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# LINEAR ORDINARY DIFFERENTIAL EQUATIONS: REVISITING THE IMPULSIVE RESPONSE METHOD USING FACTORIZATION

**Abstract.** We present an approach to the impulsive response method for solving linear ordinary differential equations based on the factorization of the differential operator. In the case of constant coefficients this approach avoids the following more advanced methods: distribution theory, Laplace transform, linear systems, the general theory of linear equations with variable coefficients and variation of parameters. The case of variable coefficients is dealt with using the result of Mammana about the factorization of a real linear ordinary differential operator into a product of first-order (complex) factors, as well as a recent generalization of this result to the case of complex-valued coefficients.

## 1. Introduction

The aim of this paper is to revisit the impulsive response method for solving linear ordinary differential equations using the factorization of the differential operator into first-order factors.

Our purpose is two fold. On the one hand, we illustrate the advantages of this approach for finding a particular solution of the non-homogeneous equation as a generalized convolution integral. This is of course elementary in the case of constant coefficients. However, the approach by factorization does not seem to be well known in the case of variable coefficients, where an old result of Mammana comes into play.

On the other hand, we obtain a representative formula for the solutions of the homogeneous equation with variable coefficients in terms of derivatives of the impulsive response kernel. This formula generalizes a well-known formula for the case of constant coefficients (see [9, p. 139], or [4, formula (26) p. 82]).

Let us give a brief overview of the main results of this paper. Suppose $L$ is a linear ordinary differential operator factored in the form

$$L=\left(\frac{d}{dx}-\alpha_1(x)\right)\left(\frac{d}{dx}-\alpha_2(x)\right)\cdots\left(\frac{d}{dx}-\alpha_n(x)\right), \tag{1}$$

where $\alpha_1,\ldots,\alpha_n$ are suitable functions defined on a common interval $I$. Then one can solve the non-homogeneous equation

$$Ly=f(x), \tag{2}$$

with $f\in C^0(I)$, in the following way. Define the impulsive response kernel $g(x,t)=g_{\alpha_1\cdots\alpha_n}(x,t)$ on $I\times I$ recursively as follows: for $n=1$ set

$$g_\alpha(x,t)=e^{\int_t^x\alpha(r)\,dr},$$

for $n \geq 2$ set

$$g_{\alpha_1 \cdots \alpha_n}(x,t) = \int_t^x g_{\alpha_n}(x,s)\, g_{\alpha_1 \cdots \alpha_{n-1}}(s,t)\, ds.$$

Then the function

$$y(x) = \int_{x_0}^x g(x,t) f(t)\, dt, \qquad (x_0, x \in I) \tag{3}$$

is the unique solution of (2) with the initial conditions

$$y(x_0) = y'(x_0) = \cdots = y^{(n-1)}(x_0) = 0.$$

This can be proved by induction on $n$, using only Fubini's theorem for interchanging the order of integration in a double integral, and the formula for solving first-order linear equations.

By induction one can also prove that, for any $t \in I$, the function $x \mapsto g(x,t)$ is the unique solution of the homogeneous equation $Ly = 0$ with the initial conditions

$$y^{(j)}(t) = 0, \text{ for } 0 \leq j \leq n-2, \qquad y^{(n-1)}(t) = 1. \tag{4}$$

Moreover, under suitable regularity assumptions on the functions $\alpha_j$ $(1 \leq j \leq n)$, one can prove that the general solution of the homogeneous equation can be written as a linear combination of partial derivatives of the kernel $g(x,t)$ with respect to the variable $t$, namely

$$y(x) = c_0 g(x,t) + c_1 \frac{\partial g}{\partial t}(x,t) + \cdots + c_{n-1} \frac{\partial^{n-1} g}{\partial t^{n-1}}(x,t), \tag{5}$$

for any $t \in I$. In other words, the $n$ functions $x \mapsto \frac{\partial^j g}{\partial t^j}(x,t)$ $(0 \leq j \leq n-1)$ form a fundamental system of solutions of the homogeneous equation for any $t \in I$. The proof is again by induction on $n$.

Consider a linear constant-coefficient differential operator of order $n$, written in the usual form

$$L = \left(\frac{d}{dx}\right)^n + a_1 \left(\frac{d}{dx}\right)^{n-1} + \cdots + a_{n-1} \frac{d}{dx} + a_n,$$

where $a_1, \ldots, a_n \in \mathbb{C}$. Then we can factor $L$ in the form (1)

$$L = \left(\frac{d}{dx} - \lambda_1\right)\left(\frac{d}{dx} - \lambda_2\right) \cdots \left(\frac{d}{dx} - \lambda_n\right),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ are the roots of the characteristic polynomial. The kernel is computed to be $g(x,t) = g(x-t)$, where $g(x) = g(x,0)$ is the impulsive response, i.e., the function defined recursively by $g_\lambda(x) = e^{\lambda x}$ $(\lambda \in \mathbb{C})$, and

$$g_{\lambda_1 \cdots \lambda_n}(x) = \int_0^x g_{\lambda_n}(x-t)\, g_{\lambda_1 \cdots \lambda_{n-1}}(t)\, dt.$$

This is the unique solution of $Ly = 0$ with the initial conditions (4) at $t = 0$.

Consider finally a linear ordinary differential operator with variable coefficients

$$L = \left(\frac{d}{dx}\right)^n + a_1(x)\left(\frac{d}{dx}\right)^{n-1} + \cdots + a_{n-1}(x)\frac{d}{dx} + a_n(x),$$

where $a_1, \ldots, a_n$ are real- or complex-valued continuous functions on an interval $I$. In the real case, Mammana [7, 8] proved that $L$ can always be factored in the form (1), with (generally) complex-valued functions $\alpha_j$ such that $\alpha_j \in C^{j-1}(I, \mathbb{C})$ $(1 \leq j \leq n)$. Recently, the result of Mammana was generalized to the case of complex-valued coefficients [1]. We can then apply the previous results to this general case as well.

In this paper we present the material outlined above in the following order. We first discuss, in Section 2, the case of constant coefficients. In this case the factorization method avoids the use of more sophisticated methods, such as distribution theory, and is accessible to anyone with a basic knowledge of calculus and linear algebra. Moreover, this method provides an elementary proof of existence, uniqueness and extendability of the solutions of the initial value problem (homogeneous or not).

In Section 3 we consider the case of variable coefficients. We first briefly review the result of Mammana and its recent generalization to the complex case. Then we prove (3) and (5). The required regularity on the coefficients $a_j$ in order for the result (5) to hold is

$$a_j \in C^{n-j-1}(I) \quad (1 \leq j \leq n-1), \qquad a_n \in C^0(I).$$

We also give the general relation between the coefficients $c_j$ in (5) and the initial data $b_j = y^{(j)}(t)$. Finally, we give another proof of (5) using the relation between the kernel $g$ and any fixed fundamental system of solutions of the homogeneous equation.

## 2. The case of constant coefficients

Consider a linear constant-coefficient non-homogeneous equation of order $n$

$$Ly = y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1}y' + a_n y = f(x), \tag{6}$$

where $a_1, \ldots, a_n$ are real or complex constants, and $f$ is a real- or complex-valued continuous function in an interval $I$. The following result is well known (see the references below for proofs involving different methods).

THEOREM 1. *Let $g$ be the solution of the homogeneous equation $Ly = 0$ satisfying the initial conditions*

$$y(0) = y'(0) = \cdots y^{(n-2)}(0) = 0, \quad y^{(n-1)}(0) = 1. \tag{7}$$

*Then the function*

$$y(x) = \int_{x_0}^{x} g(x-t)\, f(t)\, dt \qquad (x_0, x \in I) \tag{8}$$

*solves* (6) *with the initial conditions*

$$y(x_0) = y'(x_0) = \cdots = y^{(n-1)}(x_0) = 0.$$

This may be verified by differentiation under the integral sign using the formula

$$\frac{d}{dx}\int_{x_0}^{x} F(x,t)\,dt = F(x,x) + \int_{x_0}^{x} \frac{\partial F}{\partial x}(x,t)\,dt. \tag{9}$$

However, this proof is not constructive, and the origin of formula (8) remains rather obscure for $n \geq 2$. Constructive proofs are possible, based on one of the following more advanced approaches: (i) distribution theory (see [9, Proposition 14 p. 138, and formula (III,2;70) p. 139)]); (ii) the Laplace transform (see [4, formula (28) p. 82]); (iii) linear systems (see [3, chapter 3]).

One can also use the general theory of linear equations with variable coefficients and the method of variation of parameters ([2], chapter 2). However within this approach, the occurrence of the particular solution as a convolution integral (i.e., formula (8)) is rather indirect, and appears only at the end of the theory (see [2, formula (10.3) p. 86, and exercise 4 p. 89]).

We present a constructive yet elementary proof based on the factorization of the differential operator $L$ into first-order factors, namely

$$\begin{aligned} L &= \left(\frac{d}{dx}\right)^n + a_1\left(\frac{d}{dx}\right)^{n-1} + \cdots + a_{n-1}\frac{d}{dx} + a_n \\ &= \left(\frac{d}{dx} - \lambda_1\right)\left(\frac{d}{dx} - \lambda_2\right)\cdots\left(\frac{d}{dx} - \lambda_n\right), \end{aligned} \tag{10}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ are the roots of the characteristic polynomial

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

(not necessarily distinct, each counted with its multiplicity). This proof also provides a recursive formula for calculating the function $g$. Moreover, it provides existence, uniqueness and extendability of the solutions of the initial value problem with trivial initial conditions at some point. The other ingredients of the proof are the theorem of Fubini, the formula for solving first-order linear equations, and induction.

THEOREM 2. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be $n$ complex numbers (not necessarily all distinct), let $L$ be the differential operator (10), and let $f \in C^0(I)$, $I$ an interval. Then the initial value problem*

$$\begin{cases} Ly = f(x) \\ y(x_0) = y'(x_0) = \cdots = y^{(n-1)}(x_0) = 0 \end{cases} \tag{11}$$

*has a unique solution, defined on the whole of $I$, and given by formula (8), where $g = g_{\lambda_1\cdots\lambda_n}$ is the function defined recursively as follows: for $n = 1$ we set $g_\lambda(x) = e^{\lambda x}$ ($\lambda \in \mathbb{C}$), for $n \geq 2$ we set*

$$g_{\lambda_1\cdots\lambda_n}(x) = \int_0^x g_{\lambda_n}(x-t)\, g_{\lambda_1\cdots\lambda_{n-1}}(t)\,dt. \tag{12}$$

*The function $g_{\lambda_1\cdots\lambda_n}$ is the unique solution of the homogeneous problem $Ly = 0$ with the initial conditions* (7).

*Proof.* We proceed by induction on $n$. The theorem holds for $n = 1$. Indeed the solution of the first-order problem

$$\begin{cases} y' - \lambda y = f(x) \\ y(x_0) = 0 \end{cases}$$

(with $\lambda \in \mathbb{C}$) is unique and given by

$$y(x) = \int_{x_0}^{x} e^{\lambda(x-t)} f(t)\, dt.$$

Assuming the theorem holds for $n-1$, let us prove it for $n$. Consider then the problem (11) with $L$ given by (10). Letting $h = \left(\frac{d}{dx} - \lambda_n\right) y$, it is easy to check that the function $h$ solves the problem

$$\begin{cases} \left(\dfrac{d}{dx} - \lambda_1\right)\left(\dfrac{d}{dx} - \lambda_2\right) \cdots \left(\dfrac{d}{dx} - \lambda_{n-1}\right) h = f(x) \\ h(x_0) = h'(x_0) = \cdots = h^{(n-2)}(x_0) = 0. \end{cases}$$

(The initial conditions follow from $h = y' - \lambda_n y$ by computing $h', h'', \ldots, h^{(n-2)}$ and setting $x = x_0$.) By the inductive hypothesis, we have

$$h(x) = \int_{x_0}^{x} g_{\lambda_1 \cdots \lambda_{n-1}}(x-t) f(t)\, dt. \tag{13}$$

Since $y$ solves

$$\begin{cases} y' - \lambda_n y = h(x) \\ y(x_0) = 0, \end{cases}$$

we have

$$y(x) = \int_{x_0}^{x} e^{\lambda_n (x-t)} h(t)\, dt.$$

Substituting (13) into this formula, we obtain

$$\begin{aligned} y(x) &= \int_{x_0}^{x} g_{\lambda_n}(x-t) \left( \int_{x_0}^{t} g_{\lambda_1 \cdots \lambda_{n-1}}(t-s) f(s)\, ds \right) dt \\ &= \int_{x_0}^{x} \left( \int_{s}^{x} g_{\lambda_n}(x-t) g_{\lambda_1 \cdots \lambda_{n-1}}(t-s)\, dt \right) f(s)\, ds \\ &= \int_{x_0}^{x} \left( \int_{0}^{x-s} g_{\lambda_n}(x-s-t) g_{\lambda_1 \cdots \lambda_{n-1}}(t)\, dt \right) f(s)\, ds \\ &= \int_{x_0}^{x} g_{\lambda_1 \cdots \lambda_n}(x-s) f(s)\, ds. \end{aligned}$$

We have interchanged the order of integration in the second line, substituted $t$ with $t+s$ in the third, and used (12) in the last. A similar proof by induction shows that $g_{\lambda_1 \cdots \lambda_n}$ is the unique solution of $Ly = 0$ with the initial conditions (7). □

If we take in particular $f = 0$, we get that the only solution of the homogeneous problem $Ly = 0$ with all vanishing initial data at $x = x_0$ is the zero function $y = 0$. By linearity, this implies the uniqueness of the initial value problem (homogeneous or not) with arbitrary initial data.

The function $g = g_{\lambda_1 \cdots \lambda_n}$ is called the *impulsive response* of the differential operator $L$. It can be computed in terms of the exponentials $e^{\lambda_j x}$ by the recursive formula (12). For example for $n = 2$, we have $g(x) = \int_0^x e^{\lambda_2 (x-t)} e^{\lambda_1 t}\, dt$, and we obtain:

(i) if $\lambda_1 \neq \lambda_2$ ($\Leftrightarrow\ \Delta = a_1^2 - 4a_2 \neq 0$) then

$$g(x) = \frac{1}{\lambda_1 - \lambda_2} \left( e^{\lambda_1 x} - e^{\lambda_2 x} \right);$$

(ii) if $\lambda_1 = \lambda_2$ ($\Leftrightarrow\ \Delta = 0$) then

$$g(x) = x e^{\lambda_1 x}.$$

If $L$ has real coefficients and $\Delta < 0$, then $\lambda_{1,2} = \alpha \pm i\beta$ with $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$, and we get

$$g(x) = \frac{1}{\beta} e^{\alpha x} \sin(\beta x). \tag{14}$$

For generic $n$, if $\lambda_i \neq \lambda_j$ for $i \neq j$ (all distinct roots), one gets

$$g(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x},$$

where

$$c_j = \frac{1}{\prod_{i \neq j} (\lambda_j - \lambda_i)} \qquad (1 \leq j \leq n).$$

If $\lambda_1 = \lambda_2 = \cdots = \lambda_n$, then

$$g(x) = \frac{1}{(n-1)!} x^{n-1} e^{\lambda_1 x}.$$

In the general case one can prove by induction on $k$ that if $\lambda_1, \ldots, \lambda_k$ are the *distinct* roots of $p(\lambda)$, of multiplicities $m_1, \ldots, m_k$, then there exist polynomials $G_1, \ldots, G_k$, of degrees $m_1 - 1, \ldots, m_k - 1$, such that

$$g(x) = \sum_{j=1}^{k} G_j(x) e^{\lambda_j x}.$$

A recursive formula for calculating the polynomials $G_j$ for $k$ roots in terms of those for $k-1$ roots can easily be derived. For example for two distinct roots $\lambda_1, \lambda_2$, of multiplicities $m_1, m_2$, we find

$$G_1(x) = \sum_{r=0}^{m_1 - 1} \frac{(-1)^{m_1 - 1 - r}}{r!} \begin{pmatrix} m_1 + m_2 - r - 2 \\ m_2 - 1 \end{pmatrix} \frac{x^r}{(\lambda_1 - \lambda_2)^{m_1 + m_2 - r - 1}},$$

and $G_2$ is obtained from $G_1$ by interchanging $\lambda_1 \leftrightarrow \lambda_2$ and $m_1 \leftrightarrow m_2$.

Alternatively, one can use the formula for the polynomials $G_j$ based on the partial fraction expansion of $1/p(\lambda)$ (see [4, formula (21) p. 81], or [9, pp. 141–142]).

The function $g$ also provides a simple formula for the general solution of the homogeneous equation. Indeed, one can easily prove by induction on $n$ that the general solution of $Ly = 0$ can be written as

$$y(x) = \sum_{j=0}^{n-1} c_j g^{(j)}(x) \qquad (c_j \in \mathbb{C}). \tag{15}$$

In other words, the $n$ functions $g, g', g'', \ldots, g^{(n-1)}$ are linearly independent solutions of the homogeneous equation and form a basis of the vector space of its solutions (a fundamental system of solutions). If $L$ has real coefficients then $g$ is real, and the general real solution of $Ly = 0$ is given by (15) with $c_j \in \mathbb{R}$.

The relationship between the coefficients $c_j$ in (15) and the initial data at the point $x = 0$, $b_j = y^{(j)}(0)$ $(0 \le j \le n-1)$, is given by

$$\left\{\begin{array}{rcl} c_0 &=& b_{n-1} + a_1 b_{n-2} + \cdots + a_{n-2} b_1 + a_{n-1} b_0 \\ c_1 &=& b_{n-2} + a_1 b_{n-3} + \cdots + a_{n-3} b_1 + a_{n-2} b_0 \\ & \vdots & \\ c_{n-3} &=& b_2 + a_1 b_1 + a_2 b_0 \\ c_{n-2} &=& b_1 + a_1 b_0 \\ c_{n-1} &=& b_0. \end{array}\right. \tag{16}$$

This formula is easily proved from (15) by computing $y', y'', \ldots, y^{(n-1)}$ and taking $x = 0$. One gets a linear system that can be solved recursively to give (16).

If we impose the initial conditions at any point $x_0$ we can use, in place of (15), the translated formula

$$y(x) = \sum_{j=0}^{n-1} c_j g^{(j)}(x - x_0). \tag{17}$$

This follows from the fact that $L$ has constant coefficients and is therefore invariant under translations, i.e., $L(\tau_{x_0} y) = \tau_{x_0}(Ly)$, where $\tau_{x_0} y(x) = y(x - x_0)$. The relation between the coefficients $c_j$ and $b_j = y^{(j)}(x_0)$ is then the same as before.

### 3. The case of variable coefficients

Consider the linear non-homogeneous differential equation of order $n$

$$Ly = y^{(n)} + a_1(x) y^{(n-1)} + a_2(x) y^{(n-2)} + \cdots + a_{n-1}(x) y' + a_n(x) y = f(x), \tag{18}$$

where $a_1, \ldots, a_n$, $f$ are real- or complex-valued continuous functions in an interval $I$. The following result generalizes Theorem 1.

THEOREM 3. *For any $x_0 \in I$, let $x \mapsto g_{x_0}(x)$ be the solution (which exists, unique and defined on the whole of $I$) of the homogeneous equation $Ly = 0$ with the initial conditions*

$$y(x_0) = y'(x_0) = \cdots = y^{(n-2)}(x_0) = 0, \quad y^{(n-1)}(x_0) = 1.$$

*Define*

$$g : I \times I \to \mathbb{C}, \quad g(x,t) = g_t(x) \qquad (x, t \in I).$$

*Then the function*

$$y(x) = \int_{x_0}^{x} g(x,t) f(t)\, dt \qquad (x_0, x \in I)$$

*solves* (18) *with the initial conditions*

$$y(x_0) = y'(x_0) = \cdots = y^{(n-1)}(x_0) = 0.$$

The proof by direct verification (using (9)) is similar to that of Theorem 1. The analogue of (7) is given by the conditions (valid for any $t \in I$)

$$\left[\left(\frac{\partial}{\partial x}\right)^j g(x,t)\right]_{x=t} = 0 \ \text{ for } \ 0 \le j \le n-2, \qquad \left[\left(\frac{\partial}{\partial x}\right)^{n-1} g(x,t)\right]_{x=t} = 1. \tag{19}$$

We will now give a constructive proof of this result analogous to the one given in the case of constant coefficients.

Suppose first the $a_j$ are real-valued. It was proved in [7] (for $n = 2$) and in [8] (general case) that a linear ordinary differential operator

$$L = \left(\frac{d}{dx}\right)^n + a_1(x)\left(\frac{d}{dx}\right)^{n-1} + \cdots + a_{n-1}(x)\frac{d}{dx} + a_n(x), \tag{20}$$

with continuous real-valued coefficients $a_j \in C^0(I)$, can always be decomposed as a product (composition) of first-order operators

$$L = \left(\frac{d}{dx} - \alpha_1(x)\right)\left(\frac{d}{dx} - \alpha_2(x)\right)\cdots\left(\frac{d}{dx} - \alpha_n(x)\right), \tag{21}$$

where the functions $\alpha_1, \ldots, \alpha_n$ are in general complex-valued and continuous in the entire interval $I$, and such that $\alpha_j \in C^{j-1}(I, \mathbb{C})$ $(1 \le j \le n)$. (See [8, Teorema generale p. 207].)

A *local* factorization of the form (21) was already known (see, for instance, [6, p. 121]). The new point established in [7, 8] is that one can always find a *global* decomposition of the form (21) (i.e., valid on the whole of the interval $I$) if one allows the $\alpha_j$ to be complex-valued. The proof is based on the existence of a fundamental system whose complete chain of Wronskians is never zero in $I$. More specifically, let $z_1, z_2, \ldots, z_n$ be a fundamental system of solutions of the homogeneous equation $Ly = 0$ ($L$ given by (20)) with the property that the sequence of Wronskian determinants

$$w_0 = 1,\ w_1 = z_1,\ w_2 = \begin{vmatrix} z_1 & z_2 \\ z_1' & z_2' \end{vmatrix},\ \dots\ ,\ w_j = \begin{vmatrix} z_1 & z_2 & \cdots & z_j \\ z_1' & z_2' & \cdots & z_j' \\ \vdots & \vdots & & \vdots \\ z_1^{(j-1)} & z_2^{(j-1)} & \cdots & z_j^{(j-1)} \end{vmatrix}$$

(with $1 \le j \le n$) never vanishes on the interval $I$. A generic fundamental system does not have this property. Recall that $z_1, \dots, z_n$ are linearly independent solutions of $Ly = 0$ if and only if their Wronskian $w_n$ is nonzero at some point of $I$, in which case $w_n$ is nowhere zero on $I$. However, the lower-dimensional Wronskians $w_j$, $j < n$, can vanish in $I$. Mammana proves that a fundamental system with $w_j(x) \neq 0$, for all $x \in I$ and for all $j$, always exists, with $z_1$ (generally) complex-valued. The functions $\alpha_j$ in (21) are then the logarithmic derivative of ratios of Wronskians, namely

$$\alpha_j = \frac{d}{dx} \log \frac{w_{n-j+1}}{w_{n-j}} \quad (1 \le j \le n).$$

For example take $n = 2$, and let $y_1, y_2$ be two linearly independent real solutions of $Ly = y'' + a_1(x)y' + a_2(x)y = 0$ ($a_1, a_2$ real-valued). Consider the complex-valued function

$$\beta = \frac{y_1' + iy_2'}{y_1 + iy_2}.$$

This is well defined and continuous in $I$. Indeed if we had $y_1(x_0) = y_2(x_0) = 0$ for some $x_0 \in I$, then the Wronskian of $y_1, y_2$ would vanish at $x_0$. Moreover $\beta$ satisfies the Riccati equation in the interval $I$

$$\beta' + \beta^2 + a_1\beta + a_2 = 0. \tag{22}$$

It is then easy to check that

$$L = \left(\frac{d}{dx}\right)^2 + a_1\frac{d}{dx} + a_2 = \left(\frac{d}{dx} + \beta + a_1\right)\left(\frac{d}{dx} - \beta\right). \tag{23}$$

In general if $\beta$ satisfies (22) then (23) holds, and conversely. In turn this is equivalent to the existence of a solution $\alpha$ of $Ly = 0$ that vanishes nowhere in $I$. The relationship between $\alpha$ and $\beta$ is then $\beta = \alpha'/\alpha$ and $\alpha = e^{\int \beta\, dx}$. There always exists such a *complex-valued* solution, namely $\alpha = y_1 + iy_2$ as above. On the other hand, in general, there is no real-valued solution with this property. Indeed, the existence of a real-valued solution $\alpha$ with $\alpha(x) \neq 0$, for all $x \in I$, is equivalent (for $I$ open or compact) to the fact that $L$ is disconjugate on $I$, i.e., every (non trivial) real solution of $Ly = 0$ has at most one zero in $I$ (see [5, Corollary 6.1 p. 351]). In that case we get a factorization of the form (23) with real factors. In any case, if $\beta$ is a complex function satisfying (22), we get

$$\operatorname{Im}\beta(x) = \operatorname{Im}\beta(x_0)\, e^{-\int_{x_0}^{x} (2\operatorname{Re}\beta(t) + a_1(t))\, dt} \qquad (x_0, x \in I).$$

Thus the imaginary part of $\beta$ either vanishes identically on $I$, or it is always nonzero there. In the second case, the general solution of $Ly = 0$ can be written in the form [7]

$$y(x) = e^{\eta(x)}\,(c_1 \cos\omega(x) + c_2 \sin\omega(x)) \qquad (c_1, c_2 \in \mathbb{R}),$$

where

$$\eta(x) = \int_{x_0}^{x} \operatorname{Re}\beta(t)\,dt, \quad \omega(x) = \int_{x_0}^{x} \operatorname{Im}\beta(t)\,dt.$$

The function $\omega$ is strictly monotone in $I$. Moreover, the kernel $g(x,x_0)$ is given by

$$g(x,x_0) = \frac{1}{\operatorname{Im}\beta(x_0)} e^{\eta(x)} \sin\omega(x).$$

This is similar to the constant-coefficient case with complex conjugate roots of $p(\lambda)$ (cf. (14)).

Now let us go back to $L$ given by (20), and suppose the coefficients $a_j$ are complex-valued. It was proved in [1] that any linear ordinary differential operator (20) with $a_j \in C^0(I,\mathbb{C})$ admits a factorization of the form (21), with $\alpha_j \in C^{j-1}(I,\mathbb{C})$ $(1 \le j \le n)$. Again the proof consists in establishing the existence of a fundamental system with a nowhere-vanishing complete chain of Wronskians.

The following result generalizes Theorem 2 and implies Theorem 3. It also provides a recursive formula for calculating $g$ if the factorization (21) of $L$ is known.

THEOREM 4. *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be $n$ functions such that $\alpha_j \in C^{j-1}(I,\mathbb{C})$ (for $1 \le j \le n$), and let $L$ be the differential operator* (21)*. Then the initial value problem*

$$\begin{cases} Ly = f(x) \\ y(x_0) = y'(x_0) = \cdots = y^{(n-1)}(x_0) = 0 \end{cases}$$

*has a unique solution, defined on the whole of $I$, and given by the formula*

$$y(x) = \int_{x_0}^{x} g(x,t)\,f(t)\,dt, \tag{24}$$

*where $g = g_{\alpha_1 \cdots \alpha_n}$ is the function on $I \times I$ defined recursively as follows: for $n = 1$ we set*

$$g_\alpha(x,t) = e^{\int_t^x \alpha(s)\,ds},$$

*for $n \ge 2$ we set*

$$g_{\alpha_1 \cdots \alpha_n}(x,t) = \int_t^x g_{\alpha_n}(x,s)\,g_{\alpha_1 \cdots \alpha_{n-1}}(s,t)\,ds. \tag{25}$$

*The function $x \mapsto g_{\alpha_1 \cdots \alpha_n}(x,t)$ is, for any $t \in I$, the unique solution of the homogeneous problem $Ly = 0$ with the initial conditions* (19)*.*

*Proof.* The proof by induction on $n$ is entirely analogous to that of Theorem 2. The result holds for $n = 1$ since the unique solution of

$$\begin{cases} y' - \alpha(x)y = f(x) \\ y(x_0) = 0 \end{cases}$$

is

$$y(x) = \int_{x_0}^{x} e^{\int_t^x \alpha(s)\,ds} f(t)\,dt = \int_{x_0}^{x} g_\alpha(x,t) f(t)\,dt.$$

Assuming the theorem holds for $n-1$, one finds that the function $h = \left(\frac{d}{dx} - \alpha_n(x)\right) y$ is given by

$$h(x) = \int_{x_0}^{x} g_{\alpha_1 \cdots \alpha_{n-1}}(x,t) f(t)\,dt.$$

Thus

$$\begin{aligned} y(x) &= \int_{x_0}^{x} g_{\alpha_n}(x,t) h(t)\,dt \\ &= \int_{x_0}^{x} g_{\alpha_n}(x,t) \left( \int_{x_0}^{t} g_{\alpha_1 \cdots \alpha_{n-1}}(t,s) f(s)\,ds \right) dt \\ &= \int_{x_0}^{x} \left( \int_{s}^{x} g_{\alpha_n}(x,t) g_{\alpha_1 \cdots \alpha_{n-1}}(t,s)\,dt \right) f(s)\,ds \\ &= \int_{x_0}^{x} g_{\alpha_1 \cdots \alpha_n}(x,s) f(s)\,ds. \end{aligned}$$

The last part is proved again by induction in a similar way. □

The function $g(x,t)$ may be called the *impulsive response kernel* of $L$. If $g$ is known, then one can also find the general solution of the homogeneous equation as follows. Observe that in the case of constant coefficients we have

$$g(x,t) = g(x-t), \tag{26}$$

where $g(x) = g(x,0)$ is the impulsive response. This identity follows from the invariance under translations of the differential operator $L$.

In the general case this invariance breaks down. The derivatives $\frac{\partial^j g}{\partial x^j}$ $(j \geq 1)$ no longer satisfy the homogeneous equation, and (15) does not generalize in its present form. Consider, however, the translated formula (17) and notice that, using (26), we can rewrite the derivative $g^{(j)}(x-t)$ as a partial derivative of the kernel $g(x,t)$ with respect to the second variable $t$, namely

$$g^{(j)}(x-t) = (-1)^j \frac{\partial^j g}{\partial t^j}(x,t).$$

In this form, formula (17) does indeed generalize to the case of variable coefficients, under suitable assumptions of regularity on the functions $a_j$ $(1 \leq j \leq n)$.

THEOREM 5. *Let the coefficients of $L$ in* (20) *satisfy $a_j \in C^{n-j-1}(I,\mathbb{C})$ $(1 \leq j \leq n-1)$, $a_n \in C^0(I,\mathbb{C})$. Then the general solution of the homogeneous equation $Ly = 0$ can be written in the form*

$$y(x) = \sum_{j=0}^{n-1} \tilde{c}_j (-1)^j \frac{\partial^j g}{\partial t^j}(x,t) \qquad (\tilde{c}_j \in \mathbb{C}) \tag{27}$$

*for any $t \in I$. In other words, the n functions*

$$x \mapsto g(x,t),\ -\frac{\partial g}{\partial t}(x,t),\ \ldots,(-1)^{n-1}\frac{\partial^{n-1} g}{\partial t^{n-1}}(x,t) \tag{28}$$

*are a fundamental system of solutions of $Ly = 0$ for any $t \in I$.*

*Proof.* Let $L$ be factored according to (21). By equating (20) to (21), it can be verified that the conditions $a_j \in C^{n-j-1}$ $(1 \le j \le n-1)$, $a_n \in C^0$, imply

$$\alpha_n \in C^{n-1},\quad \alpha_j \in C^{n-2},\ \ \forall j = 1,\ldots,n-1. \tag{29}$$

For example for $n = 2$ we have

$$\left(\frac{d}{dx} - \alpha_1\right)\left(\frac{d}{dx} - \alpha_2\right) = \left(\frac{d}{dx}\right)^2 - (\alpha_1 + \alpha_2)\frac{d}{dx} + \alpha_1\alpha_2 - \alpha_2'.$$

Equating this to $\left(\frac{d}{dx}\right)^2 + a_1\frac{d}{dx} + a_2$ gives

$$\begin{cases} a_1 = -(\alpha_1 + \alpha_2) \\ a_2 = \alpha_1\alpha_2 - \alpha_2'. \end{cases}$$

If $a_1, a_2 \in C^0$, then we get $\alpha_2 \in C^1$ from the second equation and $\alpha_1 \in C^0$ from the first. For $n = 3$ we obtain

$$\begin{cases} a_1 = -(\alpha_1 + \alpha_2 + \alpha_3) \\ a_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 - \alpha_2' - 2\alpha_3' \\ a_3 = -\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_3' + \alpha_2\alpha_3' + \alpha_3\alpha_2' - \alpha_3''. \end{cases}$$

If $a_1 \in C^1$ and $a_2, a_3 \in C^0$, then we get $\alpha_3 \in C^2$ from the third equation, $\alpha_2 \in C^1$ from the second, and $\alpha_1 \in C^1$ from the first. In general, the coefficient $a_j$ contains the term $\alpha_j^{(j-1)}$ $(1 \le j \le n)$. Thus $a_n \in C^0$ implies $\alpha_n \in C^{n-1}$, and $a_j \in C^{n-j-1}$ implies $\alpha_j \in C^{n-2}$ $(1 \le j \le n-1)$. We also observe that under the conditions (29) the kernel $g_{\alpha_1\ldots\alpha_n}(x,t)$ has $n-1$ partial derivatives with respect to $t$. In fact from (25) one easily proves by induction on $n$ that for all $n \ge 2$,

$$\frac{\partial}{\partial t} g_{\alpha_1\cdots\alpha_n}(x,t) = -g_{\alpha_2\cdots\alpha_n}(x,t) - \alpha_1(t)\, g_{\alpha_1\cdots\alpha_n}(x,t). \tag{30}$$

Taking more derivatives with respect to $t$ and iterating, shows that for $0 \le k \le n-1$, $(\frac{\partial}{\partial t})^k g_{\alpha_1\ldots\alpha_n}(x,t)$ is a linear combination of

$$g_{\alpha_{k+1}\cdots\alpha_n}(x,t),\ g_{\alpha_k\alpha_{k+1}\cdots\alpha_n}(x,t),\ \ldots,\ g_{\alpha_1\cdots\alpha_n}(x,t), \tag{31}$$

with coefficients depending only on $t$ and involving the derivatives of the functions $\alpha_j$. The least regular coefficient is that of $g_{\alpha_1\ldots\alpha_n}(x,t)$, it contains $\alpha_1^{(k-1)}$ so it is of class

$C^{n-k-1}$ if (29) holds. It follows that $(\frac{\partial}{\partial t})^{n-1} g_{\alpha_1 \dots \alpha_n}$ exists continuous on $I \times I$, and moreover $g_{\alpha_1 \dots \alpha_n} \in C^{n-1}(I \times I)$ if (29) holds. For example for $n = 2,3,4$ we get

$$\begin{aligned}
\frac{\partial}{\partial t} g_{\alpha_1 \alpha_2}(x,t) &= -g_{\alpha_2}(x,t) - \alpha_1(t)\, g_{\alpha_1 \alpha_2}(x,t), \\
\left(\frac{\partial}{\partial t}\right)^2 g_{\alpha_1 \alpha_2 \alpha_3}(x,t) &= g_{\alpha_3}(x,t) + (\alpha_1 + \alpha_2)(t)\, g_{\alpha_2 \alpha_3}(x,t) \\
&\quad + \left(\alpha_1^2 - \alpha_1'\right)(t)\, g_{\alpha_1 \alpha_2 \alpha_3}(x,t), \\
\left(\frac{\partial}{\partial t}\right)^3 g_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}(x,t) &= -g_{\alpha_4}(x,t) - (\alpha_1 + \alpha_2 + \alpha_3)(t)\, g_{\alpha_3 \alpha_4}(x,t) \\
&\quad + \left(2\alpha_1' + \alpha_2' - \alpha_1 \alpha_2 - \alpha_1^2 - \alpha_2^2\right)(t)\, g_{\alpha_2 \alpha_3 \alpha_4}(x,t) \\
&\quad + \left(3\alpha_1 \alpha_1' - \alpha_1'' - \alpha_1^3\right)(t)\, g_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}(x,t).
\end{aligned}$$

It is also clear that the partial derivatives $y = (\frac{\partial}{\partial t})^k g_{\alpha_1 \dots \alpha_n}$ $(0 \le k \le n-1)$ solve the homogeneous equation $Ly = 0$ in the variable $x$. (Just permute the derivatives with respect to $x$ in (20) with those with respect to $t$. This is permissible as one can prove from (30) and (29) that the mixed derivatives $(\frac{\partial}{\partial x})^j (\frac{\partial}{\partial t})^k g_{\alpha_1 \dots \alpha_n}$ exist continuous on $I \times I$ for all $j,k$ with $0 \le j \le n$, $0 \le k \le n-1$. Alternatively, observe that the operator $L$ in (21) annihilates each term in (31), as easily seen.)

We will now prove the following result: let $L$ be given by (21) with $\alpha_j$ complex-valued and satisfying (29). Then the general solution of $Ly = 0$ is given by (27). This clearly implies the theorem.

We proceed by induction on $n$. For $n = 1$ the solution of $\left(\frac{d}{dx} - \alpha(x)\right) y = 0$ is

$$y(x) = c\, e^{\int_t^x \alpha(s)\, ds} = c\, g_\alpha(x,t) \qquad (c = y(t) \in \mathbb{C},\ t \in I \text{ fixed}).$$

Suppose the result holds for $n-1$. Letting $h = \left(\frac{d}{dx} - \alpha_n(x)\right) y$ in $Ly = 0$ gives

$$\left(\frac{d}{dx} - \alpha_1(x)\right) \cdots \left(\frac{d}{dx} - \alpha_{n-1}(x)\right) h = 0.$$

By the inductive hypothesis, we have

$$h(x) = \sum_{k=0}^{n-2} d_k (-1)^k \left(\frac{\partial}{\partial t}\right)^k g_{\alpha_1 \cdots \alpha_{n-1}}(x,t) \qquad (d_k \in \mathbb{C}).$$

Since $y$ solves $y' - \alpha_n y = h$, we get

$$y(x) = c\, g_{\alpha_n}(x,t) + \sum_{k=0}^{n-2} d_k (-1)^k \int_t^x g_{\alpha_n}(x,s) \left(\frac{\partial}{\partial t}\right)^k g_{\alpha_1 \cdots \alpha_{n-1}}(s,t)\, ds, \tag{32}$$

where $c = y(t)$. From (25) one easily proves by induction the following formulas for the kernel $g = g_{\alpha_1 \dots \alpha_n}$ (analogous to (19)):

$$\left[\left(\frac{\partial}{\partial t}\right)^k g(x,t)\right]_{t=x} = 0 \ \text{ for } \ 0 \le k \le n-2, \quad \left[\left(\frac{\partial}{\partial t}\right)^{n-1} g(x,t)\right]_{t=x} = (-1)^{n-1}. \tag{33}$$

Using these (with $n-1$ in place of $n$) we also find from (25)

$$\left(\frac{\partial}{\partial t}\right)^k g_{\alpha_1\cdots\alpha_n}(x,t) = \int_t^x g_{\alpha_n}(x,s)\left(\frac{\partial}{\partial t}\right)^k g_{\alpha_1\ldots\alpha_{n-1}}(s,t)\,ds, \qquad \text{for } 0\le k\le n-2.$$

We can then rewrite (32) as

$$y(x) = c\,g_{\alpha_n}(x,t) + \sum_{k=0}^{n-2} d_k(-1)^k\left(\frac{\partial}{\partial t}\right)^k g_{\alpha_1\ldots\alpha_n}(x,t).$$

To complete the proof we need to show that the term $g_{\alpha_n}(x,t)$ can be written as a linear combination of derivatives $\left(\frac{\partial}{\partial t}\right)^k g_{\alpha_1\ldots\alpha_n}(x,t)$ $(0\le k\le n-1)$, with coefficients depending only on $t$. This follows from the formula

$$(-1)^{n-1}g_{\alpha_n}(x,t) = \left(\frac{d}{dt}+\alpha_{n-1}(t)\right)\cdots\left(\frac{d}{dt}+\alpha_1(t)\right)g_{\alpha_1\cdots\alpha_n}(x,t), \quad \forall n\ge 2, \tag{34}$$

which is easily proved by induction on $n$, or equivalently, by iterating (30) rewritten as

$$\left(\frac{d}{dt}+\alpha_1(t)\right)g_{\alpha_1\cdots\alpha_n}(x,t) = -g_{\alpha_2\cdots\alpha_n}(x,t).$$

This concludes the proof of the theorem. □

It is possible to solve for the coefficients $\tilde{c}_j$ in (27) in terms of the initial data at the point $x=t$, $b_j = y^{(j)}(t)$ $(0\le j\le n-1)$. The result is as follows:

$$\tilde{c}_j = \sum_{k=j}^{n-1}(-1)^{k-j}\binom{k}{j}c_k^{(k-j)}(t) \qquad (0\le j\le n-1), \tag{35}$$

where the functions $x\mapsto c_j(x)$ are given by (16) with $a_j(x)$ in place of $a_j$, namely

$$c_j(x) = \sum_{r=0}^{n-j-1} a_r(x)b_{n-r-j-1} \qquad (a_0\equiv 1,\ 0\le j\le n-1).$$

Formula (35) can be proved by induction on $n$, though it is most easily proved using distribution theory or variation of parameters (see below). For example for $n=2,3,4$, we get

$$n=2 \ \Rightarrow\ \begin{cases} \tilde{c}_0 = b_1 + a_1(t)b_0 \\ \tilde{c}_1 = b_0, \end{cases}$$

$$n=3 \ \Rightarrow\ \begin{cases} \tilde{c}_0 = b_2 + a_1(t)b_1 + (a_2(t)-a_1'(t))b_0 \\ \tilde{c}_1 = b_1 + a_1(t)b_0 \\ \tilde{c}_2 = b_0, \end{cases} \tag{36}$$

$$n=4 \ \Rightarrow\ \begin{cases} \tilde{c}_0 = b_3 + a_1(t)b_2 + (a_2(t)-a_1'(t))b_1 + (a_3(t)-a_2'(t)+a_1''(t))b_0 \\ \tilde{c}_1 = b_2 + a_1(t)b_1 + (a_2(t)-2a_1'(t))b_0 \\ \tilde{c}_2 = b_1 + a_1(t)b_0 \\ \tilde{c}_3 = b_0. \end{cases}$$

Note that the derivatives of the $a_j$ start appearing in the $\tilde{c}_k$ as soon as $n \geq 3$. Also note that the conditions $a_j \in C^{n-j-1}$ $(1 \leq j \leq n-1)$, $a_n \in C^0$ in Theorem 5 are the minimal ones under which formula (35) makes sense. If we require the stronger conditions $a_j \in C^{n-j}$ $(1 \leq j \leq n)$, then $\alpha_j \in C^{n-1}$ $(\forall j = 1, \ldots, n)$, and the kernel $g_{\alpha_1 \cdots \alpha_n}(x,t)$ has $n$ partial derivatives with respect to $t$ (rather than $n-1$). Moreover $g_{\alpha_1 \ldots \alpha_n} \in C^n(I \times I)$, and $g_{\alpha_1 \cdots \alpha_n}$ satisfies the following adjoint equation in the variable $t$:

$$\left(\frac{d}{dt} + \alpha_n(t)\right)\left(\frac{d}{dt} + \alpha_{n-1}(t)\right) \cdots \left(\frac{d}{dt} + \alpha_1(t)\right) g_{\alpha_1 \cdots \alpha_n}(x,t) = 0,$$

with the initial conditions (33). (Just apply $\left(\frac{d}{dt} + \alpha_n(t)\right)$ to both sides of (34).)

REMARK 1. In order to make contact with the variation of parameters method, we observe the following relation between the kernel $g$ and any given fundamental system of solutions of the homogeneous equation $y_1, y_2, \ldots, y_n$:

$$g(x,t) = y_1(x)(W(t)^{-1})_{1n} + y_2(x)(W(t)^{-1})_{2n} + \cdots + y_n(x)(W(t)^{-1})_{nn}, \tag{37}$$

where $W(t)$ is the Wronskian matrix of $y_1, \ldots, y_n$, and $W(t)^{-1}$ is the inverse of $W(t)$. To prove (37) we just expand $g(\cdot\,,t)$ in terms of the $y_j$ and determine the coefficients by imposing the initial conditions (19). Recall that if $w(t) = \det W(t)$, then $(W(t)^{-1})_{jn} = w_j(t)/w(t)$ $(1 \leq j \leq n)$, where $w_j(t)$ is the determinant obtained from $w(t)$ by replacing the $j$-th column by $0, 0, \ldots, 0, 1$. Thus (24)–(37) agrees with [2, eq. (6.2) p. 123], where the variation of constants method was used, or with [3, eq. (6.15) p. 87], where linear systems were used instead. (See also [2, exercise 6 p. 125], and [3, problem 21 p. 101].)

Using (37) we can give another proof of Theorem 5 as follows. In order to show that the functions in (28) are linearly independent for all $t \in I$, it is enough to verify that their Wronskian determinant

$$\tilde{w}(x,t) = \begin{vmatrix} g & -\partial_t g & \partial_t^2 g & \cdots & (-1)^{n-1}\partial_t^{n-1} g \\ \partial_x g & -\partial_x \partial_t g & \partial_x \partial_t^2 g & \cdots & (-1)^{n-1}\partial_x \partial_t^{n-1} g \\ \vdots & \vdots & \vdots & & \vdots \\ \partial_x^{n-1} g & -\partial_x^{n-1}\partial_t g & \partial_x^{n-1}\partial_t^2 g & \cdots & (-1)^{n-1}\partial_x^{n-1}\partial_t^{n-1} g \end{vmatrix} (x,t)$$

is different from zero at some point $x \in I$, for example at $x = t$. (We are using here the notation $\partial_z = \partial/\partial z$.)

We first rewrite (37) in the following form (see [2, exercise 6, p. 125]):

$$g(x,t) = \frac{1}{w(t)} \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \cdots & y_n^{(n-2)}(t) \\ y_1(x) & y_2(x) & \cdots & y_n(x) \end{vmatrix}.$$

Using this, the formula $w'(t) = -a_1(t)w(t)$ ([2, p. 115]), and the rule for differentiating a determinant (cf. [2, p. 114]), it is easy to prove that at $x = t$ the Wronskian matrix

$$\tilde{W}(x,t)_{jk} = (-1)^k \partial_x^j \partial_t^k g(x,t) \qquad (0 \leq j,k \leq n-1)$$

has zero entries above the anti-diagonal, and all entries on this diagonal equal to 1. Thus

$$\tilde{w}(t,t) = \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 1 & & & \\ 0 & 1 & & & & \\ 1 & & & & & \end{vmatrix} = (-1)^{n(n-1)/2}.$$

The entries below the anti-diagonal in $\tilde{W}(t,t)$ can be computed by the same method. In general, these entries involve the derivatives of the coefficients $a_j$, and can be used to obtain (35). Indeed, by computing $y', y'', \ldots, y^{(n-1)}$ from (27) and imposing the initial conditions at $x = t$, $b_j = y^{(j)}(t)$ $(0 \leq j \leq n-1)$, we obtain the linear system

$$\tilde{W}(t,t) \begin{pmatrix} \tilde{c}_0 \\ \tilde{c}_1 \\ \vdots \\ \tilde{c}_{n-1} \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix}.$$

Formula (35) is obtained then by inverting the matrix $\tilde{W}(t,t)$:

$$\begin{pmatrix} \tilde{c}_0 \\ \tilde{c}_1 \\ \vdots \\ \tilde{c}_{n-1} \end{pmatrix} = \tilde{W}(t,t)^{-1} \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix}.$$

For example for $n = 2,3,4$, we get the following formulas for $\tilde{W}(t,t)$ and its inverse, thus proving (36):

$$\tilde{W}(t,t) = \begin{pmatrix} 0 & 1 \\ 1 & -a_1(t) \end{pmatrix}, \qquad \tilde{W}(t,t)^{-1} = \begin{pmatrix} a_1(t) & 1 \\ 1 & 0 \end{pmatrix},$$

$$\tilde{W}(t,t) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_1 \\ 1 & -a_1 & a_1^2 - a_2 + a_1' \end{pmatrix}(t), \quad \tilde{W}(t,t)^{-1} = \begin{pmatrix} a_2 - a_1' & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}(t),$$

$$\tilde{W}(t,t)=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -a_1 \\ 0 & 1 & -a_1 & a_1^2-a_2+2a_1' \\ 1 & -a_1 & a_1^2-a_2+a_1' & 2a_1a_2-a_3-a_1^3+a_2'-3a_1'a_1-a_1'' \end{pmatrix}(t),$$

$$\tilde{W}(t,t)^{-1}=\begin{pmatrix} a_3-a_2'+a_1'' & a_2-a_1' & a_1 & 1 \\ a_2-2a_1' & a_1 & 1 & 0 \\ a_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}(t).$$

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