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A COMPACT MINIMAL SHADOW BOUNDARY IN EUCLIDEAN SPACE IS TOTALLY GEODESIC

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Abstract. We prove that a compact minimal shadow boundary of a hypersurface in Euclidean space is totally geodesic. We show that shadow boundaries detect principal directions and umbilical points of a hypersurface. As application we deduce that every shadow boundary of a compact strictly convex surface contains at least two principal directions.

1. Introduction

The main goal of this work is to give a negative answer to Question 4.1. in [4]. Namely, we will prove the following theorem.

Theorem 1.1. *Let M be an immersed hypersurface in $\mathbb{R}^{n+1}$. If the shadow boundary $S\partial(M,d)$ is a compact minimal submanifold of M, then $S\partial(M,d)$ is totally geodesic in M.*

As a second goal we investigate the relationship between shadow boundaries of an immersed hypersurface M in Euclidean space and principal directions of the shape operator of such immersion. In Proposition 3.2, we prove that a tangent vector $d_p \in T_pM$ is a principal direction if and only if there is a shadow boundary, generated by d_p, which is orthogonal to d_p. In Corollary 3.2, we show that shadow boundaries

1991 *Mathematics Subject Classification.* Primary 53C42, 53A07.
Key words and phrases. Shadow boundary, principal direction, umbilic point.
The third author was partially supported by Conacyt.

can detect when a point p of M with non zero Gauss-Kronecker curvature is umbilic: If every shadow boundary, in direction d, through p is orthogonal to d (at p) then p is an umbilic point of M.
Finally, as an application in Corollary 3.3, we prove that every shadow boundary of a compact strictly convex surface in $\mathbb{R}^3$ contains at least two principal directions of M.

Definition 1.1. Let M be a Riemannian immersed submanifold of N, and let $Y : N \longrightarrow TN$ be a vector field in N. The *shadow boundary* of M with respect to Y is the following subset of M.

$$S\partial(M, Y) = \{x \in M \mid Y(x) \in T_xM\}. \tag{1}$$

In [1], J. Choe gave the above definition of shadow boundary of Riemannian submanifolds, calling it horizon. Using the generalized Morse index theorem, he related this concept with the index of stability of a complete minimal surface in $\mathbb{R}^3$. In our case we will work in the first two sections with the ambient $N = \mathbb{R}^{n+1}$ and our vector field Y will be a constant vector field d called a direction in $\mathbb{R}^{n+1}$. But in the last section we will use closed conformal vector fields.
In general a shadow boundary with respect to some direction is not necessarily a submanifold of M. But it is always a closed subset of M. In Proposition 3.1, we show that a shadow boundary is an embedded submanifold around its points with nonzero Gauss-Kronecker curvature.

2. Minimal shadow boundaries are totally geodesic

We remark that in this manuscript all the manifolds and submanifolds will be smooth.

In order to prove our main theorem we need the following result.

Lemma 2.1. (B. Smyth, [6] page 271). *Let $L \subset \mathbb{R}^{n+1}$ be a compact immersed submanifold of Euclidean space with mean curvature vector field denoted by H. Then, to within translation, L lies in the subspace generated by H and in no smaller subspace.*

For the sake of completeness we recall Theorem 4.1 of [4]. Let $L \subset N$ be a submanifold and let H be its mean curvature vector field. We say that L has exhaustive mean curvature vector at the point $p \in L$, if $T_pL \subset V_p$, where V_p is the vector subspace of T_pN spanned by the parallel transport of the mean curvature vector $H(x)$ along all curves in L starting at $x \in L$ and ending at $p \in L$.

Theorem 2.1. *Let $M \subset N$ be a Riemannian immersed submanifold and let us take Y a parallel vector field of N along M. Let $L \subset S\partial(M,Y)$ be a transversal helix hypersurface of M with respect to Y, and let us assume that L has exhaustive mean curvature in N. If L is minimal in M, then L is a totally geodesic submanifold of M.*

Here is the proof of **Theorem 1.1:**

Proof. By Lemma 2.1 the mean curvature of the shadow boundary $L = S\partial(M,Y)$ is exhaustive. Then by the above theorem $L = S\partial(M,Y)$ is a totally geodesic submanifold of M since we assumed $L = S\partial(M,Y)$ to be a minimal submanifold of M. □

Combined with Theorem 1.3 in [5] we obtain the following corollary.

Corollary 2.1. *Let M be a compact immersed hypersurface of $\mathbb{R}^{n+1}$. If every shadow boundary of M is a minimal submanifold of M, then M is a hypersphere.*

To assume a Differential Geometric hypotheses on every shadow boundary is also exploited in the main result of [2], it says that if every shadow boundary is transnormal, then the hypersurface is a round hypersphere. In [3], the author assumes a topological hypothesis in the main result: Let M be an immersed compact orientable surface in $\mathbb{R}^3$. If every shadow is simply connected then M is an embedded convex surface. The shadow of M in direction $d \in \mathbb{R}^3$ is the open subset of M given by $\{p \in M \mid \langle d(p), n(p)\rangle > 0\}$, where n is an unitary normal vector field on M.

3. Umbilic points vs shadow boundaries

In this section we will work with a closed conformal vector field d in $\mathbb{R}^{n+1}$ instead of a constant parallel vector field as before.

Definition 3.1. We say that $d : \mathbb{R}^{n+1} \longrightarrow T\mathbb{R}^{n+1}$ is a closed conformal vector field if there exist $\varphi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ smooth such that for every vector field X in $\mathbb{R}^{n+1}$ we have

$$D_X d = \varphi X,$$

where D is the standard Levi-Civita connection of $\mathbb{R}^{n+1}$.

For example if d is constant and parallel as before we can take $\varphi = 0$. If d is radial, the corresponding φ is a constant equal to one. In particular, the results in this section will work for constant (i.e. parallel) and radial vector fields in $\mathbb{R}^{n+1}$.

Lemma 3.1. *Let M be an immersed hypersurface in $\mathbb{R}^{n+1}$ with second fundamental form α. Let d be a closed conformal vector field in $\mathbb{R}^{n+1}$. If p is in the shadow boundary $L = S\partial(M,d)$ then $\alpha(d(p),X) = 0$ for every $X \in T_pL$.*

Proof. Let ξ be a local unitary normal vector field in $\Gamma(TM^\perp)$ around p. By the hypothesis, $p \in L = S\partial(M,d)$ or equivalently $\langle d(p), \xi\rangle = 0$. Taking the derivative with respect to any nowhere zero local vector field $X \in \Gamma(TL)$ we obtain

$$\begin{aligned} 0 &= \langle D_X d, \xi\rangle + \langle d, D_X\xi\rangle = \langle \varphi X, \xi\rangle + \langle d, -A(X)\rangle \\ &= -\langle A(d), X\rangle = -\langle \alpha(d,X), \xi\rangle, \end{aligned}$$

where D is the standard Levi-Civita connection of $\mathbb{R}^{n+1}$ and $A = A_\xi$ is the shape operator of the isometric immersion $M \subset \mathbb{R}^{n+1}$ with respect to the normal vector field ξ.
We deduce that $\alpha(d,X) = 0$ at p. □

In general a shadow boundary is just a closed subset. The next result says that a shadow boundary is a smooth submanifold when the Gauss-Kronecker curvature is non zero.

Proposition 3.1. *Let M be an oriented immersed hypersurface in $\mathbb{R}^{n+1}$. Let $p \in M$ such that the Gauss-Kronecker curvature of M at p is non zero. If d is a non zero closed conformal vector field in $\mathbb{R}^{n+1}$ such that $p \in S\partial(M,d)$, then $S\partial(M,d)$ is an embedded hypersurface of M around p.*

Proof. Let ξ be an unitary normal vector field to M. Let us define the smooth function $F : M \longrightarrow \mathbb{R}$ by $F(p) = \langle d(p), \xi\rangle$. Therefore, $S\partial(M,d) = F^{-1}(0)$. We want to prove that zero is a regular value of F in an open neighbourhood $V \subset M$ of p. We define V as an open set of M such that $p \in V$ and where the Gauss-Kronecker curvature of M is nowhere zero in V. We calculate the derivative of F with respect to a vector field of M. Let D be the standard Levi-Civita connection of M. Then

$$dF(X) = XF = \langle D_X d, \xi\rangle + \langle d, D_X\xi\rangle = -\langle \xi, \alpha(d,X)\rangle = -\langle d, A(X)\rangle,$$

where $A = A_\xi$ is the shape operator of M. The application A is a linear isomorphism of T_qM in T_qM for every $q \in V$. So, given $q \in V$ we take X such that $X(q)$ is not orthogonal to $A(d(q))$ then $dF(X) = -\langle d, A(X)\rangle = -\langle A(d), X\rangle \neq 0$ at q. This proves that 0 is a regular value of F. We can conclude that $V \cap F^{-1}(0) = V \cap S\partial(M,d)$ is an embedded hypersurface. □

Definition 3.2. Let M be an immersed hypersurface in $\mathbb{R}^{n+1}$. Given a tangent vector $d_p \in T_pM$, we say that a shadow boundary $S\partial(M,d)$ is *generated* by d_p if d is a closed conformal vector field satisfying the initial condition $d(p) = d_p$.

Corollary 3.1. *Let M be an immersed hypersurface in $\mathbb{R}^{n+1}$ with second fundamental form α. Let $p \in M$ be a point where the Gauss-Kronecker curvature of M is non zero. Then for every subspace V of T_pM of codimension one there exist a direction $d_p \in T_pM$ such that the tangent space at p of every shadow boundary $S\partial(M,d)$ generated by d_p, is equal to V.*

Proof. Since the Gauss-Kronecker curvature of M at p is non zero, the shadow boundary $S\partial(M,Y)$ is an embedded submanifold of codimension one of M around p for every $Y \in T_pM$. Let ν be an unitary normal vector in $T_pM \cap V^\perp$ and ξ a local normal unitary vector field of M around p. Let $A = A_\xi$ the shape operator of the immersion $M \subset \mathbb{R}^{n+1}$. Then $A : T_pM \longrightarrow T_pM$ is invertible at p. Let us define the direction $d_p = A^{-1}(\nu) \in T_pM$. Let $L := S\partial(M,d)$ be the shadow boundary with direction d, where d is any closed conformal vector field which takes the value d_p at p. By Lemma 3.1, $\alpha(d,X) = 0$ for every $X \in T_pL$. Therefore,

$$0 = \langle \alpha(d,X), \xi\rangle = \langle A(d), X\rangle = \langle \nu, X\rangle.$$

Therefore, ν is orthogonal to T_pL and this proves that $T_pL = V$. □

Proposition 3.2. *Let M be an immersed hypersurface in $\mathbb{R}^{n+1}$. Let $p \in M$ be any point where the Gauss-Kronecker curvature of M is non zero. A non zero direction $d_p \in T_pM$ is a principal direction of M if and only if there exists a shadow boundary $S\partial(M,d)$, generated by d_p, which is orthogonal to d at p.*

Proof. Let us assume that the shadow boundary $L_d := S\partial(M,d)$ is orthogonal to d at p, where d is a closed conformal vector field. Then $\langle A(d), X\rangle = \langle \alpha(d(p),X),\xi\rangle = 0$ for every $X \in T_pL_d$. This says that $A(d)$ is orthogonal to T_pL_d and by the hypothesis d is also orthogonal to T_pL_d. Therefore, $A(d)$ is a multiple of $d(p)$, i.e. $d(p)$ is a principal direction of A. Hence, $d_p = d(p)$ is a principal direction of M. So, $A(d_p) = \lambda d_p$.

Reciprocally, let us assume that $A(d_p) = \lambda d_p$. Since the Gauss-Kronecker curvature of M is non zero at p, we deduce that $\lambda \neq 0$. By Lemma 3.1, for any non zero vector $d_p \in T_pM$ we have that

$$\lambda\langle d, X\rangle = \langle \lambda d, X\rangle = \langle A(d), X\rangle = \langle \alpha(d,X), \xi\rangle = 0,$$

where X is any tangent vector to L_d. This proves that d is orthogonal to the shadow boundary L_d in direction d. □

Let us recall that a point p in a hypersurface M is an umbilic point if and only if every tangent vector to M at p is a principal direction of the shape operator A of M at p.

Corollary 3.2. *Let M be an immersed hypersurface in $\mathbb{R}^{n+1}$. Let $p \in M$ be any point where the Gauss-Kronecker curvature of M is non zero. The point p is an umbilic point of M if and only if for every non zero direction $d_p \in T_pM$, there exists a shadow boundary $S\partial(M,d)$, generated by d_p, which is orthogonal to d at p.*

Proof. Let us assume that the shadow boundary $L_d := S\partial(M,d)$ is orthogonal to d at p for every closed conformal vector field d. By Proposition 3.2, $d(p)$ is a principal direction of M. This finish the proof because by the hypothesis $d(p) = d_p$ is an arbitrary direction in T_pM.

□

We say that a regular curve L in a surface M contains a principal direction of M at $p \in L$ if the tangent line of L at p is generated by a principal direction, at p, of the shape operator A of $M \subset \mathbb{R}^3$.

In the next Corollaries 3.3 and 3.4, we will assume that the closed conformal vector field d is either radial or parallel (i.e constant).

Corollary 3.3. *Let M be an immersed surface in $\mathbb{R}^3$. Then every compact regular shadow boundary $L := S\partial(M,d)$ of M, with respect to a either radial or constant vector field d, contains at least two principal directions of M at two different points of L. In particular, if M is a compact strictly convex surface then every shadow boundary of M contains at least two principal directions of M.*

Proof. By Proposition 3.2, we have to prove that there are two different points p and q in L such that L is orthogonal to d at p and at q. Since L is compact, just define p and q as the closest and farthest points in L with respect to the concentric hyperspheres to d if d is radial or with respect to hyperplanes orthogonal to d if d is constant. In the first case, both concentric hyperspheres are tangent to the shadow boundary at the contact points p and q, so the radial vector field d is orthogonal to the tangent space of the hyperspheres and in particular orthogonal to the shadow boundary. Then we can apply Proposition 3.2. The second case is similar. For the second part of this Corollary, we just have to observe that by Proposition 3.1, every shadow boundary of M is a regular embedded curve of M. □

Corollary 3.4. *Let M be a compact strictly convex hypersurface in $\mathbb{R}^{n+1}$. If for every $p \in M$ and every non zero direction $d_p \in T_pM$ there exists a shadow boundary $S\partial(M, d)$, generated by d_p, which makes a constant angle with respect to d then M is a round hypersphere.*

Proof. Let us observe that M has nowhere zero Gauss-Kronecker curvature. Let p be any point in M and d be as in the hypothesis. Since M is compact and $S\partial(M, d)$ is a closed subset of M, it is compact. So, the constant angle between d and $S\partial(M, d)$ should be $\frac{\pi}{2}$. Therefore, by Corollary 3.2, we deduce that p is an umbilic point. Since M is compact, we deduce that M is a round sphere.

□

We say that a submanifold L of a hypersurface M is invariant under the shape operator A of $M \subset \mathbb{R}^{n+1}$ if for every point $p \in L$, we have that $A(T_pL) \subset T_pL$.

Corollary 3.5. *Let M be a compact strictly convex hypersurface in $\mathbb{R}^{n+1}$. If for every $p \in M$ and every non zero direction $d_p \in T_pM$, there exists a shadow boundary $S\partial(M, d)$, generated by d_p, which is invariant under the shape operator of M then M is a round hypersphere.*

Proof. Since M is strictly convex it has nowhere zero Gauss-Kronecker curvature. Equivalently, the shape operator $A = A_\xi$ is invertible $A : T_pM \longrightarrow T_pM$ for every $p \in M$. Here, ξ is an unitary normal vector field to M.
Let p be any point in M. Let d_p and d be as in the hypothesis. We will show that $S\partial(M, d)$ is orthogonal to d at p. By Lemma 3.1, $\alpha(d(p), X) = 0$ for every $X \in T_pL$, where $L := S\partial(M, d)$. Since A is invertible and L is invariant under A, then $A(L) = L$. So, for every $Y \in T_pL$ there exists $X \in T_pL$ such that $Y = A(X)$. Therefore,
$\langle Y, d(p)\rangle = \langle A(X), d(p)\rangle = \langle \alpha(X, d(p)), \xi\rangle = 0.$
Finally, by Corollary 3.2, p is an umbilic point of M. Since M is compact, we obtain that M is a round sphere.

□

Let us observe that if a shadow boundary of a compact strictly convex hypersurface M is totally geodesic in M, then such shadow boundary is invariant under the shape operator of M. So, by Corollary 3.5, if every shadow boundary of M is totally geodesic then M is a round hypersphere, this is Theorem 1.3 in [5] for the case when M is strictly convex.

Another particular case of Corollary 3.5 is when M is a strictly convex surface in $\mathbb{R}^3$ such that every shadow boundary is a line of curvature. A line of curvature is invariant under the shape operator of M, so Corollary 3.5 implies that M is a round sphere in $\mathbb{R}^3$.

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