Polynomial Chaos-Based Tolerance Analysis of Microwave Planar Guiding Structures

Paolo Manfredi, and Flavio G. Canavero

Dipartimento di Elettronica, Politecnico di Torino
10129 Torino, Italy (paolo.manfredi@polito.it)

***Abstract*— This paper focuses on the derivation of an enhanced transmission-line model allowing to describe a realistic microwave interconnect with the inclusion of external uncertainties, like tolerances or process variations. The proposed method, that is based on the expansion of the well-known telegraph equations in terms of orthogonal polynomials, turns out to be accurate and more efficient than alternative solutions like Monte Carlo method in determining the transmission-line response sensitivity to parameters variability. An application example involving the analysis of the S-parameters of a realistic PCB coplanar waveguide concludes the paper.**

***Index Terms*— Stochastic analysis, Tolerance analysis, Uncertainty, Circuit modeling, Circuit simulation, Transmission lines.**

I. Introduction

Nowadays, simulation techniques allowing for the analysis of microwave lines with the inclusion of the effects of possible uncertainties of the circuit parameters are highly desirable, in view of the urging necessity to perform right-the-first-time designs. The stochastic analysis is a tool that is extremely useful in the early design phase for the prediction of the system performance and for setting realistic design margins. A relevant example is provided by the process-induced variability that unavoidably impacts on the performance of microwave planar structures [1].

The typical resource allowing to collect quantitative information on the statistical behavior of the circuit response is based on the application of the brute-force Monte Carlo (MC) method, or possible complementary methods based on the optimal selection of the subset of model parameters in the whole design space. Such methods, however, are computationally expensive, and this fact prevents us from their application to the analysis of complex realistic structures.

Recently, an effective solution that overcomes the previous limitation, has been proposed. This methodology is based on the polynomial chaos (PC) theory and on the representation of the stochastic solution of a dynamical circuit in terms of orthogonal polynomials. For a comprehensive and formal discussion of PC theory, the reader is referred to [2], [3] and references therein. PC technique enjoys applications in several domains of Physics; we limit ourselves to mention recent results on the extension of the classical modified nodal analysis (MNA) approach to the prediction of the stochastic behavior of circuits with uncertain parameters [4]. Also, an extension of PC theory to structures described by transmission-line equations has been proposed recently [5].

This paper demonstrates the feasibility and strength of the PC approach for a realistic guiding microwave structure. This application is supported by a preliminary statistical test for the selection of the most influential design parameters.

II. Polynomial Chaos Primer

The idea underlying the PC technique is the spectral expansion of a stochastic function (intended as a given function of a random variable) in terms of a truncated series of orthogonal polynomials. Within this framework, a function H, that in our specific application will be the expression of the per-unit-length (p.u.l.) parameters and the resulting frequency-domain response of a transmission line, can be approximated by means of the following truncated series

$$H(\xi) = \sum_{k=0}^{P} H_k \cdot \phi_k(\xi), \tag{1}$$

where $\{\phi_k\}$ are suitable orthogonal polynomials expressed in terms of the random variable ξ. The above expression is defined by the class of the orthogonal bases, by the number of terms P (limited to the range $2 \div 5$ for practical applications) and by the expansion coefficients H_k. The choice of the orthogonal basis relies on the distribution of the random variables being considered. The uncertainties arising from fabrication tolerances turn out to be properly characterized in terms of gaussian variability. Therefore, in this case, the most appropiate orthogonal functions for the expansion (1) are the Hermite polynomials, the first three being $\phi_0 = 1$, $\phi_1 = \xi$ and $\phi_2 = (\xi^2 - 1)$, where ξ is the standard normal random variable, with zero mean and unity standard deviation. It is ought to remark that any random parameter in the system can be related to ξ through its mean μ and standard deviation σ; e.g., the substrate permittivity becomes $\varepsilon_r = \mu + \sigma\xi$. The orthogonality property of Hermite polynomials is expressed by

$$< \phi_k, \phi_j > = < \phi_k, \phi_k > \delta_{kj}, \tag{2}$$

where δ_{kj} is the Kronecker delta and $< \cdot, \cdot >$ denotes the inner product in the Hilbert space of the variable ξ with Gaussian weighting function, i.e.,

$$\begin{cases} <\phi_k, \phi_j> = \int_{-\infty}^{+\infty} \phi_k(\xi)\phi_j(\xi)W(\xi)d\xi \\ W(\xi) = \exp(-\xi^2/2)/(\sqrt{2\pi}). \end{cases} \quad (3)$$

With the above definitions, the expansion coefficients H_k of (1) are computed via the projection of H onto the orthogonal components ϕ_k. It is worth noting that relation (1), which is a known nonlinear function of the random variable ξ, can be used to predict the probability density function (PDF) of $H(\xi)$ via numerical simulation or analytical formulae [6]. For the sake of brevity, the formal development of PC theory for multiple variables is omitted here.

III. PC Application to Stochastic Transmision-Line Equations

This section discusses the modification of the classical transmission-line equations, as needed for incorporating the effects of the statistical variation of the p.u.l. parameters via the PC theory.

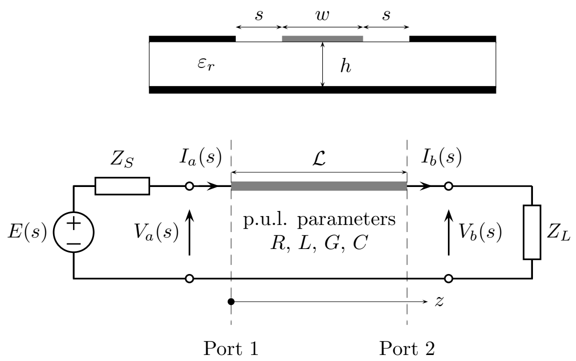

Fig. 1. Grounded coplanar waveguide test structure considered to demonstrate the proposed approach. Top panel: cross-section; bottom panel: simulation test case with port definition for S-parameters.

A. Classical Transmission-Line Model

For the sake of simplicity, the discussion is based on a two-conductor line, as the grounded coplanar waveguide (GCPW) structure shown in Fig. 1, in presence of a single random parameter. The wave propagation on the structure is governed by the telegraphers equation in the Laplace domain [7]

$$\frac{d}{dz}\begin{bmatrix} V(z,s) \\ I(z,s) \end{bmatrix} = -\begin{bmatrix} 0 & R+sL \\ G+sC & 0 \end{bmatrix}\begin{bmatrix} V(z,s) \\ I(z,s) \end{bmatrix}. \quad (4)$$

In the above equation, s is the Laplace variable, V and I are the transverse voltage and current variables in the longitudinal z direction while C, L, R and G are the p.u.l. capacitance, inductance, resistance and conductance, respectively, depending on the geometrical and material properties of the structure.

In order to account for the uncertainties affecting the guiding structure, we must consider the p.u.l. parameters as random quantities, with entries depending on the random variable ξ. In turn, (4) becomes a stochastic differential equation, leading to randomly-varying voltages and currents along the line.

B. Stochastic Transmission-Line Model

The expansion (1) of the p.u.l parameters and of the unknown voltage and current variables in terms of Hermite polynomials, yields a modified version of (4), whose second row becomes

$$\begin{aligned} &\tfrac{d}{dz}(I_0(z,s)\phi_0 + I_1(z,s)\phi_1 + I_2(z,s)\phi_2) = \\ &-[G_0\phi_0 + G_1\phi_1 + G_2\phi_2 + s(C_0\phi_0 + C_1\phi_1 + \\ &+C_2\phi_2)](V_0(z,s)\phi_0 + V_1(z,s)\phi_1 + V_2(z,s)\phi_2), \end{aligned} \quad (5)$$

where a second-order expansion (i.e., $P = 2$) is assumed; the expansion coefficients of electrical variables and of p.u.l. parameters are readily identifiable in the above equation.

Projection of (5) and of the companion relation arising from the first row of (4) on the first three Hermite polynomials leads to the following augmented system, where the random variable ξ does not appear explicitely, due to the integral projection form given in (3):

$$\frac{d}{dz}\begin{bmatrix} \mathbf{V}(z,s) \\ \mathbf{I}(z,s) \end{bmatrix} = -s\begin{bmatrix} 0 & \mathbf{R}+s\mathbf{L} \\ \mathbf{G}+s\mathbf{C} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{V}(z,s) \\ \mathbf{I}(z,s) \end{bmatrix}. \quad (6)$$

In the previous equation, vectors $\mathbf{V} = [V_0, V_1, V_2]^T$ and $\mathbf{I} = [I_0, I_1, I_2]^T$ collect the different coeffcients of the polynomial chaos expansion of the voltage and current variables. The new p.u.l. matrix $\mathbf{C}$ turns out to be

$$\mathbf{C} = \begin{bmatrix} C_0 & C_1 & 2C_2 \\ C_1 & C_0 + 2C_2 & 2C_1 \\ C_2 & C_1 & C_0 + 4C_2 \end{bmatrix} \quad (7)$$

and a similar relation holds for matrices $\mathbf{L}$, $\mathbf{R}$ and $\mathbf{G}$.

It is worth noting that (6) belongs to the same class of (4) and plays the role of the set of equations of a multiconductor transmission line with a number of conductors that is $(P+1)$ times larger than those of the original line. However, for small values of P (as typically occurs in practice), the additional overhead in handling the augmented equations is much less than the time required to run a large number of MC simulations. Extension of the procedure to the general case of a multiconductor line with multiple random parameters is straightforward.

C. Inclusion of Losses

According to [7], the p.u.l. resistance can be approximated by

$$R = R(f) = \begin{cases} R_{dc} & f < f_0 \\ R_{dc}\sqrt{\frac{f}{f_0}} & f > f_0, \end{cases} \quad (8)$$

where R_{dc} is the DC p.u.l. resistance ($= 1/(wt\sigma)$ for rectangular conductors; t, w and σ being the metal thickness, width and conductivity, respectively) and f_0 is the frequency at which the trace thickness equals two skin depths, i.e., $f_0 = 4/\pi\sigma\mu_0 t^2$.

As to the p.u.l. conductance, it can be computed as $G = \omega C \tan\delta$, where $\tan\delta$ is the loss tangent, that can be considered as the largest value in case of inhomogeneous media.

D. Boundary Conditions and Simulation

For the deterministic case, the simulation of an interconnect like the one of Fig. 1 amounts to combining the port electrical relations of the two terminal elements defining the source and load with the transmission-line equation, and solving the system. This is a standard procedure as illustrated for example in [7] (see Ch.s 4 and 5).

Similarly, when the problem becomes stochastic, the augmented transmission-line equation (6) is used in place of (4) together with the projection of the characteristics of the source and the load elements on the first $P+1$ Hermite polynomials. For the example of Fig. 1, the augmented port equations of the line terminations become

$$\begin{cases} \mathbf{V}_a(s) &= [E(s), 0, 0]^T - Z_S(s)\mathbf{I}_a(s) \\ \mathbf{V}_b(s) &= Z_L(s)\mathbf{I}_b(s), \end{cases} \tag{9}$$

where the port voltages and currents need to match the solutions of the differential equation (6) at line ends (e.g., $\mathbf{V}_a(s) = \mathbf{V}(z{=}0, s)$, $\mathbf{V}_b(s) = \mathbf{V}(z{=}\mathcal{L}, s)$). It is worth noting that in this specific example, no variability is included in the terminations, hence the augmented characteristics of the source and load turn out to have a diagonal structure.

Once the unknown voltages and currents are computed, the quantitative information on the spreading of circuit responses can be readily obtained from the analytical expression of the unknowns. As an example, the frequency-domain solution of the magnitude of voltage V_1, arising form (9) and (6) with $P=2$, leads to $|V_a(j\omega)| = |V_{a0}(j\omega) + V_{a1}(j\omega)\xi + V_{a2}(j\omega)(\xi^2 - 1)|$. The above relation can be used to compute the PDF of $|V_a(j\omega)|$, using the rules of random variable transformations given in [6].

IV. Numerical Results

In this section, the proposed technique is applied to the yield analysis of a GCPW realized on RO4350B substrate and designed to have a $50\,\Omega$ impedance, in presence of variations due to process tolerances. Referring to Fig. 1, the nominal parameters are $w = 960\,\mu$m, $s = 330\,\mu$m, $h = 500\,\mu$m, $\varepsilon_r = 3.66$ and $\mathcal{L} = 5\,$cm. As to losses, a $\tan\delta$ of 0.0031 and a DC resistance of $\approx 0.9\,\Omega$/m (arising from a trace thickness of $20\,\mu$m and the copper conductivity of $58\,$MS/m) were considered. The approximate relations given in [8] and Sec. III-C were used to compute the PC expansion of the p.u.l. parameters.

Though the PC model can handle multiple random variables, it could be useful to avoid the inclusion of scarcely influential parameters, in order to reduce the problem dimension and optimize the computational effort. To this end, as a first step, a 6-way Analysis of Variance (ANOVA) has been performed on the magnitudes of S_{11} and S_{21} at two different frequencies, $f_1 = 1$ GHz and $f_2 = 1.8$ GHz. The frequencies were chosen in order to cover two opposite behaviors of the frequency-domain response of the line: a flattening and a resonance, respectively. The trace width and conductivity, the substrate height, permittivity and loss tangent, and the separation between the signal land and the ground metalization were considered as factors, with three different levels: -5%, 0% and $+5\%$ variations with respect to the nominal value.

TABLE I

ANOVA results on the influence of geometrical and material properties on the magnitudes of S_{11} and S_{21}.

	$\|S_{11}\|$		$\|S_{21}\|$	
Factor	F @ 1 GHz	F @ 1.8 GHz	F @ 1 GHz	F @ 1.8 GHz
w	100%	25%	100%	6%
s	3%	2%	4%	2%
h	33%	14%	56%	10%
ε_r	27%	100%	58%	44%
$\tan\delta$	0	0	34%	100%
σ	0	0	6%	9%

Tab. I shows the F-test values for the six factors, computed for a confidence bound of 95%, and normalized with respect to the maximum value in each column. This helps to make comparisons within and between columns of Tab. I. The F value is related to the influence of the corresponding factor. The higher is F, the higher is the influence. According to the reported values, s and σ definitely play a negligible role in all situations. Among the remaining factors, h has always an intermediate effect, while w, ε_r and $\tan\delta$ play the most significant role in more than one case. As a consequence, only w, ε_r and $\tan\delta$ were considered as random parameters in the PC model, all with a 5% relative standard deviation. Hence, it is possible to introduce three independent standard gaussian random variables, ξ_1, ξ_2 and ξ_3, and write

$$\begin{cases} w &= 960(1 + 0.05\xi_1) \quad [\mu\text{m}] \\ \varepsilon_r &= 3.66(1 + 0.05\xi_2) \\ \tan\delta &= 0.0031(1 + 0.05\xi_3). \end{cases} \tag{10}$$

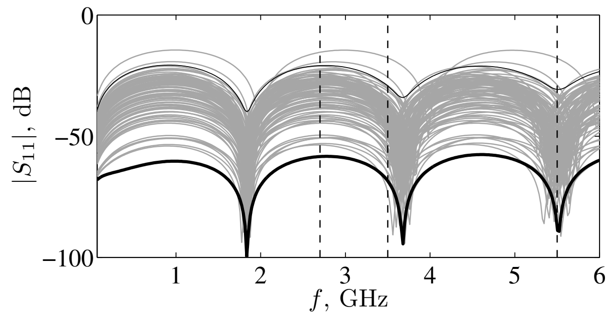

Fig. 2. Magnitude of $S_{11}(j\omega)$. Solid black thick line: deterministic response; solid black thin line: upper limit of the 3σ interval of the fourth order PC expansion; gray lines: a sample of responses obtained by means of the MC method (limited to 100 curves, for graph readability).

Figure 2 shows the frequency behavior of $|S_{11}|$ looking into the line of Fig. 1. The black thick line in Fig. 2 represents the response of the structure for the nominal values of its parameters, while the thinner black line towards the top indicates the upper limit of the 3σ bound, where σ is the standard deviation, determined from the results of the proposed technique. Finally, a qualitative set of 100 MC simulations is plotted using gray lines. Clearly, the parameter variations lead to a growing mismatch in the line impedance and, consequently, to a higher value of reflection, whose spread is well predicted by the estimated 3σ limit.

A better quantitative prediction is possible from the knowledge of the actual PDF of the network response. To this end, Figures 3 and 4 compare the PDFs of $|S_{11}(j\omega)|$ and $|S_{21}(j\omega)|$, respectively, computed for different frequencies over 20,000 MC simulations, and the distributions obtained from the analytical PC expansions. The frequencies selected for this comparison correspond to the dashed lines shown in Fig. 2. Two sets of MC simulations were performed: one (labelled as "3" in the plots) is consistent with the PC model and accounts for randomness on the three main factors suggested by the ANOVA; the other (labelled as "6") includes variability of all the six parameters.

The figures show that the inclusion of variations on s, h and σ do not significantly alter the overall behavior of the PDFs, as predicted by the preliminary sensitivity test. Moreover, the good agreement between the PDFs obtained from the PC model and the corresponding set of MC simulations confirms the potential of the proposed method. In addition, for this example, it is also clear that a PC expansion with four terms is already accurate enough to capture the dominant statistical information of the system response.

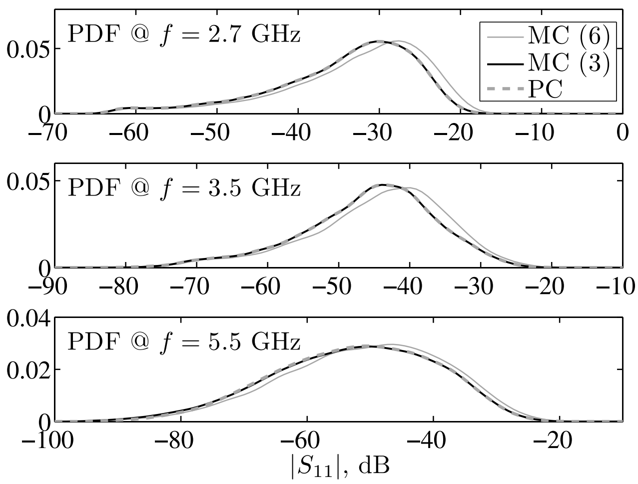

Fig. 3. Probability density function of $|S_{11}(j\omega)|$ computed at different frequencies. Curves marked MC (6) and MC (3) refer to 20,000 MC simulations, computed considering all the six involved parameters or only the three main factors as random, respectively. Finally, the distribution marked PC refers to the response obtained via a fourth-order PC expansion.

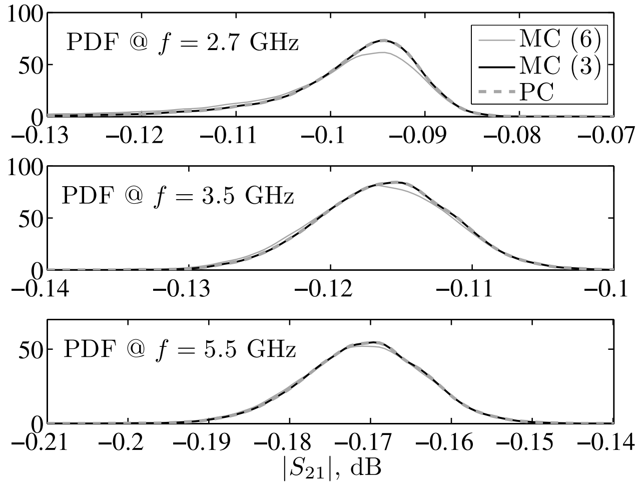

Fig. 4. Probability density function of $|S_{21}(j\omega)|$ computed at different frequencies. Same comments of Fig. 3 apply here.

V. Conclusions

The generation of an enhanced transmission-line equation describing a realistic interconnect structure with the inclusion of fabrication tolerances is addressed in this paper. The proposed method is based on the expansion of the voltage and current variables into a sum of a limited number of orthogonal basis functions, leading to an extended set of telegraph equations. Moreover, a preliminary statistical test is performed to detect the most influential design parameters. The advocated method, while providing accurate results, turns out to be more efficient than the classical Monte Carlo technique in determining the transmission-line response sensitivity to parameters variability. The strenght of the proposed technique is demonstrated by means of a realistic GCPW structure and frequency-domain analysis The speed-up observed in the proposed example is around $200\times$.

References

[1] L. R. A. X. de Menezes, A. O. Paredes, H. Abdalla, G. A. Borges, "Modeling Device Manufacturing Uncertainty in Electromagnetic Simulations," Digest of the *2008 IEEE MTT-S International Microwave Symposium,* pp. 1385–1388, Jun 15-20, 2008.

[2] R. G. Ghanen, P. D. Spanos, "Stochastic Finite Elements. A Spectral Approach," Springer-Verlag, 1991 (Ch. 2).

[3] D. Xiu, G. E. Karniadakis, "The Wiener-Askey Polynomial Chaos for Stochastic Differential Equations," SIAM, Journal of Sci. Computation, Vol. 24, No. 2, pp. 619–622, 2002.

[4] K. Strunz, Q. Su, "Stochastic Formulation of SPICE-Type Electronic Circuit Simulation with Polynomial Chaos," ACM Transactions on Modeling and Computer Simulation, Vol. 18, No. 4, Sep. 2008.

[5] P. Manfredi, I. S. Stievano, F. G. Canavero, "Parameters Variability Effects on Microstrip Interconnects via Hermite Polynomial Chaos," Proc. of the *19th Conference on Electrical Performance of Electronic Packaging and Systems,* pp. 149–152, Oct 24-27, 2010.

[6] A. Papoulis, "Probability, Random Variables and Stochastic Processes," $3^{\rm rd}$ edition, McGraw-Hill, 1991.

[7] C. R. Paul, "Analysis of Multiconductor Transmission Lines," Wiley, 1994.

[8] B. C. Wadell "Transmission Line Design Handbook," Artech House, 1991.