mathrm{st}} Y$ then we would expect $\mathrm{Corr}(\widetilde{Z}_1, \widetilde{Z}_2) \geq \mathrm{Corr}(Z_1, Z_2)$ or $\tau(\widetilde{Z}_1, \widetilde{Z}_2) \geq \tau(Z_1, Z_2)$, because $\widetilde{Y}$ seems to stochastically pull both $X_1 + \widetilde{Y}$ and $X_2 + \widetilde{Y}$ more strongly than Y would pull both $X_1 + Y$ and $X_2 + Y$. However, as the following example shows, in general this is not the case. The explanation is that the "pull" mentioned above is a change of location, and not an effect on the concordance of the underlying random variables.

Example 5.3. Let X_1, X_2 and Y be independent random variables, where X_1 and X_2 have the uniform(0,1) distribution, and Y is a Bernoulli random variable with probability of success $p \in (0,1)$. As in (1.2), let $(Z_1, Z_2) = (X_1 + Y, X_2 + Y)$.

Note that $\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = 1/12$, and $\mathrm{Var}(Y) = p(1-p)$. Thus, from (2.1) we see that

$$\mathrm{Corr}(Z_1, Z_2) = \frac{p(1-p)}{p(1-p) + \frac{1}{12}} = 1 - \frac{\frac{1}{12}}{p(1-p) + \frac{1}{12}}.$$

This expression is increasing in $p \in (0, \frac{1}{2})$ and decreasing in $p \in (\frac{1}{2}, 1)$. Now, if $\widetilde{Y}$ is Bernoulli with probability of success $\widetilde{p} \in (0,1)$, and $\widetilde{p} \geq p$, then $\widetilde{Y} \geq_{\mathrm{st}} Y$. But, as can be seen from the computations above, if $\widetilde{p} > p > 0$, then it is not true that $\mathrm{Corr}(\widetilde{Z}_1, \widetilde{Z}_2) \geq \mathrm{Corr}(Z_1, Z_2)$, where $(\widetilde{Z}_1, \widetilde{Z}_2)$ is defined in (1.3).

Now we compute $\tau(Z_1, Z_2) = \tau(X_1 + Y, X_2 + Y)$. Let (X_1', X_2', Y') be an independent copy of (X_1, X_2, Y), and define $W_1 = X_1 - X_1'$, $W_2 = X_2 - X_2'$, and $W_3 = Y - Y'$. Note that W_3 takes on only the values $-1, 0, 1$. Thus, applying (3.3) we have

$$\begin{aligned}
\tau(Z_1, Z_2) &= 2P\big\{(W_1 + W_3)(W_2 + W_3) \geq 0\big\} - 1 \\
&= 2\Big\{p(1-p)P\big\{(W_1 - 1)(W_2 - 1) \geq 0\big\} + ((1-p)^2 + p^2)P\big\{W_1 W_2 \geq 0\big\} \\
&\qquad\qquad + p(1-p)P\big\{(W_1 + 1)(W_2 + 1) \geq 0\big\}\Big\} - 1 \\
&= 2\big\{p(1-p) + ((1-p)^2 + p^2)(\tfrac{1}{4} + \tfrac{1}{4}) + p(1-p)\big\} - 1 \\
&= 2p(1-p).
\end{aligned}$$

This, like $\mathrm{Corr}(Z_1, Z_2)$, is increasing in $p \in (0, \frac{1}{2})$ and decreasing in $p \in (\frac{1}{2}, 1)$. So the analysis above applies here too. That is, $\widetilde{Y} \geq_{\mathrm{st}} Y$ does not necessarily imply that $\tau(\widetilde{Z}_1, \widetilde{Z}_2) \geq \tau(Z_1, Z_2)$. ♢

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