INVARIANT METRICS ON THE IWASAWA MANIFOLD

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Abstract. We compute the automorphism group of the complex 3-dimensional Heisenberg Lie algebra and study its action (by isometries) on the set of inner products of $\mathbb{R}^6$. As consequence we give a description of the moduli space of invariant metrics on the Iwasawa manifold. We also show that the action mentioned above is not polar and have a minimal orbit.

1. Introduction

The Iwasawa manifold M is a complex 3-dimensional nilmanifold. It is defined as a compact quotient of the $3-$dimensional complex Heisenberg group. It is well-known that M carries no Kähler metric even though it admits symplectic structures [6], [9]. The Hermitian geometry of the Iwasawa manifold was studied in [1] and [11]. In [5] was studied the topology of the quotient of the set of 2-step nilpotent Lie brackets of $\mathbb{R}^n$ under the natural action of the orthogonal group $\mathrm{O}(n)$.

Let $\mathcal{M}$ be the set of invariant metrics on the Iwasawa manifold M. Any invariant metric on the Iwasawa manifold comes from an inner product on the Lie algebra of the Heisenberg group. Thus, the space $\mathcal{M}$ can be identified with the symmetric space of inner products on $\mathbb{R}^6$.

The goal of this paper is to determine and further describe the moduli space $\mathcal{M}/\sim$ of invariant metrics up to isometries. That is to say, $[g] \in \mathcal{M}/\sim$ is the class of all invariant metrics which are isometric to g.

Let $\mathcal{D}$ be the set of symmetric positive definite 2×2 matrices and let σ be the conjugation $\sigma : \begin{pmatrix} E & F \\ F & G \end{pmatrix} \to \begin{pmatrix} E & -F \\ -F & G \end{pmatrix}$. Let $\mathcal{D}/\sigma$ be the quotient of $\mathcal{D}$ by the action of σ.

Here is our main result:

Theorem 1.1. *The moduli space $\mathcal{M}/\sim$ is homeomorphic to the product*

$$\Delta \times \mathcal{D}/\sigma \ .$$

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where Δ is the triangle $\Delta := \{(r,s) : 1 \geq r \geq s > 0\}$.
Moreover, any left invariant metric on M is isometric to a metric of the form

$$e^1 \otimes e^1 + re^2 \otimes e^2 + e^3 \otimes e^3 + se^4 \otimes e^4 + Ee^5 \otimes e^5 + 2Fe^5 \otimes e^6 + Ge^6 \otimes e^6 \ .$$

where $\omega^1 = e^1 + \mathrm{i}e^2, \omega^2 = e^3 + \mathrm{i}e^4, \omega^3 = e^5 + \mathrm{i}e^6$ are the standard left invariant forms such that $\mathrm{d}\omega^3 = \omega^2 \wedge \omega^1$.

2. Preliminaries

Let

$$H = \left\{ g = \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \ : \ z_1, z_2, z_3 \in \mathbb{C} \right\}$$

denote the complex Heisenberg group and $\mathfrak{h}$ its Lie algebra. The Iwasawa manifold is the compact quotient space $M = \Gamma \setminus H$ formed from the right cosets of the discrete subgroup Γ given by the matrices whose entries z_1, z_2, z_3 are Gaussian integers.

The forms $\omega^1 = \mathrm{d}z_1, \omega^2 = \mathrm{d}z_2, \omega^3 = \mathrm{d}z_3 - z_1\mathrm{d}z_2$ are left invariant since they are the entries of the matrix $g^{-1}\mathrm{d}g$, that is to say

$$g^{-1}\mathrm{d}g = \begin{pmatrix} 1 & -z_1 & z_1z_2 - z_3 \\ 0 & 1 & -z_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \mathrm{d}z_1 & \mathrm{d}z_3 \\ 0 & 0 & \mathrm{d}z_2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{d}z_1 & \mathrm{d}z_3 - z_1\mathrm{d}z_2 \\ 0 & 0 & \mathrm{d}z_2 \\ 0 & 0 & 0 \end{pmatrix} .$$

The real forms e^i defined by

$$\omega^1 = e^1 + \mathrm{i}e^2 , \quad \omega^2 = e^3 + \mathrm{i}e^4 , \quad \omega^3 = e^5 + \mathrm{i}e^6$$

give rise to a real basis of the dual $\mathfrak{h}^*$ of the Heisenberg Lie algebra $\mathfrak{h}$.

The group $\mathrm{Aut}(\mathfrak{h})$ is the set of all invertible linear maps such that $f[X,Y] = [fX, fY]$ for all $X, Y \in \mathfrak{h}$. Observe that $f \in \mathrm{Aut}(\mathfrak{h})$ if and only if $\mathrm{d}f^*\theta = f^*\mathrm{d}\theta$ for all one forms $\theta \in \mathfrak{h}^*$, where d is defined as $\mathrm{d}\theta(X,Y) := -\theta([X,Y])$. Recall that f acts on θ as $(f^*\theta)(X) := \theta(fX)$ for all $X \in \mathfrak{h}$.

The following lemma will be useful for the computation of $\mathrm{Aut}(\mathfrak{h})$.

Lemma 2.1. *Let V be a real vector space and let $J \in End(V)$ be a complex structure. Let $V^*_{\mathbb{C}} = \Lambda^{1,0} \oplus \Lambda^{0,1}$ be the splitting according the eigenspaces of J. Let $f \in GL(V)$ be an invertible endomorphism of V. The following conditions are equivalent:*

(i) *f is either holomorphic $f^*\Lambda^{1,0} \subset \Lambda^{1,0}$ or antiholomorphic $f^*\Lambda^{1,0} \subset \Lambda^{0,1}$;*
(ii) *$f^*\Lambda^{2,0} \subset \Lambda^{2,0} \oplus \Lambda^{0,2}$.*

Proof. (i) $\Rightarrow$ (ii) is obvious. Let $\alpha, \beta \in \Lambda^{1,0}$ be two independent forms, i.e. $\alpha \wedge \beta \neq 0$. Then $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta \neq 0$ because f is invertible. Write $f^*\alpha = A + \overline{B}$ and $f^*\beta = C + \overline{D}$, where $A, B, C, D \in \Lambda^{1,0}$. Then

$$f^*\alpha \wedge f^*\beta = (A + \overline{B}) \wedge (C + \overline{D}) = A \wedge C + A \wedge \overline{D} + \overline{B} \wedge C + \overline{B} \wedge \overline{D}$$

and (ii) implies

$$A \wedge \overline{D} + \overline{B} \wedge C = 0 \ .$$

Assume $A \neq 0$. Then wedging with $\overline{B}$ we get $A\wedge\overline{D}\wedge\overline{B} = 0$ which implies $\overline{D}\wedge\overline{B} = 0$. Then if $B \neq 0$ we get $D = \lambda B$ and inserting this in $A \wedge \overline{D} + \overline{B} \wedge C = 0$ we get

$$A \wedge \overline{\lambda B} + \overline{B} \wedge C = 0.$$

which implies $A\overline{\lambda} = C$ but then $\overline{\lambda} f^*\alpha = f^*\beta$ which contradicts $f^*\alpha \wedge f^*\beta \neq 0$. Then $B = 0$ and so $\overline{D} = 0$. This shows that if $A \neq 0$ then $f^*\alpha, f^*\beta \in \Lambda^{1,0}$ so we get (i). If $A = 0$ and $B \neq 0$ then the same argument with $\overline{f}$ at the place of f shows that $\overline{f}^*\Lambda^{1,0} \subset \Lambda^{1,0}$ and so $f^*\Lambda^{1,0} \subset \Lambda^{0,1}$. □

3. Computation of $\mathrm{Aut}(\mathfrak{h})$ by using differential forms.

Following [12, pag.15] notice that $\omega^1, \omega^2, \omega^3$ span the (1,0) space of the standard complex structure J_0. That is to say, the forms $\omega^1, \omega^2, \omega^3$ belong to the complexification of $\mathfrak{h}^*$ and

$$J_0\omega_j = \mathrm{i}\omega_j$$

for $j = 1, 2, 3$.

Let $f : \mathfrak{h} \to \mathfrak{h}$ be an automorphism of $\mathfrak{h}$. Then f has a natural extension to the complexification $\mathfrak{h}_{\mathbb{C}}$. Recall that an endomorphism $f : \mathfrak{h}_{\mathbb{C}} \to \mathfrak{h}_{\mathbb{C}}$ comes from a real endomorphism of $\mathfrak{h}$ if and only if $\overline{f(Z)} = f(\overline{Z})$ for all $Z \in \mathfrak{h}_{\mathbb{C}}$.

The following theorem gives $\mathrm{Aut}(\mathfrak{h})$ acting on the complexification $\mathfrak{h}^*_{\mathbb{C}}$ with respect to a particular basis.

Theorem 3.1. *With respect to the basis $\mathcal{B} = (\omega^1, \omega^2, \overline{\omega^1}, \overline{\omega^2}, \omega^3, \overline{\omega^3})$ an automorphism $f \in \mathrm{Aut}(\mathfrak{h})$ has one of the followings forms:*

$$f = \begin{pmatrix} a & b & 0 & 0 & p & q \\ c & d & 0 & 0 & r & s \\ 0 & 0 & \overline{a} & \overline{b} & \overline{q} & \overline{p} \\ 0 & 0 & \overline{c} & \overline{d} & \overline{s} & \overline{r} \\ 0 & 0 & 0 & 0 & ad-bc & 0 \\ 0 & 0 & 0 & 0 & 0 & \overline{ad-bc} \end{pmatrix} \tag{3.1}$$

or

$$f = \begin{pmatrix} 0 & 0 & \overline{a} & \overline{b} & p & q \\ 0 & 0 & \overline{c} & \overline{d} & r & s \\ a & b & 0 & 0 & \overline{q} & \overline{p} \\ c & d & 0 & 0 & \overline{s} & \overline{r} \\ 0 & 0 & 0 & 0 & 0 & \overline{ad-bc} \\ 0 & 0 & 0 & 0 & ad-bc & 0 \end{pmatrix} \tag{3.2}$$

with $ad - bc \neq 0$.

Proof. Recall that $f \in \mathrm{Aut}(\mathfrak{h})$ if and only if $\mathrm{d}f^*\theta = f^*\mathrm{d}\theta$ for all one forms θ. Then $f \in \mathrm{Aut}(\mathfrak{h})$ preserves the image and the kernel of the operator d. Notice that $\mathrm{Im}(\mathrm{d}) = \langle \mathrm{d}\omega^3, \overline{\mathrm{d}\omega^3}\rangle$ and $\ker(\mathrm{d}) = \langle \omega^1, \omega^2, \overline{\omega^1}, \overline{\omega^2}\rangle$. Observe that $\mathrm{d}\omega^3$ generates the $\Lambda^{2,0}$ space of the restriction to $\ker(\mathrm{d})$ of the standard complex structure J_0 . Then by using Lemma 2.1 and the fact that f comes from a real endomorphism of $\mathfrak{h}$ we get that the first 4 columns of the matrix of f are as in equations (3.1) or (3.2).

Assume that $f^*\langle\omega^1,\omega^2\rangle = \langle\omega^1,\omega^2\rangle$ and write $f^*\omega^3 = p\omega^1 + r\omega^2 + \overline{q}\overline{\omega^1} + \overline{s}\overline{\omega^2} + m\omega^3 + n\overline{\omega^3}$. Then

$$\begin{aligned}
\mathrm{d}f^*\omega^3 &= \mathrm{d}(p\omega^1 + r\omega^2 + \overline{q}\overline{\omega^1} + \overline{s}\overline{\omega^2} + m\omega^3 + n\overline{\omega^3}) = \\
&= m\mathrm{d}\omega^3 + n\mathrm{d}\overline{\omega^3} = \\
&= -m\omega^1\wedge\omega^2 - n\overline{\omega^1\wedge\omega^2} = \\
&= f^*\mathrm{d}\omega^3 = f^*(-\omega^1\wedge\omega^2) = \\
&= -f^*\omega^1\wedge f^*\omega^2 = -(a\;\omega^1 + c\;\omega^2)\wedge(b\;\omega^1 + d\;\omega^2) = \\
&= -(ad-bc)\;\omega^1\wedge\;\omega^2.
\end{aligned}$$

So $m = ad - bc$ and $n = 0$ which gives the last two columns of (3.1). If $f^*\langle\omega^1,\omega^2\rangle = \langle\overline{\omega^1},\overline{\omega^2}\rangle$ then a similar computation gives the last two columns of (3.2). In the opposite direction it is not difficult to check that any invertible matrix as in (3.1) or (3.2) is the extension to $\mathfrak{h}^*_{\mathbb{C}}$ of a map $f:\mathfrak{h}\to\mathfrak{h}$ which satisfies $\mathrm{d}f^*\omega^j = f^*\mathrm{d}\omega^j$, $(j = 1,2,3)$ and so $f\in\mathrm{Aut}(\mathfrak{h})$. $\square$

3.1. **Connected components.** Then $\mathrm{Aut}(\mathfrak{h})$ has two connected components. The set of matrices (3.1) are the connected component of the identity and is denoted $\mathrm{Aut}_0(\mathfrak{h})$. Notice that the matrices (3.2) are obtained from the matrices (3.1) by composition with the conjugation automorphism $\mathcal{C}$ of $(\mathfrak{h}, J_0)$ whose matrix with respect to the basis of Theorem 3.1 is

$$\mathcal{C} := \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{pmatrix} \tag{3.3}$$

Thus, $\mathrm{Aut}(\mathfrak{h}) = \mathrm{Aut}_0(\mathfrak{h})\bigcup\mathcal{C}(\mathrm{Aut}_0(\mathfrak{h}))$.

Observe that

$$\mathcal{C}X\mathcal{C} = \overline{X} \tag{3.4}$$

for $X\in\mathrm{Aut}(\mathfrak{h})$. Indeed,

$$\begin{aligned}
(\mathcal{C}X\mathcal{C})\omega^j &= \mathcal{C}X\overline{\omega^j}, \\
&= \overline{\mathcal{C}X\omega^j}, \\
&= \overline{X\overline{\omega^j}}, \\
&= \overline{X}\omega^j\,.
\end{aligned}$$

Remark 3.2. *The conjugation $\mathcal{C}$ should be not confused with the conjugation of complexification $\overline{X}$ of $\mathfrak{h}^*_{\mathbb{C}}$. Such a confusion induces the wrong claim that $\mathcal{C}$ must be in the center of* $\mathrm{Aut}(\mathfrak{h})$ *which is wrong as showed by equation (3.4). Actually, the conjugation $\mathcal{C}$ is a complex linear map whilst the complexification $\overline{X}$ of $\mathfrak{h}^*_{\mathbb{C}}$ is anti-complex linear map from its very definition.*

3.2. **The real description of** $\mathrm{Aut}(\mathfrak{h})$**.** Here are the matrices of $\mathrm{Aut}(\mathfrak{h})$ with respect to the basis $(e^1, e^2, e^3, e^4, e^5, e^6)$:

$$f = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 & x_{11} & x_{12} \\ -a_2\epsilon & a_1\epsilon & -b_2\epsilon & b_1\epsilon & x_{21} & x_{22} \\ c_1 & c_2 & d_1 & d_2 & x_{31} & x_{32} \\ -c_2\epsilon & c_1\epsilon & -d_2\epsilon & d_1\epsilon & x_{41} & x_{42} \\ 0 & 0 & 0 & 0 & u & v \\ 0 & 0 & 0 & 0 & -v\epsilon & u\epsilon \end{pmatrix} \tag{3.5}$$

where $\epsilon = \pm 1$, x_{ij} are arbitrary real numbers and $a = a_1 + \mathrm{i}a_2, b = b_1 + \mathrm{i}b_2, c = c_1 + \mathrm{i}c_2, d = d_1 + \mathrm{i}d_2$ are complex such that $ad - bc = u + \mathrm{i}v \neq 0$. Indeed, this follows by taking the real and imaginary parts of the forms $f^*\omega^i = f^*e^{2i-1} + \mathrm{i}f^*e^{2i}$, $i = 1, 2, 3$. The matrices with $\epsilon = 1$ corresponds to $\mathrm{Aut}_0(\mathfrak{h})$.

Here is the Lie algebra of $\mathrm{Aut}(\mathfrak{h})$:

$$f = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 & x_{11} & x_{12} \\ -a_2 & a_1 & -b_2 & b_1 & x_{21} & x_{22} \\ c_1 & c_2 & d_1 & d_2 & x_{31} & x_{32} \\ -c_2 & c_1 & -d_2 & d_1 & x_{41} & x_{42} \\ 0 & 0 & 0 & 0 & u & v \\ 0 & 0 & 0 & 0 & -v & u \end{pmatrix} \tag{3.6}$$

where x_{ij} are arbitrary real numbers and $a = a_1 + \mathrm{i}a_2, b = b_1 + \mathrm{i}b_2, c = c_1 + \mathrm{i}c_2, d = d_1 + \mathrm{i}d_2$ are complex such that $a + d = u + \mathrm{i}v$.

3.3. **Twisted semidirect product.** Notice that $\mathrm{Aut}_0(\mathfrak{h})$ looks like the semidirect product $GL(2, \mathbb{C}) \ltimes \mathbb{C}^{2,2}$. Indeed, a matrix f as in (3.1) can be written as

$$\begin{pmatrix} A & 0 & P \\ 0 & \overline{A} & P' \\ 0 & 0 & \Delta(A) \end{pmatrix}$$

where $A \in GL(2, \mathbb{C})$, $P \in \mathbb{C}^{2,2}$, $\Delta(A) = \begin{pmatrix} \det(A) & 0 \\ 0 & \overline{\det(A)} \end{pmatrix}$ and P' is the matrix obtained from P after conjugation and swapping the columns.

Then there is a one-one correspondence between $\mathrm{Aut}_0(\mathfrak{h})$ and $GL(2, \mathbb{C}) \times \mathbb{C}^{2,2}$. Here is the product rule in terms of pairs $(A, P), (B, Q) \in GL(2, \mathbb{C}) \times \mathbb{C}^{2,2}$:

$$(A, P)(B, Q) = (AB, AQ + P\Delta(B)) \ . \tag{3.7}$$

Thus $\mathrm{Aut}_0(\mathfrak{h})$ looks like the semidirect product $GL(2, \mathbb{C}) \ltimes \mathbb{C}^{2,2}$ but it is not exactly this semidirect product. Notice, for example, that the center of the semidirect product $GL(2, \mathbb{C}) \ltimes \mathbb{C}^{2,2}$ is not trivial.

3.4. **The center is trivial.**

Proposition 3.3. *The center of* $\mathrm{Aut}_0(\mathfrak{h})$ *is trivial.*

Proof. If (A_0, P_0) is in the center of $\mathrm{Aut}_0(\mathfrak{h})$ then

$$(A_0, P_0)(B, Q) = (A_0B, A_0Q + P_0\Delta(B)) =$$
$$(B, Q)(A_0, P_0) = (BA_0, BP_0 + Q\Delta(A_0)).$$

So A_0 is in the center of $GL(2,\mathbb{C})$, that is to say $A_0 = \lambda Id$. Taking $Q = 0$ we get

$$P_0\Delta(B) = BP_0\,.$$

This implies $P_0 = 0$ since the above equation means that the columns of P_0 are eigenvectors of B. Finally, $\lambda Q = Q\Delta(\lambda Id)$ implies $\lambda^2 = \lambda$, and so $\lambda = 1$. $\square$

Corollary 3.4. *The center of* $\mathrm{Aut}(\mathfrak{h})$ *is trivial.*

Proof. Let $A \neq Id \in \mathrm{Aut}(\mathfrak{h})$ be a non trivial element in the center. Since $\mathrm{Aut}(\mathfrak{h})$ has two connected components we have $A = \mathcal{C}A_0$ with $A_0 \in \mathrm{Aut}_0(\mathfrak{h})$. Then

$$\mathcal{C}A_0\mathcal{C}X = A_0X = \overline{X}A_0\ .$$

This implies that A_0 commutes with all the real matrices in $\mathrm{Aut}_0(\mathfrak{h})$. Now the same argument as in previous proposition shows that $A_0 = Id$. Since $\mathcal{C}$ is not in the center we conclude that $A = Id$. $\square$

3.5. **Inner automorphisms.** An inner automorphism is given by $\exp(ad_X)$ where $X \in \mathfrak{h}$.

Proposition 3.5. *The inner automorphisms in* $\mathrm{Aut}(\mathfrak{h})$ *are given by*

$$(Id, \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix})$$

where $a, b \in \mathbb{C}$. *So the subgroup of inner automorphisms is isomorphic to* $\mathbb{C}^2$.

4. GL$(k,\mathbb{C})$ equivalents inner products on $\mathbb{R}^{2k}$

It is well-known that a symmetric positive definite matrix can be diagonalized by using an orthogonal transformation. Here we show that also by using a transformation in GL$(k,\mathbb{C})$ it is possible to diagonalize a symmetric positive definite matrix.

Let J be the complex structure of $\mathbb{R}^{2k}$ given by the identification with $\mathbb{C}^k$.

Theorem 4.1. *Let g be any inner product on $\mathbb{R}^{2k}$. Then there exist k g-orthogonal complex lines. That is to say, there exist g-unitary vectors $e_1, e_2, \cdots, e_k \in \mathbb{R}^{2k}$ such that*

$$\mathbb{R}^{2k} = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \cdots \oplus \mathbb{C}e_k$$

where $\mathbb{C}e_j$ is g-orthogonal to $\mathbb{C}e_i$ for $i \neq j$. Moreover, we can assume $g(e_i, Je_i) = 0$ for $i = 1, \cdots, k$.

Proof. By induction it is enough to show that there exists a 2-dimensional complex subspace whose g-orthogonal complement is also complex. Let $J = S + A$ be the decomposition of J into self-adjoint part S and skew part A with respect to g. The identity $J^2 = -Id$ implies:

$$\begin{cases} i)\, SA = -AS, \\ ii)\, S^2 + A^2 = -I. \end{cases}$$

Indeed, $SA + AS$ is skew and $-I = J^2 = S^2 + A^2 + SA + AS$ imply $SA + AS$ is self-adjoint. Then $SA + AS$ must vanish.

Let v be an eigenvector of S. Then $i)$ and $ii)$ imply that the $span(v, Av)$ is a J-invariant, 2-dimensional subspace whose orthogonal complement is also

J-invariant.$\square$

Here is another proof due to Simon Salamon.

Proof. Let K be the g-transpose of J. Then KJ is a self-adjoint map with respect to g. If v is an eigenvector of KJ with eigenvalue λ then Jv is also an eigenvector of KJ with eigenvalue $\frac{1}{\lambda}$. Thus we can group the eigenvectors into pairs to get the required orthogonal decomposition.$\square$

4.1. **The action of $\mathrm{GL}(2,\mathbb{C})$ on the inner products of $\mathbb{R}^4$.** Let $\mathcal{M}_4$ be the set of inner products of $\mathbb{R}^4$. By fixing a basis (e_1, e_2, e_3, e_4) we identify $\mathcal{M}_4$ with the symmetric space $\mathrm{GL}(4,\mathbb{R})/\mathrm{O}(4)$. We also identify $\mathbb{R}^4$ with $\mathbb{C}^2$ by

$$(z, w) = (z_1 + \mathrm{i}z_2, w_1 + \mathrm{i}w_2) \cong z_1e_1 + z_2e_2 + w_1e_3 + w_2e_4\,.$$

Denote with J the complex structure on $\mathbb{R}^4$ due the above identification.

The subgroup $\mathrm{GL}(2,\mathbb{C}) \subset \mathrm{GL}(4,\mathbb{R})$ acts naturally on $\mathcal{M}_4$ and Theorem 4.1 can be interpreted by saying that the submanifold

$$\Sigma = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{pmatrix} \,:\, r, s \in \mathbb{R}^+ \right\}.$$

meets all $\mathrm{GL}(2,\mathbb{C})$-orbits. When $r = s = 1$ we get the standard Hermitian metric g_0.

Let us identify Σ with the first quadrant

$$\{(r, s) \in \mathbb{R}^2 : r, s > 0\}$$

and let $\Gamma \subset Diff(\Sigma)$ be the finite group with 8 elements generated by the following diffeomorphisms

$$\begin{cases} r(x, y) := (y, x), \\ i(x, y) := (\frac{1}{x}, y), \\ j(x, y) = (x, \frac{1}{y})\,. \end{cases}$$

By swapping the order of the complex lines $\mathbb{C}e_1$ and $\mathbb{C}e_3$ and by re-scaling the generators we get the following result.

Proposition 4.2. *The quotient space $\mathcal{M}_4/\mathrm{GL}(2,\mathbb{C})$ is homeomorphic to the quotient Σ/Γ.*

Proof. Let $g \in \mathcal{M}_4$ be an inner product and let $(r, s) \in \Sigma$ be a pair corresponding to g. As we observed in the second proof of Theorem 4.1 the positive numbers r, s are eigenvalues of the g-self-adjoint map KJ. Then the group Γ acts on such pairs because the set of eigenvalues of KJ is $\{r, \frac{1}{r}, s, \frac{1}{s}\}$. Reciprocally, any of the generators of Γ acts as an element of $\mathrm{GL}(2,\mathbb{C})$. Indeed, the map r corresponds to swapping the complex lines and the maps i, j are the respective re-scalings in each complex line.$\square$

Notice that Σ/Γ is homeomorphic to the triangle

$$\Delta := \{(r, s) : 1 \geq r \geq s > 0\}\,.$$

The symmetric space $\mathcal{M}_4$ carries a Riemannian metric with respect to $\mathrm{GL}(4,\mathbb{R})$ acts by isometries. Then $\mathrm{GL}(2,\mathbb{C})$ acts by isometries on $\mathcal{M}_4$. It is well-known that any isolated orbit of an isometry group is a minimal submanifold. Thus, we get the following corollary.

Corollary 4.3. *The* $\mathrm{GL}(2,\mathbb{C})$*-orbit through the canonical Hermitian metric* g_0 *is a minimal submanifold of* $\mathcal{M}$*. Moreover, by using the identification* $\Sigma/\Gamma \cong \Delta$*, the principal orbits correspond to the open triangle*

$$\{(r,s) : 1 > r > s > 0\}\,.$$

5. Inner products on the Heisenberg algebra

Theorem 4.1 implies that any inner product g on the Heisenberg algebra is conjugated by the group of automorphism to a metric whose matrix with respect to the basis $\mathcal{B} = (\omega^1, \omega^2, \overline{\omega^1}, \overline{\omega^2}, \omega^3, \overline{\omega^3})$ is

$$g = \begin{pmatrix} 1-r & 0 & 1+r & 0 & A & B \\ 0 & 1-s & 0 & 1+s & C & D \\ 1+r & 0 & 1-r & 0 & \overline{B} & \overline{A} \\ 0 & 1+s & 0 & 1-s & \overline{D} & \overline{C} \\ A & C & \overline{B} & \overline{D} & M & t \\ B & D & \overline{A} & \overline{C} & t & \overline{M} \end{pmatrix} \tag{5.1}$$

where $r, s \in \mathbb{R}^+, t \in \mathbb{R}, A, B, C, D \in \mathbb{C}$.

Now an straightforward computation shows that there exists an unique $P \in \mathbb{C}^{2,2}$ such that the automorphism $f = (Id, P)$ (see 3.7) which sends g to the matrix

$$g' = \begin{pmatrix} 1-r & 0 & 1+r & 0 & 0 & 0 \\ 0 & 1-s & 0 & 1+s & 0 & 0 \\ 1+r & 0 & 1-r & 0 & 0 & 0 \\ 0 & 1+s & 0 & 1-s & 0 & 0 \\ 0 & 0 & 0 & 0 & M & t \\ 0 & 0 & 0 & 0 & t & \overline{M} \end{pmatrix} \tag{5.2}$$

Let $\mathcal{M}$ be the set of inner products on the Heisenberg algebra. By taking real and imaginary parts we have the following result.

Theorem 5.1. *Let* $g \in \mathcal{M}$ *be any inner product on the Heisenberg algebra* $\mathfrak{h}$*. Then* g *is equivalent by an automorphism of* $\mathrm{Aut}_0(\mathfrak{h})$ *to*

$$e^1 \otimes e^1 + re^2 \otimes e^2 + e^3 \otimes e^3 + se^4 \otimes e^4 + Ee^5 \otimes e^5 + 2Fe^5 \otimes e^6 + Ge^6 \otimes e^6\ .$$

Moreover, the moduli space $\mathcal{M}/\mathrm{Aut}(\mathfrak{h})$ *is homeomorphic to the product*

$$\Delta \times (\mathcal{M}_2/\sigma)\,,$$

where $\Delta := \{(r,s) : 1 \geq r \geq s > 0\}$*,* $\mathcal{M}_2$ *is the set of inner products of* $\mathbb{R}^2$*,* σ *is the conjugation* $\sigma : (x,y) \to (x,-y)$ *and* $\mathcal{M}_2/\sigma$ *denotes the quotient by the action of* σ*.*

6. Proof of Theorem 1.1

In [13] E. Wilson showed that two left invariant metrics g_1, g_2 on a nilmanifold M are isometric if and only if g_1, g_2 are conjugated under the automorphism group of the nilpotent Lie algebra associated to M.

Now Theorem 1.1 is a direct application of Theorem 5.1.

7. The geometry of the action of $\mathrm{Aut}(\mathfrak{h})$.

Here we study the action of $\mathrm{Aut}(\mathfrak{h})$ on $\mathcal{M}$. We refer to the book [4] for details about Submanifold Geometry.

As we explain in the introduction $\mathcal{M}$ can be identified with the set of 6×6 positive definite matrices. More precisely, by fixing the basis $e^1, e^2, e^3, e^4, e^5, e^6$ we identify $\mathcal{M}$ with the set of left cosets $\mathrm{GL}(6,\mathbb{R})/\mathrm{O}(6)$. The canonical metric $g_0 = \sum_i (e^i)^2$ is identified with the identity matrix I_6. The tangent space $\mathrm{T}_{g_0}\mathcal{M}$ is identified with the set of 6×6 symmetric matrices $\mathcal{S}_6$ and the symmetric Riemannian metric $\langle , \rangle$ at $\mathrm{T}_{g_0}\mathcal{M}$ is given by

$$\langle A, B\rangle := \mathrm{trace}(AB)\ .$$

Since $\mathrm{Aut}(\mathfrak{h}) \subset \mathrm{GL}(6,\mathbb{R})$ it follows that $\mathrm{Aut}(\mathfrak{h})$ acts by isometries on $\mathcal{M}$.

Here is the tangent space to the orbit $\mathrm{Aut}(\mathfrak{h})\mathrm{I}_6$ at I_6:

$$\mathrm{T}_{\mathrm{I}_6}\mathrm{Aut}(\mathfrak{h})\mathrm{I}_6 = \left\{ \begin{pmatrix} 2a_1 & 0 & c_1 & -c_2 & x_{11} & x_{12} \\ 0 & 2a_1 & c_2 & c_1 & x_{21} & x_{22} \\ c_1 & c_2 & 2d_1 & 0 & x_{31} & x_{32} \\ -c_2 & c_1 & 0 & 2d_1 & x_{41} & x_{42} \\ x_{11} & x_{21} & x_{31} & x_{41} & 2a_1+2d_1 & 0 \\ x_{12} & x_{22} & x_{32} & x_{42} & 0 & 2a_1+2d_1 \end{pmatrix} \right\} . \tag{7.1}$$

Indeed, this follows from the description of the Lie algebra of $\mathrm{Aut}(\mathfrak{h})$ given in equation (3.6) and the fact that the Killing vector field $X \in \mathfrak{gl}(6,\mathbb{R})$ is represented by $X^t + X$ as a vector at $\mathrm{T}_{g_0}\mathcal{M}$.

The isotropy group $\mathrm{Aut}(\mathfrak{h})_{\mathrm{I}_6}$ at I_6 is isomorphic to $\mathrm{U}(2)$. Here is its Lie algebra:

$$f = \begin{pmatrix} 0 & a_2 & b_1 & b_2 & 0 & 0 \\ -a_2 & 0 & -b_2 & b_1 & 0 & 0 \\ -b_1 & b_2 & 0 & d_2 & 0 & 0 \\ -b_2 & -b_1 & -d_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_2+d_2 \\ 0 & 0 & 0 & 0 & -a_2-d_2 & 0 \end{pmatrix} . \tag{7.2}$$

Here is the normal space $\nu_{\mathrm{I}_6}(\mathrm{Aut}(\mathfrak{h})\mathrm{I}_6)$:

(7.3)
$$\nu_{\mathrm{I}_6}(\mathrm{Aut}(\mathfrak{h})\mathrm{I}_6) = \left\{ \begin{pmatrix} n_1 & n_2 & m_1 & m_2 & 0 & 0 \\ n_2 & n_3 & m_2 & -m_1 & 0 & 0 \\ m_1 & m_2 & p_1 & p_2 & 0 & 0 \\ m_2 & -m_1 & p_2 & -p_1+n_1+n_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & y \\ 0 & 0 & 0 & 0 & y & -x-n_1-n_3 \end{pmatrix} \right\} .$$

The normal space $\nu_{\mathrm{I}_6}(\mathrm{Aut}(\mathfrak{h})\mathrm{I}_6)$ is invariant under the action of the isotropy group $\mathrm{Aut}(\mathfrak{h})_{\mathrm{I}_6}$ and splits into irreducible invariant subspaces as:

$$\nu_{\mathrm{I}_6}(\mathrm{Aut}(\mathfrak{h})\mathrm{I}_6) = \mathbb{R} \oplus \mathbb{R}^2 \oplus \mathbb{R}^6 \tag{7.4}$$

where

$$\mathbb{R} = \mathbb{R}.\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\mathbb{R}^2 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & y \\ 0 & 0 & 0 & 0 & y & -x \end{pmatrix} \right\}$$

$$\mathbb{R}^6 = \left\{ \begin{pmatrix} n_1 & n_2 & m_1 & m_2 & 0 & 0 \\ n_2 & -n_1 & m_2 & -m_1 & 0 & 0 \\ m_1 & m_2 & p_1 & p_2 & 0 & 0 \\ m_2 & -m_1 & p_2 & -p_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

7.1. **Non Polar action.** Let M be a connected complete Riemannian manifold and G a closed subgroup of isometries of M. A complete, embedded and closed submanifold Σ of M is called *section* if Σ intersects each orbit of G and is perpendicular to orbits at the intersection points. If there exists a section in M, then the action is called *polar.*

Theorem 7.1. *The action of* $\mathrm{Aut}(\mathfrak{h})$ *on* $\mathrm{GL}(6,\mathbb{R})/\mathrm{O}(6)$ *is not polar.*

Proof. By contradiction assume that the action of $\mathrm{Aut}(\mathfrak{h})$ on $\mathrm{GL}(6,\mathbb{R})/\mathrm{O}(6)$ is polar. Then by [4, Page 43, Proposition 3.2.2] the representation of the isotropy group $\mathrm{Aut}(\mathfrak{h})_{\mathrm{I}_6}$ at I_6 on the normal space is polar. That is to say, the U(2)-action on $\mathbb{R}\oplus\mathbb{R}^2\oplus\mathbb{R}^6$ described above is polar. Then [7, Theorem 4] imply that the U(2)-action in $\mathbb{R}^6$ is also polar. Observe that the U(2)-action in $\mathbb{R}^6$ is irreducible. Indeed, this representation can be regarded as a complex representation of SU(2) on $\mathbb{C}^3$. Since the SU(2)-action has not fixed point it follows that it is irreducible. Thus, the U(2)-action in $\mathbb{R}^6$ is irreducible. Now by Dadok's Theorem [7, Proposition 6] it follows that U(2)-action in $\mathbb{R}^6$ is orbit equivalent to a irreducible representation of a Riemannian symmetric space. Since U(2) has dimension 4 the only possibility is that the Riemannian symmetric space is of rank 2. Indeed, by the classification of irreducible Riemannian symmetric spaces it follows that in dimension 6 they are either of rank 1 or rank 2. Then the principal U(2)-orbits in $\mathbb{R}^6$ are 4-dimensional. Since the U(2)-action on the $\mathbb{R}^2$-factor is non trivial it follows from [7, Theorem 4, (ii)] that the principal orbits of the polar U(2)-action on $\mathbb{R}^2\oplus\mathbb{R}^6$ are of dimension greater than 4 which is a contradiction with the fact that U(2) has dimension 4. □

7.2. **Orbit types.** The goal of this subsection is to compute the isotropy group (up to conjugation) of each orbit of the action of $\mathrm{Aut}(\mathfrak{h})$ on $\mathrm{GL}(6,\mathbb{R})/\mathrm{O}(6)$.

As proved in Theorem 1.1 any $\mathrm{Aut}(\mathfrak{h})$-orbit has a representative of the form

$$g=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & r & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & s & 0 & 0\\ 0 & 0 & 0 & 0 & E & F\\ 0 & 0 & 0 & 0 & F & G \end{pmatrix}.$$

In order to compute the Lie algebra of the isotropy group at g we will write $g=\begin{pmatrix} D & 0\\ 0 & m\end{pmatrix}$ where $D=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & r & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & s\end{pmatrix}$ and $m=\begin{pmatrix} E & F\\ F & G\end{pmatrix}$.

Let $K=\begin{pmatrix} A & B\\ 0 & z\end{pmatrix}$ be vector in the Lie algebra of $\mathrm{Aut}(\mathfrak{h})$ as in equation (3.6). Then K is in the Lie algebra of the isotropy group at g if and only if the following conditions hold:

$$\begin{cases} A^tD+DA=0\,,\\ DB=0\\ z^tm+mz=0\end{cases} \tag{7.5}$$

A direct computation shows that the above conditions are equivalent to the following system

$$\begin{cases} B=0\,,\\ F(a_2+d_2)=0\,,\\ (E-G)(a_2+d_2)=0\,,\\ a_1=0\,,\\ a_2(1-r)=0\,,\\ b_1+c_1=0\,,\\ b_2-c_2s=0\,,\\ c_2-b_2r=0\,,\\ b_1r+c_1s=0\,,\\ d_1=0\,,\\ d_2(1-s)=0\end{cases} \tag{7.6}$$

By using Theorem 1.1 we get the following classification of isotropy types:

$$\begin{cases} r=s=1\Rightarrow \begin{cases} F=0, E=G\Rightarrow \mathrm{U}(2)\ ,\\ F\neq 0 \text{ or } E\neq 0\Rightarrow \mathrm{SU}(2)\ ,\end{cases} & ,\\ \\ rs\neq 1\Rightarrow \begin{cases} r=s\Rightarrow \mathrm{SO}(2)\ ,\\ r\neq s\Rightarrow \{Id\}\ ,\end{cases} \end{cases} \tag{7.7}$$

7.3. **A minimal orbit.** A submanifold $N \subset M$ of a Riemannian manifold is called *minimal* if its mean curvature vector H vanishes. The mean curvature vector is the trace of the second fundamental form α of the submanifold.

Here we show that the $\mathrm{Aut}(\mathfrak{h})$-orbit through I_6 is a minimal submanifold of $\mathrm{GL}(6,\mathbb{R})/\mathrm{O}(6)$.

Let $X \in \mathrm{T}_{\mathrm{I}_6}\mathrm{Aut}(\mathfrak{h})\mathrm{I}_6$ be a tangent vector and and $\xi \in \nu_{\mathrm{I}_6}(\mathrm{Aut}(\mathfrak{h})\mathrm{I}_6)$ be a normal vector. Then [3, Proposition 2.2.] implies

$$\langle \alpha(X,X), \xi \rangle = \langle [\xi, X]^*, X^* \rangle \tag{7.8}$$

where X^* indicates the Killing vector field induced by the vector X in the Lie algebra. In our case $X^* = X^t + X$.

Notice that I_6 represents the metric g used in [1, 2] so the following proposition shows that the metric g has also an interesting property inside the space of left-invariant metrics.

Proposition 7.2. *The orbit through* I_6 *of the action of* $\mathrm{Aut}(\mathfrak{h})$ *on* $\mathrm{GL}(6,\mathbb{R})/\mathrm{O}(6)$ *is a minimal submanifold of* $\mathrm{GL}(6,\mathbb{R})/\mathrm{O}(6)$.

Proof. Let H_0 be the mean curvature vector of the orbit $\mathrm{Aut}(\mathfrak{h}).\mathrm{I}_6$ at I_6. We are going to show that $\langle H_0, \xi^* \rangle = 0$ where $\xi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$

As follows from equation (7.1) the following 12 vectors are a (non orthonormal) basis of $\mathrm{T}_{\mathrm{I}_6}\mathrm{Aut}(\mathfrak{h})\mathrm{I}_6$:

$$A_1^* = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, D_1^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$C_1^* = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, C_2^* = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{11}^* = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \cdots, X_{42}^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \end{pmatrix}$$

An straightforward computation shows that

$$[\xi, A_1] = [\xi, D_1] = [\xi, C_1] = [\xi, C_2] = 0 \ .$$

So by using equation (7.8) we get

$$\begin{cases} \langle \alpha(A_1, A_1), \xi^* \rangle = 0 \,, \\ \langle \alpha(D_1, D_1), \xi^* \rangle = 0 \,, \\ \langle \alpha(C_1, C_1), \xi^* \rangle = 0 \,, \\ \langle \alpha(C_2, C_2), \xi^* \rangle = 0 \,, \end{cases}$$

Another simple computation shows that the matrices

$$[\xi, X_{11}], \cdots, [\xi, X_{42}]$$

are skew-symmetric. So by using equation (7.8) we get that the mean curvature vector H_0 has no component in the ξ^* direction.

Since the mean curvature vector H_0 of a orbit is invariant by the isotropy representation we get, according to the decomposition (7.4) of the normal space, that H_0 must be a multiple of ξ^*. Thus, $H_0 = 0$ and this completes the proof. □

Remark 7.3. *It is well-known that an 'isolated' orbit $G.p$ (i.e. there is not orbit of the same type around it* [10]*) of an isometry action is a minimal submanifold. Notice that the orbit* $\mathrm{Aut}(\mathfrak{h}).\mathrm{I}_6$ *through* I_6 *of the action of* $\mathrm{Aut}(\mathfrak{h})$ *on* $\mathrm{GL}(6,\mathbb{R})/\mathrm{O}(6)$ *is* **not isolated**.

8. Appendix: Polar irreducible representations of $\mathrm{U}(n)$

As a direct consequence of [8] we get the following result.

Proposition 8.1. *Let ρ be an irreducible representation of* $\mathrm{U}(n)$*. Assume that ρ is a polar representation.*

If ρ is faithful then ρ is one of the following representations of $\mathrm{U}(n)$*:*

(i) $\mathrm{U}(n) \cong S(U(n) \times U(1))$ *acting naturally on* $\mathbb{C}^n \otimes \mathbb{C}$,
(ii) $\mathrm{U}(n)$ *acting naturally on* $\Lambda^2(\mathbb{C}^n)$,
(iii) $\mathrm{U}(n)$ *acting naturally on* $S^2(\mathbb{C}^n)$.

If ρ is not faithful and ρ restricts to an irreducible representation of $\mathrm{SU}(n)$ *then ρ is one of the following representations of* $\mathrm{SU}(n)$*:*

(i) $\mathrm{SU}(n) \cong SU(n) \times SU(1))$ *acting naturally on* $\mathbb{C}^n \otimes \mathbb{C}$,
(ii) $\mathrm{SU}(n)$ *acting naturally on* $\Lambda^2(\mathbb{C}^n)$,

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