TOPOLOGY AND HOMOCLINIC TRAJECTORIES OF DISCRETE DYNAMICAL SYSTEMS

Dedicated to Petr P. Zabrejko

Abstract. We show that nontrivial homoclinic trajectories of a family of discrete, nonautonomous, asymptotically hyperbolic systems parametrized by a circle bifurcate from a stationary solution if the asymptotic stable bundles $E^s(+\infty)$ and $E^s(-\infty)$ of the linearization at the stationary branch are twisted in different ways.

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1. Introduction

In this paper we will investigate the birth of homoclinic trajectories of discrete nonautonomous dynamical systems from the point of view of topological bifurcation theory. This means that instead of proving the existence of a homoclinic trajectory for a single dynamical system we will consider a one parameter family of discrete nonautonomous dynamical systems on $\mathbb{R}^N$ having $x \equiv 0$ as a stationary trajectory and show that, under appropriate conditions, the dynamical systems with parameter values close to a given point, must have trajectories homoclinic to 0. Values of the parameter for which this occurs are called bifurcation points.

Bifurcation theory for various types of bounded solutions of discrete nonautonomous dynamical systems have been studied in [10, 21] and more recently in [18, 19]. However our approach is different. We will not look for homoclinics bifurcating at a value of the parameter given apriori but instead we will discuss the appearance of homoclinic solutions forced by the asymptotic behavior of the family of linearized equations at 0.

When the family is asymptotically hyperbolic, the asymptotic stable and unstable subspaces of the linearized equations form vector bundles over the parameter space which might be nontrivial when the parameter space carries some nontrivial topology. We will show that homoclinic trajectories bifurcate from a stationary

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solution if the asymptotic stable bundles $E^s(+\infty)$ and $E^s(-\infty)$ of the linearization along the stationary branch are "twisted" in different ways.

Our results require methods going beyond the classical Lyapunov-Schmidt reduction and spectral analysis at a potential bifurcation point. While similar results can be proved for general parameter spaces using more sophisticated technology from algebraic topology, here we will concentrate to on the simplest topologically nontrivial parameter space, the circle. This is equivalent to considering families of dynamical systems parametrized by an interval $[a, b]$ with the assumption that the systems at a and b are the same.

Roughly speaking, we will first translate the problem of bifurcation of homoclinic trajectories into a problem of bifurcation from a trivial branch of zeroes for a parametrized family of C^1-Fredholm maps. Then we will consider the *index bundle* of the family of linearizations at points of the trivial branch given by the stationary solutions of the equation. The index bundle of a family of Fredholm operators is a refinement of the ordinary index of a Fredholm operator which takes into account the topology of the parameter space. A special "homotopy variance" property of the topological degree for C^1-Fredholm maps, constructed in [17] relates the nonorientability of the index bundle to bifurcation of zeroes. On the other hand, an elementary index theorem, Theorem 4.1, allows us to compute the index bundle in terms of the asymptotic stable bundles of the linearized problem, relating in this way the appearance of homoclinics to the asymptotic behavior of coefficients of the linearized equations. The precise result is stated in Theorem 2.3 of Section 2. An analogous approach applied to nonautonomous differential equations can be found in [14, 15].

The paper is organized as follows. In the next section we introduce the problem and state our main result. In Section 3 we recall the concept of the index bundle and discuss its orientability. In the fourth section, we compute the index bundle of the family of operators associated to a family of asymptotically hyperbolic nonautonomous dynamical systems. In Section 5, we discuss the *parity* of a path of Fredholm operators of index 0, and we recall the construction in [17] of a topological degree theory for C^1-Fredholm maps of index 0 extending to proper Fredholm maps the well known Leray-Schauder degree. In Section 6, using the computation of the index bundle and the homotopy property of the topological degree constructed in [17] we prove Theorem 2.3. In the seventh section, we illustrate Theorem 2.3 with a non-trivial example. Section 8 is devoted to comments and possible extensions of our results. The appendix collects the proofs of various properties of the index bundle used in the article.

2. The main result

A nonautonomous discrete dynamical system on $\mathbb{R}^N$ is defined by a doubly infinite sequence of maps $\mathbf{f} = \{f_n : \mathbb{R}^N \to \mathbb{R}^N \mid n \in \mathbb{Z}\}$. A trajectory of the system $\mathbf{f}: \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}^N$ is a sequence $\mathbf{x} = (x_n)$ such that

$$x_{n+1} = f_n(x_n). \tag{1}$$

In the terminology of [18] (1) is a nonautonomous difference equation whose solutions are trajectories of the corresponding dynamical system.

In what follows we will always assume that the f_n are C^1 and that $f_n(0) = 0$. Under this assumption the system has a stationary trajectory $\mathbf{x} = \mathbf{0}$, where $\mathbf{0}$ is a sequence of zeroes. A trajectory $\mathbf{x} = (x_n)$ of $\mathbf{f}$ is called *homoclinic* to $\mathbf{0}$, or simply a homoclinic trajectory, if $\lim_{n\to\pm\infty} x_n = 0$. Under our assumptions the system $\mathbf{f}$ has always a trivial homoclinic trajectory. Namely, the stationary trajectory $\mathbf{0}$. We will look for nontrivial homoclinic trajectories.

A natural function space for the study of homoclinic trajectories is the Banach space

$$\mathbf{c}(\mathbb{R}^N) := \{\mathbf{x}\colon \mathbb{Z} \to \mathbb{R}^N \mid \lim_{|n|\to\infty} x_n = 0\}$$

equipped with the norm $\|\mathbf{x}\| := \sup_{k\in\mathbb{Z}} |x_n|$. Any homoclinic trajectory of $\mathbf{f}$ is naturally an element of this space. Moreover, each dynamical system $\mathbf{f}$ induces a nonlinear Nemytskii (substitution) operator

$$F\colon \mathbf{c}(\mathbb{R}^N) \to \mathbf{c}(\mathbb{R}^N) \tag{2}$$

defined by $F(\mathbf{x}) = (f_n(x_n))$. Under some natural assumptions (see below) F becomes C^1-map such that $F(\mathbf{0}) = \mathbf{0}$. In this way nontrivial homoclinic trajectories become the nontrivial solutions of the equation $S\mathbf{x} - F(\mathbf{x}) = \mathbf{0}$, where

$$S\colon \mathbf{c}(\mathbb{R}^N) \to \mathbf{c}(\mathbb{R}^N)$$

is the shift operator $S(\mathbf{x}) = (x_{n+1})$.

The linearization of the system $\mathbf{f}$ at the stationary solution $\mathbf{0}$ is the nonautonomous linear dynamical system $\mathbf{a}\colon \mathbb{Z}\times\mathbb{R}^N \to \mathbb{R}^N$ defined by the sequence of matrices $(a_n) \in \mathbb{R}^{N\times N}$, with $a_n = Df_n(0)$. The corresponding linear difference equation is

$$x_{n+1} = a_n x_n, \tag{3}$$

If $f_n = f$, for all $n \in \mathbb{Z}$, the system is called autonomous. We will deal only with discrete nonautonomous dynamical systems whose linearization at $\mathbf{0}$ is asymptotic for $n \to \pm\infty$ to an autonomous linear dynamical system associated to a hyperbolic matrix. We will call systems with this property *asymptotically hyperbolic*.

Let us recall that an invertible matrix a is called *hyperbolic* if a has no eigenvalues of norm one, i.e., $\sigma(a) \cap \{|z| = 1\} = \emptyset$. The spectrum $\sigma(a)$ of an hyperbolic matrix a consists of two disjoint closed subsets $\sigma(a) \cap \{|z| < 1\}$ and $\sigma(a) \cap \{|z| > 1\}$, so $\mathbb{R}^N$ has the a-invariant spectral decomposition $\mathbb{R}^N = E^s(a) \oplus E^u(a)$, where $E^s(a)$ (respectively $E^u(a)$) is the real part of sum of the generalized eigenspaces corresponding to the part of the spectrum of a inside the unit disk (respectively outside of the unit disk). It is easy to see that $\zeta \in E^s(a)$ if and only if $\lim_{n\to\infty} a^n\zeta = 0$. The unstable subspace $E^u(a)$ has a similar characterization, i.e., $\zeta \in E^u(a)$ if and only if $\lim_{n\to\infty} a^{-n}\zeta = 0$.

When the linearized system is asymptotically hyperbolic, the map $G = S - F$ becomes a Fredholm map (at least in a neighborhood of $\mathbf{0}$) which will allow us to apply the results of general bifurcation theory for Fredholm maps to our problem by relating the corresponding bifurcation invariants to the asymptotic behavior of the linearization at $\pm\infty$.

Let us describe precisely our setting and assumptions.

A *continuous family of C^1-dynamical systems parametrized by the unit circle S^1* is a sequence of maps

$$\mathbf{f} = \{f_n\colon S^1 \times \mathbb{R}^N \to \mathbb{R}^N \mid n \in \mathbb{Z}\} \tag{4}$$

such that f_n is differentiable with respect to the second variable and, for all $n \in \mathbb{Z}$, $0 \le j \le 1$, the map $(\lambda, x) \mapsto \dfrac{\partial^j f_n}{\partial x^j}(\lambda, x)$ is continuous.

To put it shortly, a continuous family of C^1-dynamical systems is a continuous map

$$\mathbf{f}\colon \mathbb{Z} \times S^1 \times \mathbb{R}^N \to \mathbb{R}^N,$$

differentiable in the third variable and such that all the partials depend continuously on (λ, x). We will use $\mathbf{f}_\lambda$ to denote the dynamical system corresponding to the parameter value λ.

Remark 2.1. Alternatively one can think of $\mathbf{f}$ as a double infinite sequence of maps $f_n\colon [a,b] \times \mathbb{R}^N \to \mathbb{R}^N$, such that $f_n(a,x) = f_n(b,x)$ for all $n \in \mathbb{Z}$.

Pairs $(\lambda, \mathbf{x})$ which solve the parameter-dependent difference equation:

$$x_{n+1} = f_n(\lambda, x_n), \quad \text{for all } n \in \mathbb{Z}, \tag{5}$$

will be called *homoclinic solutions.* Equivalently, $(\lambda, \mathbf{x})$ is a homoclinic solution of (5) if $\mathbf{x} = (x_n)$ is a homoclinic trajectory of the dynamical system $\mathbf{f}_\lambda$.

Homoclinic solutions of (5) of the form $(\lambda, \mathbf{0})$ are called trivial and the set $S^1 \times \{\mathbf{0}\}$ is called the *trivial or stationary branch.* We are interested in nontrivial homoclinic solutions.

We will assume that the family $\mathbf{f}\colon \mathbb{Z} \times S^1 \times \mathbb{R}^N \to \mathbb{R}^N$ of dynamical systems satisfies the following conditions:

(A0) For all $\lambda \in S^1$ and $n \in \mathbb{Z}$, $f_n(\lambda, 0) = 0$.

(A1) For any $M > 0$ and $\varepsilon > 0$ there exists a $\delta > 0$ such for all $(\lambda, x), (\mu, y) \in S^1 \times \bar{B}(0, M)$ ([1]) with $d\big((\lambda, x), (\mu, y)\big) < \delta$ and all j, $0 \le j \le 1$,

$$\sup_{n\in\mathbb{Z}} \left\| \frac{\partial^j f_n}{\partial x^j}(\lambda, x) - \frac{\partial^j f_n}{\partial x^j}(\mu, y) \right\| < \varepsilon.$$

Here d is the product distance in the metric space $S^1 \times \mathbb{R}^N$.

(A2) For all bounded $\Omega \subset S^1 \times \mathbb{R}^N$ one has

$$\sup_{(n,\lambda,x)\in\mathbb{Z}\times\Omega} \left\| \frac{\partial f_n}{\partial x}(\lambda, x) \right\| < \infty.$$

(A3) Let $a_n(\lambda) := \dfrac{\partial f_n}{\partial x}(\lambda, 0)$. As $n \to \pm\infty$ the family of matrices $a_n(\lambda)$ converges uniformly to a family of hyperbolic matrices $a(\lambda, \pm\infty)$. Moreover, for some, and hence for all $\lambda \in S^1$, $a(\lambda, +\infty)$ and $a(\lambda, -\infty)$ have the same number of eigenvalues (counting algebraic multiplicities) inside of the unit disk.

(A4) There exists $\lambda_0 \in S^1$ such that

$$x_{n+1} = a_n(\lambda_0) x_n, \tag{6}$$

admits only the trivial solution $(x_n \equiv 0)_{n\in\mathbb{Z}}$.

By $(A3)$ the map $\lambda \to a(\lambda, \pm\infty)$ is a continuous family of hyperbolic matrices. Since there are no eigenvalues of $a(\lambda, \pm\infty)$ on the unit circle, the projectors to the spectral subspaces corresponding to the spectrum inside and outside the unit disk depend continuously on the parameter λ (see [12]). It is well known that the images of a continuous family of projectors form a vector bundle over the parameter space [13]. Therefore, the vector spaces $E^s(\lambda, \pm\infty)$ and $E^u(\lambda, \pm\infty)$ whose elements are the generalized real eigenvectors of $a(\lambda, \pm\infty)$ corresponding to the eigenvalues with

[1] Given a normed space $\mathbb{E}$, $\bar{B}(x, r)$ and $B(x, r)$, where $x \in \mathbb{E}$ and $r > 0$, denote the closed and open ball around x of radius r in $\mathbb{E}$, respectively.

absolute value smaller (respectively greater) than 1 are fibers of a pair of vector bundles $E^s(\pm\infty)$ and $E^u(\pm\infty)$ over S^1 which decompose the trivial bundle $\Theta(\mathbb{R}^N)$ with fiber $\mathbb{R}^N$ into a direct sum:

$$E^s(\pm\infty) \oplus E^u(\pm\infty) = \Theta(\mathbb{R}^N). \tag{7}$$

In what follows $E^s(\pm\infty)$ and $E^u(\pm\infty)$ will be called *stable and unstable* asymptotic bundles at $\pm\infty$.

Our main theorem relates the appearance of homoclinic solutions to the topology of the asymptotic stable bundles $E^s(\pm\infty)$. Due to relation (7) the consideration of the unstable bundles would give the same result.

In what follows, for notational reasons, it will be convenient for us to work with the multiplicative group $\mathbb{Z}_2 = \{1, -1\}$ instead of the standard additive $\mathbb{Z}_2 = \{0, 1\}$.

A vector bundle over S^1 is orientable if and only if it is trivial, i.e., isomorphic to a product $S^1 \times \mathbb{R}^k$. Moreover, whether a given vector bundle E over S^1 is trivial or is not is determined by a topological invariant $w_1(E) \in \mathbb{Z}_2$.

In order to define $w_1(E)$ let us identify S^1 with the quotient of an interval $I = [a, b]$ by its boundary $\partial I = \{a, b\}$. If $p\colon [a, b] \to S^1 = I/\partial I$ is the projection, the pullback bundle $p^*E = E'$ is the vector bundle over I with fibers $E'_t = E_{p(t)}$. Since I is contractible to a point, E' is trivial and the choice of an isomorphism between E' and the product bundle provides E' with a frame, i.e., a basis $\{e_1(t), ..., e_k(t)\}$ of E'_t continuously depending on t. Since $E'_a = E_{p(a)} = E_{p(b)} = E'_b$, $\{e_i(a) \mid 1 \leq i \leq k\}$ and $\{e_i(b) \mid 1 \leq i \leq k\}$ are two bases of the same vector space. We define $w_1(E) \in \mathbb{Z}_2$ by

$$w_1(E) := \operatorname{sign} \det C, \tag{8}$$

where C is the matrix expressing the basis $\{e_i(b) \mid 1 \leq i \leq k\}$ in terms of the basis $\{e_i(a) \mid 1 \leq i \leq k\}$.

It is easy to see that $w_1(E)$ is independent from the choice of the frame. We claim that $w_1(E) = 1$ if an only if E is trivial. The if part is an immediate consequence of the definition of $w_1(E)$. On the other hand, if $w_1(E) =$, then$\det C > 0$ and there exists a path $C(t)$ with $C(a) = C$ and $C(b) = \operatorname{Id}$. Now, $f_i(t) = C(t)e_i(t)$ is a frame such that $f_i(a) = f_i(b)$ and hence $\Phi(t, x_1, \ldots, x_k) = (t, \sum x_i f_i(t))$ is an isomorphism between the product bundle $S^1 \times \mathbb{R}^k$ and E. Thus E is trivial.

Remark 2.2. Under the isomorphism $H^1(S^1; \mathbb{Z}_2) \cong \mathbb{Z}_2$, $w_1(E)$ can be identified with the first Stiefel-Whitney class of E.

Our main result is:

Theorem 2.3. *If the system* (5) *verifies* (A0)–(A4) *and if*

$$w_1(E^s(+\infty)) \neq w_1(E^s(-\infty)), \tag{9}$$

then for all ε *small enough there is a homoclinic solution* $(\lambda, \mathbf{x})$ *of* (5) *with* $\|\mathbf{x}\| = \varepsilon$.

The proof will be presented in Section 6.

A point $\lambda_* \in S^1$ is a *bifurcation point* for homoclinic solutions of (5) from the stationary branch $(\lambda, \mathbf{0})$ if in every neighborhood of $(\lambda_*, \mathbf{0})$ there is a point a nontrivial homoclinic solution $(\lambda, \mathbf{x})$ of $x_{n+1} = f_n(\lambda, x_n)$.

By Theorem 2.3 we can find a sequence of nontrivial homoclinic solutions $(\lambda_n, \mathbf{x}_n)$ of (5) such that $\|\mathbf{x}_n\| \to 0$. Since S^1 is compact λ_n possesses a subsequence converging to some $\lambda_* \in S^1$. Hence we obtain:

Corollary 2.4. *Under the assumptions of Theorem 2.3 there exists at least one bifurcation point $\lambda_* \in S^1$ of nontrivial homoclinic solutions from the branch of stationary solutions. In other words there exists a $\lambda_* \in S^1$ and a sequence $(\lambda_k, \mathbf{x}_k)$ such that $\lambda_k \to \lambda_*$ and $\mathbf{x}_k \neq \mathbf{0}$ is a nontrivial homoclinic trajectory of $\mathbf{f}$.*

Let us observe that if $\mathbf{f}\colon \mathbb{Z}\times S^1 \times \mathbb{R}^N \to \mathbb{R}^N$ verifies the assumptions of Theorem 2.3 and $\tilde{\mathbf{f}}\colon \mathbb{Z}\times S^1 \times \mathbb{R}^N \to \mathbb{R}^N$ is defined by $\tilde{\mathbf{f}} = \mathbf{f} + \mathbf{h}$, where

$$\mathbf{h} = (h_n)\colon \mathbb{Z}\times S^1 \times \mathbb{R}^N \to \mathbb{R}^N,$$

verifies $(A0)$–$(A2)$ and moreover

$(A3')$ $\dfrac{\partial h_n}{\partial x}(\lambda, 0) \to 0$ as $n \to \pm\infty$ uniformly on λ,

$(A4')$ $\sup_{n\in\mathbb{Z}} \left\| \dfrac{\partial h_n}{\partial x}(\lambda_0, 0) \right\|$ is small enough,

then also $\tilde{\mathbf{f}}$ verifies Assumptions $(A0)$–$(A4)$. Indeed, $(A3')$ for $\mathbf{h}$ implies $(A3)$ for $\tilde{\mathbf{f}}$. On the other hand, it is shown in the proof of Theorem 2.3 that $(A3)$ and $(A4)$ together imply that the operator $L_{\lambda_0}\colon \mathbf{c}(\mathbb{R}^N) \to \mathbf{c}(\mathbb{R}^N)$ defined by

$$L_{\lambda_0}\mathbf{x} = (x_{n+1} - a_n(\lambda_0)x_n)$$

is invertible. Now, that $\tilde{\mathbf{f}}$ verifies $(A4)$ follows from $(A4')$ and the fact that the set of all invertible operators is open.

Summing up we have:

Corollary 2.5. *If $\mathbf{f}$ verifies the assumptions of Theorem 2.3 then any perturbation $\tilde{\mathbf{f}} = \mathbf{f} + \mathbf{h}$ as above must have nontrivial homoclinic solutions bifurcating from the stationary branch at some point of the parameter space.*

3. The index bundle

Let us recall that a bounded operator $T \in \mathcal{L}(X,Y)$ ([2]) is Fredholm if it has finite dimensional kernel and cokernel. The index of a Fredholm operator is by definition $\operatorname{ind} T := \dim \operatorname{Ker} T - \dim \operatorname{Coker} T$. The Fredholm operators will be denoted by $\Phi(X,Y)$ and those of index 0 by $\Phi_0(X,Y)$.

The index bundle generalizes to the case of families of Fredholm operators the concept of index of a single Fredholm operator. If a family L_λ of Fredholm operators depends continuously on a parameter λ belonging to some topological space Λ and if the kernels $\operatorname{Ker} L_\lambda$ and cokernels $\operatorname{Coker} L_\lambda$ form two vector bundles $\operatorname{Ker} L$ and $\operatorname{Coker} L$ over Λ, then, roughly speaking, the index bundle is $\operatorname{Ker} L - \operatorname{Coker} L$ where one has to give a meaning to the difference by working in an appropriate group generalizing $\mathbb{Z}$. We will first define such a group and then will see how to handle the case where the kernels do not form a vector bundle.

If Λ is a compact topological space, the Grothendieck group $KO(\Lambda)$ is the group completion of the abelian semigroup $\operatorname{Vect}(\Lambda)$ of all isomorphisms classes of real vector bundles over Λ. In other words, $KO(\Lambda)$ is the quotient of the semigroup $\operatorname{Vect}(\Lambda) \times \operatorname{Vect}(\Lambda)$ by the diagonal sub-semigroup. The elements of $KO(\Lambda)$ are called virtual bundles. Each virtual bundle can be written as a difference $[E] - [F]$ where E, F are vector bundles over Λ and $[E]$ denotes the equivalence class of $(E, 0)$. Moreover, one can show that $[E] - [F] = 0$ in $KO(\Lambda)$ if and only if the two vector

[2] By $\mathcal{L}(X,Y)$ we will denote the space of bounded linear operators between two Banach spaces X and Y.

bundles become isomorphic after the addition of a trivial vector bundle to both sides. Taking complex vector bundles instead of the real ones leads to the complex Grothendieck group denoted by $K(\Lambda)$. In what follows the trivial bundle with fiber $\Lambda \times V$ will be denoted by $\Theta(V)$. The trivial bundle, $\Theta(\mathbb{R}^N)$, will be simplified to Θ^N.

Let X, Y be real Banach spaces and let $L\colon \Lambda \to \Phi(X,Y)$ be a continuous family of Fredholm operators. As before $L_\lambda \in \Phi(X,Y)$ will denote the value of L at the point $\lambda \in \Lambda$. Since $\operatorname{Coker} L_\lambda$ is finite dimensional, using compactness of Λ, one can find a finite dimensional subspace V of Y such that

$$\operatorname{Im} L_\lambda + V = Y \text{ for all } \lambda \in \Lambda. \tag{10}$$

Because of the transversality condition (10) the family of finite dimensional subspaces $E_\lambda = L_\lambda^{-1}(V)$ defines a vector bundle over Λ with total space

$$E = \bigcup_{\lambda \in \Lambda} \{\lambda\} \times E_\lambda.$$

Indeed, the kernels of a family of surjective Fredholm operators form a finite dimensional vector bundle [13]. Denoting with π the canonical projection of Y onto Y/V, from (10) it follows that the operators πL_λ are surjective with $\operatorname{Ker} \pi L_\lambda = E_\lambda$, which shows that $E \in Vect(\Lambda)$.

We define the *index bundle* $\operatorname{Ind} L$ by:

$$\operatorname{Ind} L = [E] - [\Theta(V)] \in KO(\Lambda). \tag{11}$$

Notice that the index bundle of a family of Fredholm operators of index 0 belongs to the reduced Grothendieck group $\widetilde{KO}(\Lambda)$ which, by definition, is the kernel of the rank homomorphism $rk\colon KO(\Lambda) \to \mathbb{Z}$ given by

$$\operatorname{rk}([E] - [F]) = \dim E_\lambda - \dim F_\lambda.$$

We will mainly, but not always, work with families of Fredholm operators of index 0. If $\Lambda = pt$ consists of just one point, then the rank homomorphism rk is an isomorphism and the index bundle coincides with the ordinary numerical index $\operatorname{ind} L = \dim \operatorname{Ker} L - \dim \operatorname{Coker} L$ of a Fredholm operator L. The index bundle enjoys the same nice properties of the ordinary index. Namely, homotopy invariance, additivity with respect to directs sums, logarithmic property under composition of operators. Clearly it vanishes if L is a family of isomorphisms. We will use these properties in the sequel. The precise statements and proofs can be found in Appendix A (Proposition 9.1).

It can be shown that any element $\eta \in \widetilde{KO}(\Lambda)$ can be written as $[E] - [\Theta^N]$. Moreover, $[E] - [\Theta^N] = [E'] - [\Theta^M]$ in $\widetilde{KO}(\Lambda)$ if and only if there exist two trivial bundles Θ and Θ' such that $E \oplus \Theta$ is isomorphic to $E' \oplus \Theta'$, (see [11, Theorem 3.8]).

The obstruction $w_1(E)$ to the triviality of vector bundle E over S^1 defined in Section 2 induces a well defined homomorphism $w_1\colon \widetilde{KO}(S^1) \to \mathbb{Z}_2$ by putting

$$w_1([E] - [F]) = w_1(E)w_1(F). \tag{12}$$

Indeed, taking $\Lambda = S^1$ we observe that $w_1(E)$ remains unmodified under addition of a trivial vector bundle which, on the basis of the above discussion, proves that (12) is well defined.

Proposition 3.1. *The homomorphism $w_1\colon \widetilde{KO}(S^1) \to \mathbb{Z}_2$ is an isomorphism.*

Proof. This follows again from the above discussion and the fact that $w_1(E) = 1$ implies that E is a trivial vector bundle over S^1. □

4. The index bundle of the family of operators associated to linear asymptotically hyperbolic systems

In this section we will deal only with linear asymptotically hyperbolic systems $\mathbf{a}\colon \mathbb{Z} \times S^1 \to GL(N)$, where $GL(N)$ is the set of all invertible matrices in $\mathbb{R}^{N\times N}$. This means:

(a) As $n \to \pm\infty$ the sequence $\mathbf{a}(\lambda) = (a_n(\lambda))$ converges uniformly with respect to $\lambda \in S^1$ to a family of matrices $a(\lambda, \pm\infty)$.
(b) $a(\lambda, \pm\infty) \in GL(N)$ is hyperbolic for all $\lambda \in S^1$.

Given a family $\mathbf{a}$ of asymptotically hyperbolic systems parametrized by S^1 let us consider the family of linear operators

$$L = \{L_\lambda \colon \mathbf{c}(\mathbb{R}^N) \to \mathbf{c}(\mathbb{R}^N); \lambda \in S^1\}$$

defined by $L_\lambda = S - A_\lambda$, where S is the shift operator and

$$A_\lambda \colon \mathbf{c}(\mathbb{R}^N) \to \mathbf{c}(\mathbb{R}^N)$$

is defined by $A_\lambda \mathbf{x} := (a_n(\lambda)x_n)$.

Since the sequence $(a_n(\lambda))$ converges uniformly, it is bounded, from which follows immediately that A_λ and L_λ are well defined bounded operators. Moreover it is easy to see that the map $A\colon S^1 \to \mathcal{L}(\mathbf{c}(\mathbb{R}^N), \mathbf{c}(\mathbb{R}^N))$ defined by $A(\lambda) := A_\lambda$ is continuous with respect to the norm topology of $\mathcal{L}(\mathbf{c}(\mathbb{R}^N), \mathbf{c}(\mathbb{R}^N))$. Hence the same holds for the family L.

Clearly, $\mathbf{x} = (x_n) \in \mathbf{c}(\mathbb{R}^N)$ verifies a linear difference equation $x_{n+1} = a_n(\lambda)x_n$ if and only if $L_\lambda \mathbf{x} = 0$.

By the discussion in the previous section the families $a(\lambda, \pm\infty) \in GL(N)$ define two vector bundles $E^s(\pm\infty)$ over S^1. The next theorem relates the index bundle of the family L to $E^s(\pm\infty)$.

Theorem 4.1. *Let $\boldsymbol{a}\colon S^1 \times \mathbb{Z} \to GL(N)$ be a continuous map verifying (a) and (b). Then the family $L\colon S^1 \to \mathcal{L}(\mathbf{c}(\mathbb{R}^N), \mathbf{c}(\mathbb{R}^N))$ verifies:*

(i) L_λ is a Fredholm operator for all $\lambda \in S^1$.
(ii) $\operatorname{Ind} L = [E^s(+\infty)] - [E^s(-\infty)] \in KO(S^1)$.

Remark 4.2. In the proof of Theorem 4.1 we will also compute the index of L_λ in terms of dimensions of the stable spaces at $\pm\infty$. This is far from being new, and similar computations using exponential dichotomies can be found in many places, e.g., [5, 22]. Here we are not interested in the index but rather in the index bundle and our theorem can be considered an extension to the case of families of the computations quoted above.

Proof. Let $\bar{\mathbf{a}}\colon S^1 \times \mathbb{Z} \to GL(N)$ be defined by

$$\bar{\mathbf{a}}(\lambda, n) = (a_n(\lambda)) = \begin{cases} a(\lambda, +\infty) & \text{if } n \geq 0, \\ a(\lambda, -\infty) & \text{if } n < 0. \end{cases} \tag{13}$$

Put $X := \mathbf{c}(\mathbb{R}^N)$. Fix $\lambda \in S^1$ and denote by $\bar{A}_\lambda \in \mathcal{L}(X, X)$ the operator associated to $\bar{\mathbf{a}}_\lambda$. We claim that the operator $K_\lambda = A_\lambda - \bar{A}_\lambda$ is a compact operator. To this end, we will show that K_λ is the limit (in the norm topology of $\mathcal{L}(X, X)$) of a sequence of operators $\tilde{K}_\lambda^m$ with finite dimensional range. We observe that K_λ is defined by $K_\lambda \mathbf{x} = (k_n(\lambda)x_n)$, where $k_n(\lambda) = a_n(\lambda) - \bar{a}_n(\lambda)$ and define

$$\tilde{K}_\lambda^m \mathbf{x} = \begin{cases} k_n(\lambda)x_n & \text{if } \ |n| \le m, \\ 0 & \text{if } \ |n| > m. \end{cases} \tag{14}$$

Clearly $\operatorname{Im} \tilde{K}_\lambda^m$ is finite dimensional. We are to prove that

$$\sup_{\|\mathbf{x}\|=1} \|(K_\lambda - \tilde{K}_\lambda^m)\mathbf{x}\| \xrightarrow[m\to\infty]{} 0, \tag{15}$$

for $\mathbf{x} \in X$. Observe that

$$\|(K_\lambda - \tilde{K}_\lambda^m)\mathbf{x}\| = \sup_{|n|>m} \|k_n(\lambda)x_n\| \ge \sup_{|n|>m+1} \|k_n(\lambda)x_n\| = \|(K_\lambda - \tilde{K}_\lambda^{m+1})\mathbf{x}\|, \tag{16}$$

for all $m \in \mathbb{N}$. Since

$$\lim_{|n|\to\infty} k_n(\lambda) = 0,$$

we infer that for all $\varepsilon > 0$ there exists $n_0 > 0$ such that for all $|n| > n_0$ and $\|\mathbf{x}\| = 1$ one has

$$\|k_n(\lambda)x_n\| < \varepsilon.$$

Consequently, for all $\varepsilon > 0$ there exists $n_0 > 0$ such that

$$\sup_{\|\mathbf{x}\|=1} \|(K_\lambda - \tilde{K}_\lambda^{n_0})\mathbf{x}\| \le \varepsilon. \tag{17}$$

Now taking into account (16) and (17), we deduce that for all $\varepsilon > 0$ there exists $n_0 > 0$ such that for all $m \ge n_0$ one has

$$\sup_{\|\mathbf{x}\|=1} ||(K_\lambda - \tilde{K}_\lambda^m)\mathbf{x}|| \le \varepsilon, \tag{18}$$

which proves (15) and the compactness of the operator K_λ.

Let $\bar{L}_\lambda = S - \bar{A}_\lambda$. Then $L_\lambda - \bar{L}_\lambda = K_\lambda$ and hence the family L differs from the family $\bar{L}$ by a family of compact operators. Therefore L_λ is Fredholm if and only if $\bar{L}$ is Fredholm and moreover the homotopy invariance of the index bundle applied to the homotopy $H(\lambda, t) = \bar{L}_\lambda + tK_\lambda$ shows that $\operatorname{Ind} \bar{L} = \operatorname{Ind} L$. Hence in order to prove the theorem we can assume without loss of generality that $\mathbf{a}$ has already the special form of (13), which we will do from now on. Let

$$\mathbf{c}_k^+ = \{\mathbf{x} \in \mathbf{c}(\mathbb{R}^N) \mid x_i = 0 \ \text{ for } \ i < k\},$$
$$\mathbf{c}_k^- = \{\mathbf{x} \in \mathbf{c}(\mathbb{R}^N) \mid x_i = 0 \ \text{ for } \ i > k\}.$$

Both $\mathbf{c}_k^\pm$ are closed subspaces of $\mathbf{c}(\mathbb{R}^N)$. The space $\mathbf{c}_k^+$ can be isometrically identified with

$$\mathbf{c}_k(\mathbb{R}^N) := \{\mathbf{x} \colon [k, \infty) \cap \mathbb{Z} \to \mathbb{R}^N \mid \lim_{n\to\infty} x_n = 0\}$$

and similarly for $\mathbf{c}_k^-$.

Put $X^+ = Y^+ = \mathbf{c}_0^+$ and $X^- = \mathbf{c}_0^-$, $Y^- = \mathbf{c}_{-1}^-$. Let us consider four linear operators $I \colon Y^- \oplus Y^+ \to X$, $J \colon X \to X^- \oplus X^+$, $L_\lambda^+ \colon X^+ \to Y^+$ and $L_\lambda^- \colon X^- \to Y^-$

defined respectively by

$$I(\mathbf{x},\mathbf{y}) = \mathbf{x}+\mathbf{y},$$

$$J(\mathbf{x})(n) = \begin{cases} (x_0, x_0) & \text{if } \ n = 0, \\ (x_n, 0) & \text{if } \ n < 0, \\ (0, x_n) & \text{if } \ n > 0, \end{cases}$$

$$(L^+_\lambda \mathbf{x})(n) = \begin{cases} x_{n+1} - a(\lambda, +\infty)x_n & \text{for } \ n \geq 0, \\ 0 & \text{for } \ n < 0, \end{cases}$$

$$(L^-_\lambda \mathbf{x})(n) = \begin{cases} 0 & \text{for } \ n > -1, \\ x_{n+1} - a(\lambda, -\infty)x_n & \text{for } \ n \leq -1. \end{cases}$$

We decompose $L_\lambda\colon X \to X$ via the following commutative diagram:

$$\begin{array}{ccc} X^- \oplus X^+ & \xrightarrow{L^-_\lambda \oplus L^+_\lambda} & Y^- \oplus Y^+ \\ \uparrow J & & \downarrow I \\ X & \xrightarrow[L_\lambda]{} & X. \end{array} \tag{19}$$

The commutativity of diagram (19) is easy to check.

Indeed, one has

$$I(L^-_\lambda \oplus L^+_\lambda)J\mathbf{x}(n) = L^-_\lambda J\mathbf{x}(n) + L^+_\lambda J\mathbf{x}(n) = \begin{cases} (L^+_\lambda \mathbf{x})(n) & \text{if } \ n \geq 0, \\ (L^-_\lambda \mathbf{x})(n) & \text{if } \ n < 0, \end{cases}$$

which is the same as

$$(L_\lambda \mathbf{x})(n) = \begin{cases} x_{n+1} - a(\lambda, +\infty)x_n & \text{if } \ n \geq 0, \\ x_{n+1} - a(\lambda, -\infty)x_n & \text{if } \ n < 0. \end{cases} \tag{20}$$

Next, we will show that $L^\pm_\lambda\colon X^\pm \to Y^\pm$ are Fredholm and we will compute the index bundles of $L^\pm$.

For L^+_λ this is the content of the following Lemma:

Lemma 4.3. ([2, Lemma 2.1]) *Let $a \in GL(N)$ be an hyperbolic matrix. Then the operator $S - A\colon \boldsymbol{c}^+_0 \to \boldsymbol{c}^+_0$, defined by*

$$((S-A)\mathbf{x})(n) = \begin{cases} x_{n+1} - ax_n & \text{if } \ n \geq 0, \\ 0 & \text{if } \ n < 0, \end{cases}$$

is surjective with

$$\ker(S-A) = \{\mathbf{x} \in \boldsymbol{c}^+_0 \mid x_{n+1} = a^n x_0 \textit{ for all } n \geq 0 \textit{ and } x_0 \in E^s(a)\}.$$

This lemma was proved in [2, Lemma 2.1] by constructing an explicit right inverse to the operator $S - A\colon \mathbf{c}^+_0 \to \mathbf{c}^+_0$. □

By Lemma 4.3

$$\operatorname{Ker} L^+_\lambda = \{\mathbf{x} \in X^+ \mid x_n = a(\lambda, +\infty)^n x_0 \ \text{ and } \ x_0 \in E^s(\lambda, +\infty)\}. \tag{21}$$

Hence the transformation $\mathbf{x} \mapsto x_0$ defines an isomorphism between $\operatorname{Ker} L^+$ and $E^s(\lambda, +\infty)$, which is finite dimensional. Being $\operatorname{Coker} L_\lambda = 0$, L^+_λ is Fredholm with $\operatorname{ind} L^+_\lambda = \dim E^s(\lambda, +\infty)$. Clearly the index bundle $\operatorname{Ind} L^+ = [E^s(+\infty)]$.

We will reduce the calculation of $\operatorname{Ind} L^-$ to Lemma 4.3 as follows:

Put $Y^- := \mathbf{c}_{-1}^-$ and $X^- := \mathbf{c}_0^-$ and consider the family of isomorphisms $B = \{B_\lambda \colon Y^- \to Y^-\}$ defined by

$$(B_\lambda \mathbf{x})(n) = \begin{cases} 0 & \text{for } n > -1, \\ -a^{-1}(\lambda, -\infty)x_n & \text{for } n \leq -1. \end{cases}$$

We compose $L_\lambda^- \colon X^- \to Y^-$ on the right with the isomorphism $B_\lambda \colon Y^- \to Y^-$ followed by the negative shift S^{-1} viewed as an operator from Y^- to X^-. Since both operators are isomorphisms the composition does not affect the Fredholm property. On the other hand considering S^{-1} as a constant family of isomorphisms, by logarithmic property of the index bundle, $\operatorname{Ind} S^{-1}BL^- = \operatorname{Ind} L^-$. Hence the index bundle of L^- coincides with the index bundle of the family $D = S^{-1}BL^-$. Observe now that, if $\mathbf{x} \in Y^-$, then

$$(B_\lambda L_\lambda^-)\mathbf{x})(n) = \begin{cases} 0 & \text{for } n > -1, \\ x_n - a^{-1}(\lambda, -\infty)x_{n+1} & \text{for } n \leq -1. \end{cases}$$

But since $S^{-1}\mathbf{x} = (x_{n-1})$, one obtains

$$(D_\lambda \mathbf{x})(n) = \begin{cases} 0 & \text{for } n > 0, \\ x_{n-1} - a^{-1}(\lambda, -\infty)x_n & \text{for } n \leq 0. \end{cases}$$

Thus $D_\lambda \colon X^- \to X^-$ is the same type of operator as L_λ^+ but with n going from 0 to $-\infty$. By Lemma 4.3, each D_λ is surjective. Moreover,

$$\operatorname{Ker} D_\lambda = \operatorname{Ker} L_\lambda^- = \{\mathbf{x} \in X^- \mid x_n = a(\lambda, -\infty)^n x_0 \text{ and } x_0 \in E^u(\lambda, -\infty))\} \tag{22}$$

is isomorphic to $E^u(\lambda, -\infty)$.

Summing up, we have obtained that $\operatorname{Ind} L^+ = [E^s(+\infty)]$ and $\operatorname{Ind} L^- = [E^u(-\infty)]$. In particular we have

$$\operatorname{ind} L_\lambda^+ = \dim E^s(\lambda, +\infty) \quad \text{and} \quad \operatorname{ind} L_\lambda^- = \dim E^u(\lambda, -\infty). \tag{23}$$

With this at hand we can compute the index bundle of L completing the proof of the theorem. Let us notice firstly that I and J are Fredholm operators. Indeed, $I \colon Y^- \oplus Y^+ \to X$ is clearly an isomorphism, and the map $J \colon X \to X^- \oplus X^+$ is a monomorphism whose image is given by $\operatorname{Im} J = \{(\mathbf{a}, \mathbf{b}) \in X^- \oplus X^+ \mid a_0 = b_0\}$. Putting $P \colon X^- \oplus X^+ \to \mathbb{R}^N$ by $P(\mathbf{a}, \mathbf{b}) := a_0 - b_0$, for $\mathbf{a} \in X^-$ and $\mathbf{b} \in X^+$, one obtains that $\operatorname{Im} J = \operatorname{Ker} P$. But since P is an epimorphism, we deduce that $\operatorname{Coker} J = X^- \oplus X^+ / \operatorname{Ker} P \simeq \mathbb{R}^N$ and therefore J is Fredholm of index $-N$. From the commutativity of diagram (19) and (23) it follows that $L_\lambda = I(L_\lambda^- \oplus L_\lambda^+)J$ is Fredholm and

$$\begin{aligned} &\operatorname{ind}(L_\lambda) = \operatorname{ind}(I) + \operatorname{ind}(L_\lambda^- \oplus L_\lambda^+) + \operatorname{ind}(J) = \\ &\dim E^s(\lambda, +\infty) + \dim E^u(\lambda, -\infty) - N = \\ &\dim E^s(\lambda, +\infty) - \dim E^s(\lambda, -\infty). \end{aligned} \tag{24}$$

As for (ii), considering I and J as constant families of Fredholm operators, $\operatorname{Ind} I = 0$, $\operatorname{Ind} J = -[\Theta(\mathbb{R}^N)]$. Using the logarithmic and direct sum properties of the index bundle together with (7), we obtain

$$\operatorname{Ind} L = [E^u(-\infty)] + [E^s(+\infty)] - [\Theta(\mathbb{R}^N)] = [E^s(+\infty)] - [E^s(-\infty)],$$

which proves (ii). $\square$

Remark 4.4. Notice that from (21), (22) and (20) it follows that in the case of systems of the special form (13) elements of $\operatorname{Ker} L_\lambda$ are sequences $(x_n) \in X$ such that $x_0 \in E^s(\lambda, +\infty) \cap E^u(\lambda, -\infty)$ and

$$x_n = a(\lambda, +\infty)^n x_0, \text{ for } n \geq 0 \text{ and } x_n = a(\lambda, -\infty)^n x_0, \text{ for } n \leq 0.$$

5. Parity and topological degree of C^1-Fredholm maps

In order to deal with the nonlinear aspects of the problem we will use an extension of the well known Leray-Schauder degree to proper Fredholm maps of index 0 introduced in [17] under the name of *base point degree.* This construction uses a homotopy invariant of paths of Fredholm operators of index 0 called *parity* which is closely related to the index bundle. We will briefly review the concept of parity and the construction of the base point degree. We are specially interested in the particular form of the homotopy property of the base point degree since it represents the main argument in our proof of Theorem 2.3.

From now on we will consider only Fredholm operators of index 0.

Given a continuous map $L\colon [a,b] \to \Phi_0(X,Y)$, a *regular parametrix* (or regularizator) for the path L is a path of isomorphisms $P\colon [a,b] \to Iso(Y,X)$ such that $L_tP_t = \operatorname{Id}_Y - K_t$ and $P_tL_t = \operatorname{Id}_X - K'_t$, where K_t, K'_t are operators of finite rank.

Every path in $\Phi_0(X,Y)$ possesses at least one parametrix. Below we describe a construction related to the index bundle (see [9] for details):

Given $L\colon [a,b] \to \Phi_0(X,Y)$, arguing as in the construction of the index bundle (see Section 3), we take a finite dimensional subspace V of Y and consider the vector bundle

$$E = \bigcup_{t\in[a,b]} \{t\} \times L_t^{-1}(V). \tag{25}$$

It is easy to see that $\dim E_t = \dim V$, where $E_t := L_t^{-1}(V)$. Since E is a trivial bundle there is a vector bundle isomorphism $T\colon E \to \Theta(V) = [a,b] \times V$. Let Q_t be a family of projectors of X with $\operatorname{Im} Q_t = E_t$, let Q' be a projector with $\operatorname{Ker} Q' = V$ and let $A_t = Q'L_t + T_tQ_t$. It is easy to see that A_t is an isomorphism for any $t \in [a,b]$. Its inverse $P_t := A_t^{-1}$ is a *regular parametrix* for L because, as it is easy to see, $L_tP_t = \operatorname{Id}_Y - K_t$ with $\operatorname{Im} K_t \subset V$ and $P_tL_t = \operatorname{Id}_X - K'_t$ with $\operatorname{Im} K'_t \subset E_t$.

Let now $L\colon [a,b] \to \Phi_0(X,Y)$ be a path such that both L_a and L_b are invertible operators. Let P be a parametrix for L. Then $L_tP_t = \operatorname{Id}_Y - K_t$ is invertible for $t = a$ and $t = b$, and so are its restrictions $C_t\colon V \to V$ to any finite dimensional subspace V containing the images of K_t.

The *parity* of the path L is the element $\sigma(L) \in \mathbb{Z}_2 = \{1, -1\}$ defined by

$$\sigma(L) = \operatorname{sign}\det C(a) \operatorname{sign}\det C(b).$$

It is easy to see that this definition is independent of the choices involved and that the parity is invariant under homotopies of paths with invertible end points. Moreover it has the following multiplicative property: if $\{I_k, 1 \leq k \leq m\}$ is a partition of $I = [a,b]$ then

$$\sigma(L) = \prod_{k=1}^{m} \sigma(L_{|I_k}). \tag{26}$$

It can be shown that $\sigma(L) = 1$ if and only if L can be deformed to a family of invertible operators by a homotopy which keeps the end points invertible (see [9]).

If the path L is closed, i.e., $L_a = L_b$, then, via the identification $S^1 \simeq [a,b]/\{a,b\}$ we can consider the path L as a map $L\colon S^1 \to \Phi_0(X,Y)$ and relate the parity of a closed path with the obstruction to triviality $w_1\colon \widetilde{KO}(S^1) \to \mathbb{Z}_2$.

Lemma 5.1. *Under the above assumptions,*

$$\sigma(L) = w_1(\operatorname{Ind} L). \tag{27}$$

Proof. Since, by Proposition 3.1, w_1 is an isomorphism of $\widetilde{KO}(S^1)$ with $\mathbb{Z}_2$ it is enough to check that $\sigma(L) = 1$ if and only if $\operatorname{Ind} L = 0$. Let us recall that two bundles are *stably equivalent* if they become isomorphic after addition of trivial bundles on both sides. It is well known [11] that stable equivalence classes form a group isomorphic to the reduced Grothendieck group $\widetilde{KO}(\Lambda)$.

Since the index bundle of a family of Fredholm operators of index 0 belongs to $\widetilde{KO}(\Lambda)$, it follows that $\operatorname{Ind} L$ can be identified with the stable equivalence class of the vector bundle $E = \bigcup_{\lambda\in\Lambda} \{\lambda\} \times L_\lambda^{-1}(V)$ arising in the construction (10).

If $\operatorname{Ind} L = 0 \in \widetilde{KO}(S^1)$, then, for some $k \geq 0$, $E \oplus \Theta(\mathbb{R}^k)$ is isomorphic to the trivial bundle $\Theta(V \oplus \mathbb{R}^k)$, where V is as in (10). Taking in the definition of the index bundle in (11) a larger subspace V' such that $V'/V \cong \mathbb{R}^k$ we can assume that E itself is trivial. If we use such a V' in the construction of a parametrix for $L\colon [a,b] \to \Phi_0(X,Y)$ as described above, then we get $\sigma(L) = 1$.

On the other hand, if $\sigma(L) = 1$, then one can modify any parametrix of L on $[a,b]$ to a parametrix P with $P_a = P_b$, which defines P on S^1. Then for any $t \in S^1$ we have $P_t L_t = \operatorname{Id}_Y - K_t$ with K_t compact and therefore

$$H(t,s) = P_t^{-1}(\operatorname{Id}_Y - sK_t)$$

is a homotopy in $\Phi_0(X,Y)$ between L and a family of isomorphisms, which implies that $\operatorname{Ind} L = 0$ by Proposition 9.1. □

Now let us sketch the construction of the base point degree in [17].

Let $\mathcal{O} \subset X$ be an open simply connected set and let $f\colon \mathcal{O} \to Y$ be a C^1-Fredholm map of index 0 that is proper on closed bounded subsets of the domain (recall that a C^1-map $f\colon \mathcal{O} \to Y$ is Fredholm of index 0 if the Fréchet derivative $Df(x)$ of f at x is a Fredholm operator of index 0, for all $x \in \mathcal{O}$). Using the parity we can assign to each regular point ([3]) of the map f an orientation $\epsilon(x) = \pm 1$ with similar properties to the sign of the Jacobian determinant in finite dimensions. For this we choose a fixed regular point b of f, called *base point*, and then the corresponding orientation $\epsilon_b(x)$ at any regular point x is uniquely defined by the requirement $\epsilon_b(x) = \sigma(Df \circ \gamma)$, where γ is any path in $\mathcal{O}$ joining b to x. Since $\mathcal{O}$ is simply connected, the independence from the choice of the path follows from the homotopy invariance of the parity.

Let Ω be an open bounded set whose closure is contained in $\mathcal{O}$ such that 0 is a regular value of the restriction of f to Ω and such that $0 \notin f(\partial\Omega)$. Then the base point degree of f in Ω is defined by

$$\deg_b(f,\Omega,0) = \sum_{x\in f^{-1}(0)} \epsilon_b(x). \tag{28}$$

In the above definition we use the convention that a sum over the empty set is 0.

It is proved in [17] that this assignment extends to an integral-valued degree theory for C^1-Fredholm maps defined on simply connected sets that are proper on

[3] p is a regular point of f if $Df(p)$ is an isomorphism.

closed bounded subsets of its domain. The base point degree is invariant under homotopies only up to sign and, as a matter of fact, since the identity map of a (separable) Hilbert space can be connected to an isomorphism of the form the identity map plus a compact map whose Leray-Schauder degree is -1, no degree theory for general Fredholm maps extending the Leray-Schauder degree can be homotopy invariant.

The main reason for introducing the base point degree is that the change in sign along a homotopy can be determined using the parity.

An admissible homotopy in our setting is a continuous family of C^1-Fredholm maps $h\colon [0,1]\times\mathcal{O}\to Y$ parametrized by $[0,1]$ which is proper on closed bounded subsets of $[0,1]\times\mathcal{O}$. As usual, continuous family of C^1-maps means that h is continuous, differentiable in the second variable with the derivative continuously depending on (t,x).

Our proof of the main result will be based on the following *homotopy variation property* of the base point degree (see [16, Lemma 2.3.1]):

Lemma 5.2. *Let $h\colon [0,1]\times\mathcal{O}\to Y$ be an admissible homotopy, and let Ω be an open bounded subset of X such that $0\notin h([0,1]\times\partial\Omega)$. If $b_i\in\mathcal{O}$ is a base point for $h_i =: h(i,-); i=0,1,$ then*

$$\deg_{b_0}(h_0,\Omega,0)=\sigma(M)\deg_{b_1}(h_1,\Omega,0), \tag{29}$$

where $M\colon[0,1]\to\Phi_0(X,Y)$ is the path $L\circ\gamma$, where $L(t,x)=Dh_t(x)$ and γ is any path joining $(0,b_0)$ to $(1,b_1)$ in $[0,1]\times\mathcal{O}$.

6. Proof of Theorem 2.3

Let $\mathbf{f}\colon\mathbb{Z}\times S^1\times\mathbb{R}^N\to\mathbb{R}^N$ be a continuous family of nonautonomous dynamical systems verifying $(A0)$–$(A2)$. Take $X:=\mathbf{c}(\mathbb{R}^N)$ and let $F\colon S^1\times X\to X$ be defined by

$$F(\lambda,\mathbf{x})=(f_n(\lambda,x_n)),\ \text{for } \mathbf{x}\in X \text{ and } \lambda\in S^1. \tag{30}$$

Firstly we observe that $F\colon S^1\times X\to X$ is well defined. Indeed, given $\mathbf{x}\in X$, taking into account Assumption $(A2)$, we deduce that

$$C_\lambda:=\sup_{(n,s)\in\mathbb{Z}\times[0,1]}\left\|\frac{\partial f_n}{\partial x}(\lambda,sx_n)\right\|<\infty,$$

for all $\lambda\in S^1$. Hence using the mean value estimate we get

$$\|f_n(\lambda,x_n)\|=\|f_n(\lambda,x_n)-f_n(\lambda,0)\|\le\sup_{s\in[0,1]}\left\|\frac{\partial f_n}{\partial x}(\lambda,sx_n)\right\|\cdot\|x_n\|\le C_\lambda\|x_n\|.$$

Thus $f_n(\lambda,x_n)\to 0$ as $n\to\pm\infty$, which proves that the map $F\colon S^1\times X\to X$ is well defined. Furthermore, the same argument allows us to define the family of linear bounded operators $T\colon S^1\times X\to\mathcal{L}(X,X)$ by

$$T(\lambda,\mathbf{x})\mathbf{y}:=\left(\frac{\partial f_n(\lambda,x_n)}{\partial x}y_n\right), \tag{31}$$

for $\mathbf{x}=(x_n),\mathbf{y}=(y_n)\in X$ and $\lambda\in S^1$.

Lemma 6.1.

i) If $\mathbf{f}$ *verifies* $(A1)$ *and* $(A2)$*, then the map* $F\colon S^1 \times X \to X$ *defined by* (30) *is a continuous family of* C^1*-maps parametrized by* S^1*. Moreover* $DF_\lambda(\mathbf{x}) = T(\lambda, \mathbf{x})$*.*

ii) If also $(A3)$ *holds, then there exists a closed neighborhood* $D = \bar{B}(\mathbf{0}, \delta)$ *of* $\mathbf{0}$ *in* X *such that the restriction of* F *to* $S^1 \times D$ *is a proper continuous family of* C^1*-Fredholm maps of index* 0*. Namely,* $F\colon S^1 \times D \to X$ *is continuous and proper. Moreover, for any* $\lambda \in S^1$*, the map* $F\colon S^1 \times D \to X$ *is differentiable in the second variable and* $DF_\lambda(\mathbf{x})$ *is a Fredholm operator of index* 0 *continuously depending on* $(\lambda, \mathbf{x})$*.*

Proof. The proof of $i)$ follows the lines of [20, Lemma 2.3]. We sketch it below for convenience of the reader, since our setting is slightly more general than the one in [20]. Notice that $(A1)$ tells that the sequence, for $j = 0, 1$, $\frac{\partial^j f_n}{\partial x^j}$ is uniformly equicontinuous, while $(A2)$ means that the restriction of the same sequence to bounded subsets of the domain is equibounded.

The continuity of F and the map T defined above follows easily from the equicontinuity assumption $(A1)$.

For fixed $\mathbf{x} \in X$ and $\lambda \in S^1$ we will show that $DF_\lambda(\mathbf{x}) = T(\lambda, \mathbf{x})$. To this end, let

$$R(\mathbf{x}, \mathbf{h}; \lambda) := \|F(\lambda, \mathbf{x} + \mathbf{h}) - F(\lambda, \mathbf{x}) - T(\lambda, \mathbf{x})\mathbf{h}\|, \tag{32}$$

where $\mathbf{h} \in \mathbf{c}(\mathbb{R}^N)$ and $\lambda \in S^1$. We are to show that $\frac{R(\mathbf{x},\mathbf{h};\lambda)}{\|\mathbf{h}\|} \to 0$ as $\|\mathbf{h}\| \to 0$. Let

$$c_n(\mathbf{h}; \lambda) := \sup_{s \in [0,1]} \left\| \frac{\partial f_n(\lambda, x_n + s h_n)}{\partial x} - \frac{\partial f_n(\lambda, x_n)}{\partial x} \right\|,$$

for $n \in \mathbb{Z}$. Then Assumptions $(A2)$ and $(A1)$ imply that

$$c_n(\mathbf{h}; \lambda) < \infty \text{ and } \sup_{n \in \mathbb{Z}} c_n(\mathbf{h}; \lambda) \to 0 \text{ as } \|\mathbf{h}\| \to 0. \tag{33}$$

Then

$$\begin{aligned}
&\left\| f_n(\lambda, x_n + h_n) - f_n(\lambda, x_n) - \frac{\partial f_n(\lambda, x_n)}{\partial x} h_n \right\| = \\
&\left\| \int_0^1 \frac{\partial f_n(\lambda, x_n + s h_n)}{\partial x} h_n ds - \frac{\partial f_n(\lambda, x_n)}{\partial x} h_n \right\| \le \\
&\int_0^1 \left\| \frac{\partial f_n(\lambda, x_n + s h_n)}{\partial x} - \frac{\partial f_n(\lambda, x_n)}{\partial x} \right\| ds \cdot \|h_n\| \le \\
&\int_0^1 \sup_{n \in \mathbb{Z}} c_n(\mathbf{h}; \lambda) ds \cdot \|h_n\| = \|h_n\| \cdot \sup_{n \in \mathbb{Z}} c_n(\mathbf{h}; \lambda) \le \|\mathbf{h}\| \cdot \sup_{n \in \mathbb{Z}} c_n(\mathbf{h}; \lambda).
\end{aligned}$$

Hence

$$0 \le R(\mathbf{x}, \mathbf{h}; \lambda) \le \|\mathbf{h}\| \sup_{n \in \mathbb{Z}} c_n(\mathbf{h}; \lambda), \tag{34}$$

which implies that $\frac{R(\mathbf{x},\mathbf{h};\lambda)}{\|\mathbf{h}\|} \to 0$ as $\|\mathbf{h}\| \to 0$. This completes the proof of $i)$ since we already know that T is continuous.

Let us prove $ii)$. By the previous considerations the map $G(\lambda, \mathbf{x}) = S\mathbf{x} - F(\lambda, \mathbf{x})$ is a continuous family of C^1-maps. Since $a_n(\lambda) = \frac{\partial f_n}{\partial x}(\lambda, 0)$, it follows from (31) that $DG_\lambda(\mathbf{0})$ is the operator $L_\lambda\colon X\to X$ defined by

$$L_\lambda \mathbf{x} = (x_{n+1} - a_n(\lambda)x_n). \tag{35}$$

Being $\mathbf{a}$ asymptotically hyperbolic, by Theorem 4.1, the operator L_λ is Fredholm with index given by (24). Thus $\operatorname{ind} L_\lambda = 0$, since by $(A3)$ the stable subspaces at $\pm\infty$ have the same dimension.

Since $\Phi_0(X,X)$ is an open subset of $\mathcal{L}(X,X)$, by continuity of $DG_\lambda(\mathbf{x})$ and compactness of S^1, there exists a $\delta>0$ such that the restriction of G to $S^1\times B(\mathbf{0},\delta)$ is a continuous family of C^1-Fredholm maps of index 0. Using compactness of S^1 again, we can find an eventually smaller δ such that for all $\lambda\in S^1$ the restriction of G to $S^1\times\bar B(\mathbf{0},\delta)$ becomes proper, because continuous families of C^1-Fredholm maps are locally proper ([7, Lemma 3.5]). □

Now we can finalize the proof of Theorem 2.3 using the homotopy variance property of base point degree. By $(A4)$, L_{λ_0} is injective, and hence it follows from the Fredholm alternative that L_{λ_0} must be invertible. The inverse function theorem implies that for $\delta>0$ small enough $\mathbf{0}$ is the only solution of $G_{\lambda_0}(\mathbf{x})=\mathbf{0}$ in $B(\mathbf{0},\delta)$. Moreover we can take δ so small that $G\colon S^1\times\bar B(\mathbf{0},\delta)\to X$ verifies $ii)$ of the above Lemma. To simplify notation we suppose that $\lambda_0=1\in S^1$.

Assume that for $\varepsilon<\delta$ there are no homoclinic solutions $(\lambda,\mathbf{x})$ of (5) with $\|\mathbf{x}\|=\varepsilon$, then $G_\lambda(\mathbf{x})\neq\mathbf{0}$ on $\partial B(\mathbf{0},\varepsilon)$. Consider the homotopy $H\colon[0,1]\times\bar B(\mathbf{0},\varepsilon)\to X$ defined by $H(t,\mathbf{x})=G(\exp(2\pi i t),\mathbf{x})$. Then H is an admissible homotopy with $H_0=G_1=H_1$. Furthermore, we can take $b=\mathbf{0}$ as the base point for both H_0 and H_1. Since $\mathbf{0}\notin H\big([0,1]\times\partial B(\mathbf{0},\varepsilon)\big)$ by Lemma 5.2, with $\gamma(t)=(t,0)$,

$$\deg_{\mathbf{0}}(H_1,B(\mathbf{0},\varepsilon),\mathbf{0})=\sigma(M)\deg_{\mathbf{0}}(H_0,B(\mathbf{0},\varepsilon),\mathbf{0}),$$

where $M(t)=L_{\exp(2\pi i t)}$. By definition of the degree for a regular value (28) we have

$$\deg_{\mathbf{0}}(H_j,B(\mathbf{0},\varepsilon),\mathbf{0})=\deg_{\mathbf{0}}(G_1,B(\mathbf{0},\varepsilon),\mathbf{0})=1,\ j=0,1. \tag{36}$$

Thus $\sigma(M)=1$. But $\sigma(M)$ coincides with the parity of the closed path L. Hence, by Lemma 5.1, Theorem 4.1 and (12)

$$1=\sigma(L)=w_1(\operatorname{Ind} L)=w_1(E^s(+\infty))w_1(E^s(-\infty)),$$

which contradicts our assumption. □

7. An example

In this section we are going to illustrate the content of Theorem 2.3 comparing our result with the standard theory.

For $\lambda=\exp(i\theta)$, $0\le\theta\le 2\pi$, we put

$$a(\lambda)=a(\exp i\theta):=\begin{pmatrix}1/2+(3/2)\sin^2\theta/2 & -(3/4)\sin\theta\\ -(3/4)\sin\theta & 1/2+(3/2)\cos^2\theta/2\end{pmatrix}$$

and consider the linear nonautonomous system $\mathbf{a}=(a_n(\lambda))\colon\mathbb{Z}\times S^1\to GL(N)$ defined by

$$a_n(\lambda)=\begin{cases}a(\lambda) & \text{if } n\ge 0,\\ a(1) & \text{if } n<0.\end{cases} \tag{37}$$

Notice that system $\mathbf{a}$ has the special "jump" form (13), used in the proof of Theorem 4.1.

Since independently of $\lambda \in S^1$ the matrix $a(\lambda)$ has two eigenvalues $1/2$ and 2, the system $\mathbf{a}$ is asymptotically hyperbolic.

We will apply our results to nonlinear perturbations of $\mathbf{a}$. We compute the asymptotic stable bundles of $\mathbf{a}$ at $\pm\infty$:

$$E^s(+\infty) = \{(\lambda, u) \in S^1 \times \mathbb{R}^2 \mid u = t(\cos(\theta/2), \sin(\theta/2)), \lambda = \exp(i\theta), t \in \mathbb{R}\},$$
$$E^s(-\infty) = \{(\lambda, u) \in S^1 \times \mathbb{R}^2 \mid u = (t, 0), t \in \mathbb{R}\}.$$

Thus $E^s(-\infty)$ is a trivial bundle and hence $w_1(E^s(-\infty)) = 1$. In order to compute $w_1(E^s(+\infty))$ we notice that $v_\theta = (\cos(\theta/2), \sin(\theta/2))$ is a basis for $E^s_\theta(+\infty)$ which is the fiber of the pullback E' of $E^s(+\infty)$ by the map $p\colon [0, 2\pi] \to S^1$ defined by $p(\theta) = \exp(i\theta)$.

Since $v_0 = (1, 0)$ and $v_{2\pi} = (-1, 0)$, the determinant of the matrix C arising in (8) is -1. Hence $w_1(E^s(+\infty)) = -1 \neq w_1(E^s(-\infty))$. Notice that $E^s(+\infty)$ is an infinite Moebius band while $E^s(-\infty)$ is an infinite cylinder.

If $\mathbf{h}$ is any nonlinear perturbation of $\mathbf{a}$ verifying $(A0)$–$(A2)$ and $(A3')$, $(A4')$ then by Corollary 2.5 the family $\mathbf{f} = \mathbf{a} + \mathbf{h}$ must have nontrivial homoclinic solutions bifurcating from the stationary branch at some $\lambda_* \in S^1$.

On the other hand, let us consider the family L of operators L_λ defined by

$$L_\lambda(\mathbf{x})(n) = \begin{cases} x_{n+1} - a(\lambda)x_n & \text{if } \; n \geq 0, \\ x_{n+1} - a(1)x_n & \text{if } \; n < 0. \end{cases}$$

By Remark 4.4 $\operatorname{Ker} L_\lambda$ is isomorphic to $E^s(\lambda, +\infty) \cap E^u(\lambda, -\infty)$.

But $E^u(-\infty) = \{(\lambda, u) \in S^1 \times \mathbb{R}^2 \mid u = (0, t), t \in \mathbb{R}\}$ and hence a nontrivial intersection arises only for $\theta = \pi$, i.e., $\lambda = -1$. Thus $\operatorname{Ker} L_\lambda \neq 0$ only if $\lambda = -1$.

Now, if $\mathbf{h}$ verifies, in a neighborhood of $\lambda = -1$, that $\frac{\|h_n(\lambda, x)\|}{\|x\|} \to 0$ uniformly in (n, λ), we can use the classical approach based on Lyapunov-Schmidt reduction in order to obtain the existence of a branch of homoclinics bifurcating from the stationary branch at this point.

Indeed, if the above condition holds true, the family $H\colon S^1 \times \mathbf{c}(\mathbb{R}^N) \to \mathbf{c}(\mathbb{R}^N)$, induced on function spaces, verifies $H(\lambda, \mathbf{x}) = o(\|\mathbf{x}\|)$ as $\mathbf{x} \to \mathbf{0}$. Being $\operatorname{Ker} L_{-1}$ one dimensional, we can check the hypothesis of the Crandal-Rabinowitz Bifurcation Theorem in order to find at $\lambda = -1$ a bifurcating branch of homoclinics [8].

Instead, by using Corollary 2.5, we lost any information about the position of the bifurcation point, but we proved the appearance of nontrivial homoclinic trajectories for a rather general class of perturbations.

To some extent, the use in bifurcation theory of elliptic invariants "at large" in place of the Lyapunov-Schmidt method parallels the use of the topological degree instead of the multiplicity of an isolated solution in continuation problems. This observation is formulated more precisely in [16]. Let us point out however, that the relation between the birth of homoclinics and the topology of asymptotic stable bundles is interesting by itself and goes beyond the formal aspects of the abstract theory.

8. Comments

It should be noted that our results can be proved under weaker assumptions. Mainly, it suffices to assume that the nonautonomous difference system admits an exponential dichotomy [5, 18, 20]. This becomes useful in dealing with difference equations in Banach spaces. On the other hand our method can be easily adapted in order to study bifurcation of homoclinics on manifolds. Following [2], to each finite dimensional manifold M and discrete dynamical system $\mathbf{f}$ on M having $x \in M$ as a stationary trajectory we can associate the Banach manifold $\mathbf{c}_x(M)$ which is a natural place for the study of trajectories of the dynamical system $\mathbf{f}$ homoclinic to x.

There are two interesting problems which were not considered here. Namely, global bifurcation, which studies the existence of connected branches of solutions and their behavior, and the existence of large homoclinic trajectories using bifurcation from infinity. These will be treated in a forthcoming paper of the present authors.

9. Appendix: Properties of the index bundle

Proposition 9.1. *The element* $\operatorname{Ind} L \in KO(\Lambda)$ *defined by* (11) *is independent from the choice of* V*. Moreover the index bundle* $\operatorname{Ind} L$ *verifies:*

- (i) Functoriality: *If* $L\colon \Lambda \to \Phi(X,Y)$ *is a family of Fredholm operators and* $\alpha\colon \Sigma \to \Lambda$ *is a continuous map between compact spaces, then*
$$\operatorname{Ind} L \circ \alpha = \alpha^*(\operatorname{Ind} L),$$
where $\alpha^*\colon KO(\Lambda) \to KO(\Sigma)$ *is the homomorphism induced by* α.
- (ii) Homotopy invariance: *Let* $H\colon [0,1] \times \Lambda \to \Phi(X,Y)$ *be a homotopy, then* $\operatorname{Ind} H_0 = \operatorname{Ind} H_1$. *In particular,* $\operatorname{Ind}(L+K) = \operatorname{Ind} L$, *if* K *is a family of compact operators.*
- (iii) Additivity: $\operatorname{Ind}\big(L \oplus M\big) = \operatorname{Ind} L + \operatorname{Ind} M$.
- (iv) Logarithmic property: $\operatorname{Ind}\big(LM\big) = \operatorname{Ind} L + \operatorname{Ind} M$.
- (v) Normalization: *If* L *is homotopic to a family in* $GL(X,Y)$, *then* $\operatorname{Ind} L = 0$. *Moreover, the converse holds if* Y *is a Kuiper space.*

We will first show that the index bundle is well defined: If V_1 and V_2 are two subspaces verifying the transversality condition (10) and E, F are the corresponding vector bundles, we can suppose without loss of generality that $V_1 \subset V_2$ and hence that E is a subbundle of F. The restriction of the family L to F induces an isomorphism of F/E with the trivial bundle with fiber V_2/V_1. Since exact sequences of vector bundles split, it follows that F is isomorphic to a direct sum of E with a trivial bundle and hence $E - \Theta(V_1)$ and $F - \Theta(V_2)$ define the same class in $KO(\Lambda)$. This shows that $\operatorname{Ind} L$ is well defined.

Clearly $\operatorname{Ind} L = 0$ if L is homotopic to a family of invertible operators. Taking the same subspace V in the definition of the index bundle for both L and $L \circ \alpha$, property (i) follows plainly from the definition of $\alpha^*(E)$. Now, (ii) follows from (i) applied to the top and bottom inclusions of Λ in $[0,1] \times \Lambda$. The proof of (iii) is straightforward. In order to prove (iv) we observe that in the construction of the index bundle one can replace the finite dimensional subspace V of Y with a finite dimensional subbundle of $\Lambda \times Y$ transverse to L. Now, if $\Theta(V)$ is transverse to LM, then $\Theta(V)$ is transverse to L and $E = L^{-1}\Theta(V)$ is transverse to M. Then, denoting by $F = M^{-1}E$, in $KO(\Lambda)$ we have
$$\operatorname{Ind}\big(LM\big) = [F] - [\Theta(V)] = ([F] - [E]) + ([E] - [\Theta(V)]) = \operatorname{Ind} L + \operatorname{Ind} M.$$
The proof of (v) can be found in [9, Theorem 1.6.3].

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