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A B S T R A C T

This work presents the buckling analysis of laminated composite thin walled structures by the 1D finite element based unified higher-order models obtained within the framework of the Carrera Unified Formulation (CUF). In the present study, the refined beam theories are obtained on the basis of Taylor-type expansions. The finite element analysis has been chosen to easily handle arbitrary geometries as well as boundary conditions. Buckling behavior of laminated composite beam and flat panels are analyzed to illustrate the efficacy of the present formulation and various types of buckling modes are observed depending on the geometrical and material parameters. It is observed that the lower order models are unable to deal with torsion.

## 1. Introduction

Composite laminated structures are increasingly being used in the design of load-carrying members for the aerospace, civil and modern engineering applications. These composites are often very susceptible to buckle in various modes. Moreover, the classical theories have been shown to underpredict the deflections and to overpredict the buckling loads [1,2] of these structures. Since much emphasis is given in achieving an optimized ratio of the structure's weight to strength ratio in modern structures, the accurate prediction of their stability limit state is of fundamental importance in the design of thin-walled laminated composite structures. Further, for the case of flat panels, the torsional modes usually occur very close to the critical buckling loads so, use to refined higher order theory which accounts for out of plane displacements is also important.

The first available work in open literature on flexural torsional buckling seems to due to Michell [3] and Prandtl [4] on solid rectangular beams. Significant work on homogeneous beams, based on thin tube theory assumptions, has been carried out by Timoshenko and Gere [5], Hodges and Peters [6], Reissner [7], Hodges [8]. I t i s apt to make a mention here that [7] was the first one who incorporated the transverse shear in the analysis. Studies pertaining to the flexural torsional buckling of composite laminated structures, taking in to account of the shear deformation effect have been carried out in the last two decades. Some of the recent studies are [9–12]. A recent study was also carried out by Sapountzakis and Dourakopoulos [13] using the boundary element method (BEM) in which the assumption of thin walled theory has been avoided. However, it can be noted that these studies are limited to Timoshenko beams with uniform and constant cross sections. To the best of the author's knowledge, publications on the solution to the flexural–torsional buckling analysis of generalized beams of arbitrarily shaped composite cross-section do not exist in open literature. In this investigation, the flexural torsional buckling has been carried out using an efficient hierarchical approach with variable kinematic 2D models successfully developed by Carrera [14,16] and Demasi and Carrera [15]. Carrera Unified Formulation (CUF) permits a systematic assessment of a large number of plate models, whose accuracy has been demonstrated to range from classic 2D models to quasi-3D descriptions for both the dynamic, stability and static stress analyses [17–21]. It may be noted that CUF is a hierarchical formulation which considers the order of the model as a free-parameter of the analysis, in other words, refined models are obtained with no need for *ad hoc* formulations. However, the 2D models used for composite structures are complex and computational expensive and thereby, a need for a much simpler model was brought out.

Because of their simplicity and computational efficiency over the two dimensional theories, the 1D higher order beam theories are also being used to analyze the structural behavior of slender bodies. Studies using the 1D finite element based hierarchical beam theory which incorporates the different expansion orders within the framework of Carrera Unified Formulation (CUF) have been carried out recently. This unified theory was first proposed by Carrera and Giunta [22]. Displacement-based theories that account for non-classical effects, such as transverse shear, in- and out-of-plane warping of the cross-section, can be formulated without the need of any assumption for warping functions. Classical models, such as Euler-Bernoulli and Timoshenko beam theories

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can be retrieved as particular cases. The authors found that the results of one-dimensional CUF models match very well with that obtained using the three-dimensional models. Carrera et al. [23] studied the systematic implementation of the finite elements in the CUF and found out that the finite element based CUF models were able to estimate accurate displacement/strain/stress distribu-tions over the cross-section as well as over particular geometrical features such as corners and voids. Carrera and Petrolo [24] investigated the influence of higher order terms in the refined beam theories for static analyses and recommended the use of full CUF for practical purposes and for nonclassical geometries and loading conditions. Carrera et al. [25] investigated the free vibration analysis of beams with arbitrary geometries using the CUF. A brief review about developments in finite element formulations for vibration analysis of thin and thick laminated beams was also provided. The authors found out that the hierarchical beam models based on CUF were capable of detecting 3-D effects on the vibration modes as well as predicting shell-type vibration modes in case of thin walled beam sections.

This work presents the buckling analysis of thin walled structures by the 1D finite element higher-order models obtained within the framework of the CUF. In the present study, the refined beam theories are obtained on the basis of Taylor-type expansions. The finite element analysis has been chosen to easily handle arbitrary geometries as well as boundary conditions. Several beams are analyzed to illustrate the efficacy of the present formulation. In particular, buckling behavior of laminated composite beam and laminated flat panels are investigated and various types of buckling modes are observed.

## 2. Refined beam models and related FE formulations

### 2.1. Definitions

The adopted coordinate frame is presented in Fig. 1. The beam boundaries over $y$ are $0 \leqslant y \leqslant L$. The displacement vector is:

$$\mathbf{u}(x,y,z) = \{ u_x \quad u_y \quad u_z \}^T \tag{1}$$

The superscript "$T$" represents the transposition operator. Stress, $\boldsymbol{\sigma}$, and strain, $\boldsymbol{\epsilon}$, components are grouped as follows:

$$\boldsymbol{\sigma}_p = \{ \sigma_{zz}\ \sigma_{xx}\ \sigma_{zx} \}^T, \quad \boldsymbol{\epsilon}_p = \{ \epsilon_{zz}\ \epsilon_{xx}\ \epsilon_{zx} \}^T$$
$$\boldsymbol{\sigma}_n = \{ \sigma_{zy}\ \sigma_{xy}\ \sigma_{yy} \}^T, \quad \boldsymbol{\epsilon}_n = \{ \epsilon_{zy}\ \epsilon_{xy}\ \epsilon_{yy} \}^T \tag{2}$$

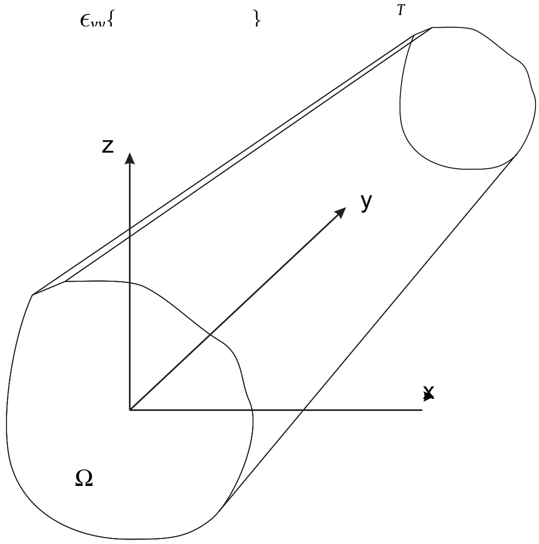


**Fig. 1.** Coordinate frame of the beam model.

The subscript "$n$" stands for terms laying on the cross-section, while "$p$" stands for terms laying on planes which are orthogonal to $\Omega$. Linear strain–displacement relations are used:

$$\boldsymbol{\epsilon}_p = \boldsymbol{D}_p \mathbf{u}$$
$$\boldsymbol{\epsilon}_n = \boldsymbol{D}_n \mathbf{u} = (\boldsymbol{D}_{n\Omega} + \boldsymbol{D}_{ny})\mathbf{u} \tag{3}$$

with:

$$\boldsymbol{D}_p = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & 0 & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}, \quad \boldsymbol{D}_{n\Omega} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & 0 \end{bmatrix}, \quad \boldsymbol{D}_{ny} = \begin{bmatrix} 0 & \frac{\partial}{\partial y} & 0 \\ \frac{\partial}{\partial y} & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial y} \end{bmatrix} \tag{4}$$

The Hooke law is exploited:

$$\boldsymbol{\sigma} = \tilde{\mathbf{C}}\boldsymbol{\epsilon} \tag{5}$$

According to Eq. (2), the previous equation becomes:

$$\boldsymbol{\sigma}_p = \tilde{\boldsymbol{C}}_{pp}\boldsymbol{\epsilon}_p + \tilde{\boldsymbol{C}}_{pn}\boldsymbol{\epsilon}_n$$
$$\boldsymbol{\sigma}_n = \tilde{\boldsymbol{C}}_{np}\boldsymbol{\epsilon}_p + \tilde{\boldsymbol{C}}_{nn}\boldsymbol{\epsilon}_n \tag{6}$$

The material matrices $\tilde{\boldsymbol{C}}_{pp}$, $\tilde{\boldsymbol{C}}_{nn}$, $\tilde{\boldsymbol{C}}_{pn}$ and $\tilde{\boldsymbol{C}}_{np}$ are:

$$\tilde{\boldsymbol{C}}_{pp} = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{16} \\ \tilde{C}_{12} & \tilde{C}_{22} & \tilde{C}_{26} \\ \tilde{C}_{16} & \tilde{C}_{26} & \tilde{C}_{66} \end{bmatrix}, \quad \tilde{\boldsymbol{C}}_{nn} = \begin{bmatrix} \tilde{C}_{55} & \tilde{C}_{45} & 0 \\ \tilde{C}_{45} & \tilde{C}_{44} & 0 \\ 0 & 0 & \tilde{C}_{33} \end{bmatrix},$$
$$\tilde{\boldsymbol{C}}_{pn} = \tilde{\boldsymbol{C}}^T_{np} = \begin{bmatrix} 0 & 0 & \tilde{C}_{13} \\ 0 & 0 & \tilde{C}_{23} \\ 0 & 0 & \tilde{C}_{36} \end{bmatrix} \tag{7}$$

For the sake of brevity, the dependence of coefficients $[\tilde{C}]_{ij}$ versus Young's modulus ($E$) and Poisson's ratio ($\nu$) is not reported here. It can be found in standard texts [26].

In the framework of the CUF, the displacement field is assumed as an expansion in terms of generic functions, $F_\tau$:

$$\mathbf{u} = F_\tau \mathbf{u}_\tau, \qquad \tau = 1, 2, \ldots, M \tag{8}$$

where $F_\tau$ are functions of coordinates $x$ and $z$ on the cross-section. $\mathbf{u}_\tau$ is the displacement vector and $M$ stands for the number of terms of the expansion. According to the Einstein notation, the repeated subscript $\tau$ indicates summation. In the next subsections, advanced theories based on Taylor series expansion for cross sectional displacement coupled with finite elements representing the translational displacement field and the theory based on Lagrange polynomials are presented.

### 2.2. Taylor 1D CUF

Using the Maclaurin expansion that uses as basis the 2D polynomials $x^i z^j$, where $i$ and $j$ are positive integers. Considering the expansion up to the quadratic terms, Eq. (8) can be written as:

**Table 1** First three dimensionless bending buckling loads ($P = P_o \frac{12}{\pi^2}\left(\frac{L}{a}\right)^2$, where $P_o$ is the actual buckling load) for different refined beam theories using 10 B4 elements for an isotropic beam ($L/a$ = 20).

| Critical load ($P$) | $P_1$ | $P_2$ | $P_3$ |
|---|---|---|---|
| Ref. [32] | 0.9919 | 3.873 | 8.387 |
| EBBT | 0.995 | 3.956 | 8.813 |
| TBT | 0.9900 | 3.875 | 8.422 |
| $N$ = 1 | 0.9925 | 3.884 | 8.437 |
| $N$ = 2 | 0.9927 | 3.885 | 8.444 |
| $N$ = 3 | 0.9918 | 3.873 | 8.387 |

**Table 2**
First buckling load for different refined beam theories using 10 B4 elements for the eight layered symmetric cross ply (0°/90°/0°/90°/90°/0 °/90°/0°) laminated composite beam.

| Critical load ($P$, $\times 10^5 N$) | EBBT | TBT | $N = 1$ | $N = 2$ | $N = 3$ | $N = 4$ | Ref. [2] |
|---|---|---|---|---|---|---|---|
| $P_1$ | 2.57 | 1.84 | 1.59 | 1.58 | 1.45 | 1.45 | 1.52 |

**Table 3**
First five buckling loads for different refined beam theories using 10 B4 elements for clamped clamped laminated composite three (0°/90°/0°) layered cross ply beam.

| Critical load ($P$, $\times 10^4 N$) | EBBT | TBT | $N = 1$ | $N = 2$ | $N = 3$ |
|---|---|---|---|---|---|
| $P_1$ | 1.53 | 1.45 | 1.51 | 1.51 | 1.51 |
| $P_2$ | 3.12 | 2.97 | 3.08 | 3.09 | 3.07 |
| $P_3$ | 6.10 | 5.76 | 5.99 | 6.0 | 5.93 |
| $P_4$ | 9.21 | 8.67 | 9.0 | 9.01 | 8.85 |
| $P_5$ | 13.70 | 12.79 | 13.28 | 13.30 | 12.97 |

**Table 4**
First three bending and torsional buckling loads for clamped clamped laminated composite three (0°/90°/0°) layered cross ply beams using 10 B4 elements and $N = 3$.

| Critical load ($P$, $\times 10^4 N$) | Bending | Torsional |
|---|---|---|
| $P_1$ | 1.51 | 13.40 |
| $P_2$ | 3.07 | 14.87 |
| $P_3$ | 5.93 | 17.63 |

$$
\begin{aligned}
u_x &= u_{x_1} + x u_{x_2} + z u_{x_3} + x^2 u_{x_4} + xz u_{x_5} + z^2 u_{x_6}\\
u_y &= u_{y_1} + x u_{y_2} + z u_{y_3} + x^2 u_{y_4} + xz u_{y_5} + z^2 u_{y_6}\\
u_z &= u_{z_1} + x u_{z_2} + z u_{z_3} + x^2 u_{z_4} + xz u_{z_5} + z^2 u_{z_6}
\end{aligned}
\tag{9}
$$

It may be noted that the quadratic model is reported as an example and that any-order models can be obtained. The Timoshenko beam model (TBT) can be obtained by acting on the $F_\tau$ expansion as shown in [23].

### 2.3. Finite element implementation and stiffness matrix

Introducing the shape functions, $N_i$, and the nodal displacement vector, $\mathbf{q}_{\tau i}$:

$$
\mathbf{q}_{\tau i} = \left\{ q_{u_{x_{\tau i}}} \quad q_{u_{y_{\tau i}}} \quad q_{u_{z_{\tau i}}} \right\}^T \tag{10}
$$

The displacement vector becomes:

$$
\mathbf{u} = N_i F_\tau \mathbf{q}_{\tau i} \tag{11}
$$

For the sake of brevity, the shape functions are not reported here. They can be found in many books, for instance in [31]. Elements with four nodes (B4) are herein formulated, that is, cubic approximations along the $y$ axis are adopted. It has to be highlighted that, while the order of the beam model is related to the expansion on the cross-section, the number of nodes per each element is related to the approximation along the longitudinal axis. These two parameters are totally free and not related to each others. An $N$-order

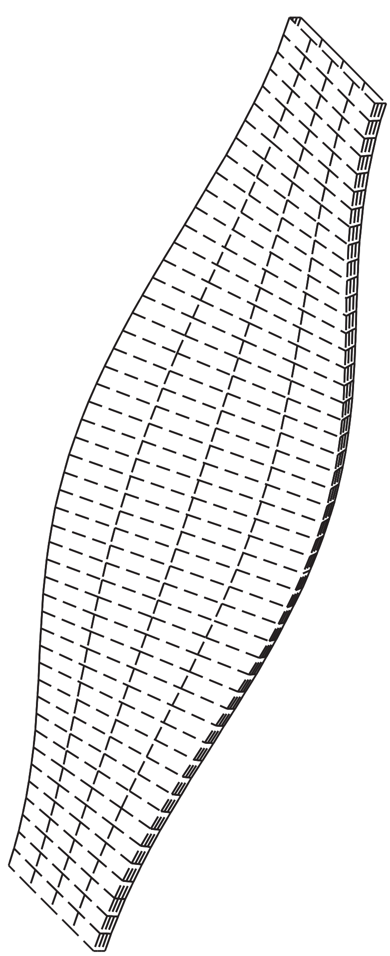

**Fig. 2.** First torsional buckling mode for three layered cross ply beam ($P = 13.4 \times 10^4 N$; $N = 3$).

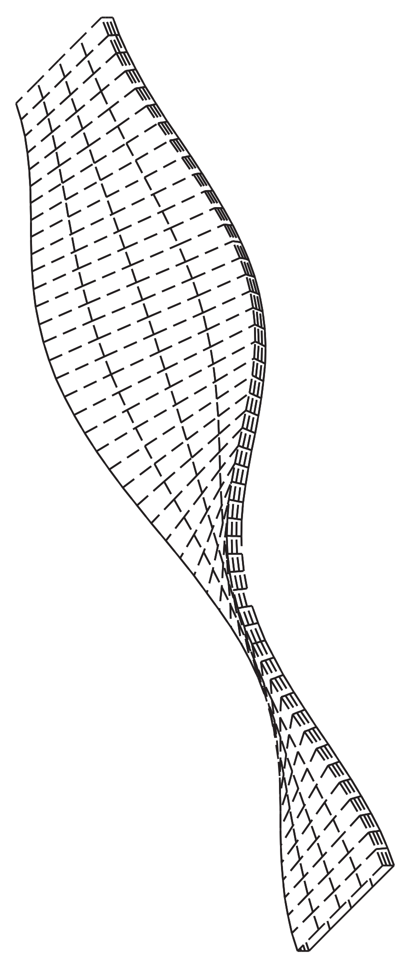

**Fig. 3.** Second torsional buckling mode for three layered cross ply beam ($P = 14.87 \times 10^4 N$; $N = 3$).

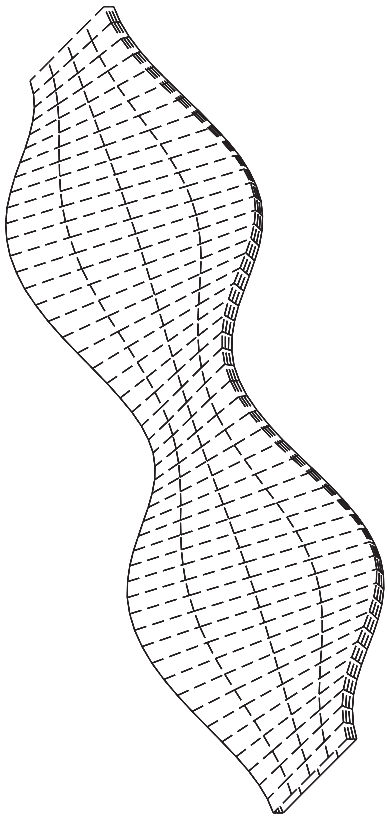

**Fig. 4.** Third torsional buckling mode for three layered cross ply beam ($P = 17.63 \times 10^4 N$; $N = 3$).

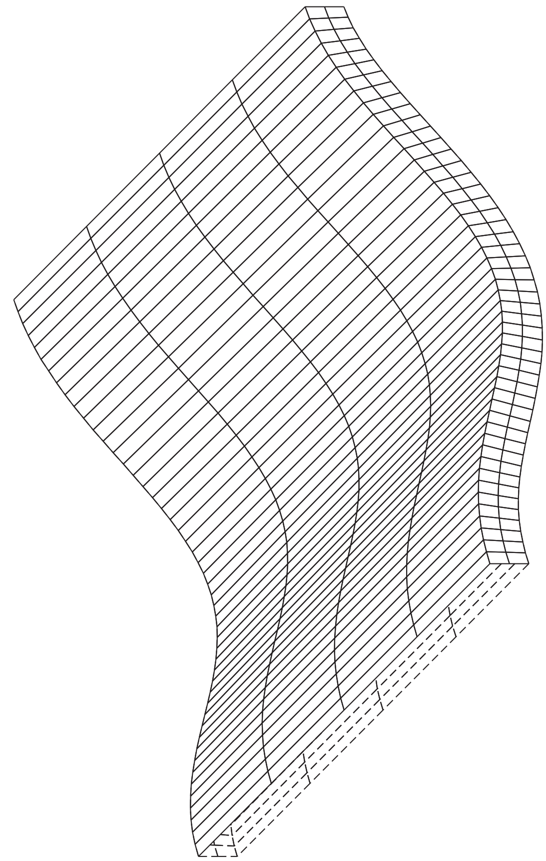

**Fig. 5.** First mode of five layered ($\theta°/-\theta°/0°/-\theta°/\theta°$) composite panel (a) ($P = 12.93 \times 10^5 N$; $N = 6$).

beam model is therefore a theory which exploits an $N$-order polynomial to describe the kinematics of the cross-section.

The stiffness matrix of the elements and the external loadings, which are consistent with the model, are obtained via the Principle of Virtual Displacements:

$$\delta L_{int} = \int_V \left( \delta \boldsymbol{\epsilon}_p^T \boldsymbol{\sigma}_p + \delta \boldsymbol{\epsilon}_n^T \boldsymbol{\sigma}_n \right) dV = \delta L_{ext} \tag{12}$$

where $L_{int}$ stands for the strain energy, and $L_{ext}$ is the work of the external loadings. $\delta$ stands for the virtual variation. The virtual variation of the strain energy is rewritten using Eqs. (3), (6) and (11), i n a compact format it becomes:

$$\delta L_{int} = \delta \mathbf{q}_{\tau i}^T \mathbf{K}^{ij\tau s} \mathbf{q}_{sj} \tag{13}$$

where $\mathbf{K}^{ij\tau s}$ is the stiffness matrix in the form of the fundamental nucleus. The first component of the fundamental nucleus can be written as:

$$K_{xx}^{ij\tau s} = \widetilde{C}_{22} \int_\Omega F_{\tau,x} F_{s,x} \, d\Omega \int_l N_i N_j \, dy + \widetilde{C}_{66} \int_\Omega F_{\tau,z} F_{s,z} \, d\Omega \int_l N_i N_j \, dy + \widetilde{C}_{44} \int_\Omega F_\tau F_s \, d\Omega \int_l N_{i,y} N_{j,y} \, dy$$

The detailed expansion of fundamental nucleus can be seen in [29,30,33]. It should be noted that no assumptions on the approxima-tion order have been done. It is therefore possible to obtain refined beam models without changing the formal expression of the nucleus components. This is the key-point of CUF which permits, with only nine FORTRAN statements, to implement any-order beam theories. The shear locking is corrected through a selective integration [31].

## 3. Governing equations for the linearized stability analysis

The buckling equations are obtained according to Euler's method of adjacent equilibrium states. It consists of a linearized stability analysis of an undeformed equilibrium configuration, whose criti-

**Table 5**
First five buckling loads for different refined beam theories using 10 B4 elements for panel (a) five layered ($\theta°/-\theta°/0°/-\theta°/\theta°$) plate.

| Critical load ($P$, $\times 10^5 N$) | EBBT | TBT | $N = 1$ | $N = 2$ | $N = 3$ | $N = 4$ | $N = 5$ | $N = 6$ |
|---|---|---|---|---|---|---|---|---|
| $P_1$ | 19.14[a] | 13.39[a] | 13.36[a] | 13.38[a] | 13.14[a] | 13.14[a] | 12.94[a] | 12.93[a] |
| $P_2$ | 38.82[a] | 26.23[a] | 26.07[a] | 26.05[a] | 25.04[a] | 25.02[a] | 24.23[a] | 24.18[a] |
| $P_3$ | 74.73[a] | 48.31[a] | 47.68[a] | 33.89[b] | 33.01[b] | 31.35[b] | 30.68[b] | 29.79[b] |
| $P_4$ | 111.17[a] | 68.23[a] | 66.99[a] | 44.81[b] | 42.30[b] | 40.09[b] | 38.78[b] | 37.47[b] |
| $P_5$ | 161.78[a] | 93.93[a] | 91.68[a] | 47.90[b] | 46.20[b] | 45.51[b] | 43.69[b] | 43.46[b] |

[a] Bending mode.
[b] Torsional mode.

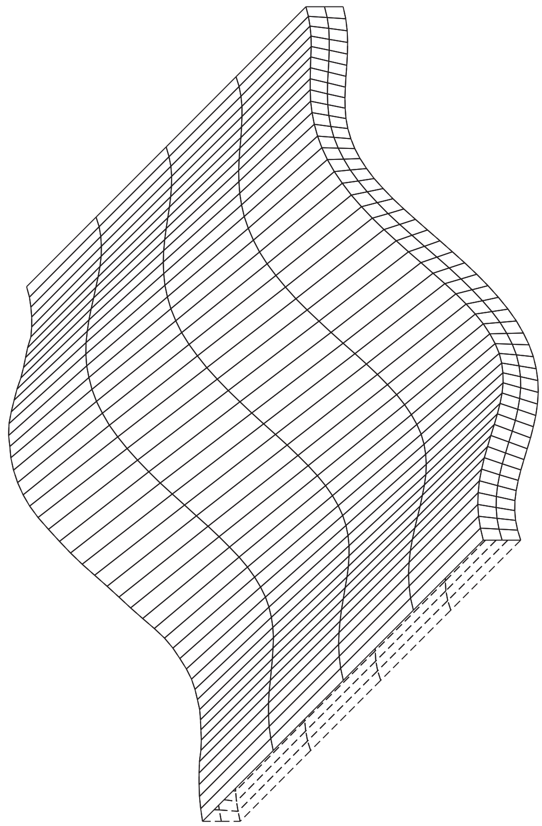

**Fig. 6.** Second mode of five layered ($\theta°/-\theta°/0°/-\theta°/\theta°$) composite panel (a) ($P = 24.18 \times 10^5 N$; $N = 6$).

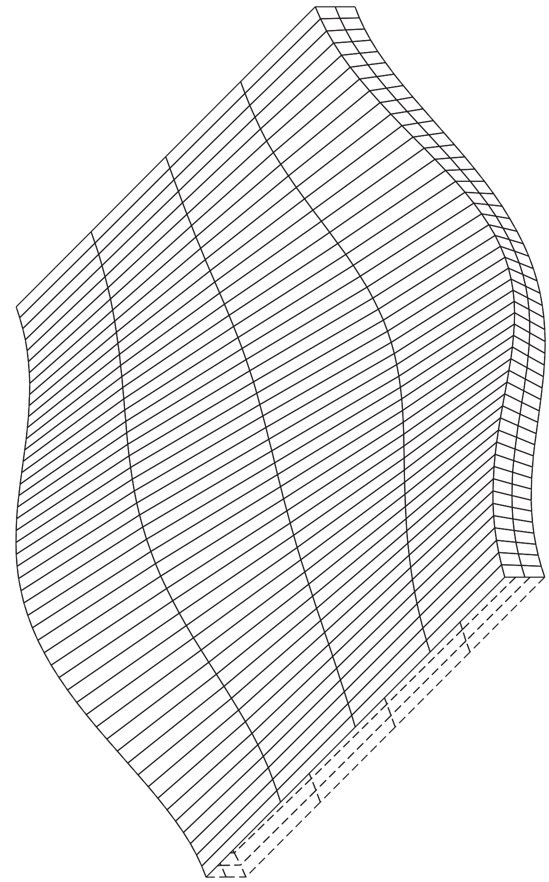

**Fig. 7.** Third mode of five layered ($\theta°/-\theta°/0°/-\theta°/\theta°$) composite panel (a) ($P = 29.79 \times 10^5 N$; $N = 6$).

cal condition is defined by a proportionally scaled load in combination with a geometric or initial stress stiffness built up from the geometric nonlinearities. The following assumptions are thus necessarily introduced.

1. The prebuckling deformation can be neglected.
2. The initial stress $\sigma^0$ remains constant and varies neither in magnitude nor in direction during buckling.
3. At bifurcation, the equilibrium states are infinitesimally adjacent so that a linearization is possible.

The buckling load can then be defined via a scalar load factor $\lambda$ as the load $\sigma = \lambda\sigma^0$ for which an equilibrium configuration $\mathbf{u} \neq \mathbf{0}$ exists such that

$$\delta\mathbf{u}^T[\mathbf{K} + \lambda\mathbf{K}_\sigma(\sigma^0)]\mathbf{u} = \mathbf{0}$$

The symbol $\delta$ denotes the virtual variation and $\mathbf{K}$ is the usual linear stiffness matrix reported elsewhere. $\mathbf{K}_\sigma$ is the geometric stiffness matrix that is obtained from the expression of the work done by the virtual nonlinear strains with the actual initial stresses:

$$\int_\Omega \int_{hk} \delta\epsilon_{yk}^{nl}\sigma_{yk}^{o}\, dy_k \tag{14}$$

where the nonlinear direct in-plane strains can be expressed as:

$$\epsilon_y^{nl} = \frac{1}{2}\left(u_{y,y}^2 + u_{y,y}^2 + u_{y,y}^2\right) \tag{15}$$

Subsequently, the non-zero terms of the geometric stiffness matrix $\mathbf{K}_\sigma$ can be written as:

$$K_{\sigma xx}^{ij\tau s} = K_{\sigma yy}^{ij\tau s} = K_{\sigma zz}^{ij\tau s} = \int_\Omega F_\tau F_s\, d\Omega \int_l N_{i,y}N_{j,y}\, dy$$

## 4. Numerical analysis and discussion

Validation of the present approach is carried out first by considering simply supported beams with square cross sections. Geometrical and Material properties are: Length ($L$) to thickness ($a$) ratio $L/a = 20$; Young's modulus $E = 71.7$ (GPa); Poisson ratio $\nu = 0.3$. The results are compared with that of Matsunaga [32]. Table 1 presents the first three buckling loadings using the third order expansion models and discretized by 10 B4 elements in longitudinal directions. It can be noted that the $N = 2$ model gives a higher buckling load than $N = 1$. It is important to underline that this is due to the Poisson Locking Correction that artificially improve the linear solution [27,28].The present results are in good agreement with those given in literature [32] obtained using the analytical approach.

In the subsequent sections, buckling analysis of laminated composite rectangular beam and flat panels has been carried out. In all the cases, fixed boundary conditions on the shorter edges and long edges free are considered for the beam and panels unless specified otherwise.

### 4.1. Composite beam

The buckling of composite beam is studied in this section. An eight layered symmetric cross ply (0/90/0/90/90/0/90/0) laminated composite beam is considered first. The material properties are given by: $[(E_1, E_2, E_3, G_{12}, G_{23}, G_{31}),\ \ \nu_{12}, \nu_{23}, \nu_{31}] = [(1.344, 0.1034, 0.1034, 0.05, 0.02, 0.05)$ GPa; $0.33, 0.33, 0.33]$. The beam has a length $L = 0.127$ m; width $b = 0.0127$ m; total thickness $a = 0.01016$ m ($L/a = 12.5$).

**Table 6**
First five buckling loads for different refined beam theories using 10 B4 elements for panel (b) five layered (15°/75°/0°/75°/15°) plate.

| Critical load ($P$, $\times 10^5 N$) | EBBT | TBT | $N = 1$ | $N = 2$ | $N = 3$ | $N = 4$ | $N = 5$ | $N = 6$ |
|---|---|---|---|---|---|---|---|---|
| $P_1$ | 8.18[a] | 5.50[a] | 5.43[a] | 5.30[a] | 5.01[a] | 4.86[a] | 4.62[a] | 4.54[a] |
| $P_2$ | 16.59[a] | 10.64[a] | 10.35[a] | 7.41[b] | 6.04[b] | 5.85[b] | 5.67[b] | 5.46[b] |
| $P_3$ | 31.19[a] | 19.30[a] | 18.47[a] | 12.53[b] | 10.75[b] | 9.67[b] | 9.23[b] | 8.77[b] |
| $P_4$ | 47.48[a] | 26.78[a] | 25.17[a] | 14.02[b] | 11.13[b] | 10.28[b] | 9.26[b] | 9.07[b] |
| $P_5$ | 69.09[a] | 36.24[a] | 33.37[a] | 20.02[b] | 16.67[b] | 14.67[b] | 13.98[b] | 12.79[b] |

[a] Bending mode.
[b] Torsional mode.

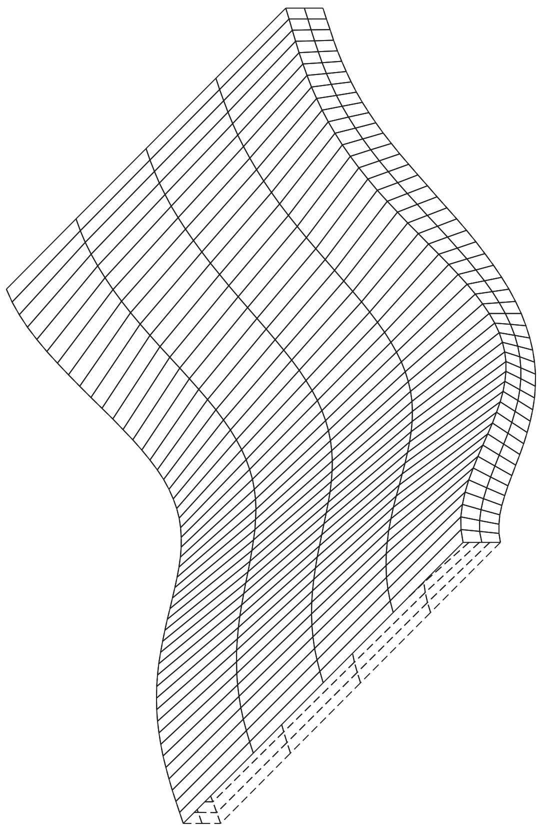

**Fig. 8.** First mode of five layered (15°/75°/0°/75°/15°) composite panel (b) ($P = 4.54 \times 10^5 N$; $N = 6$).

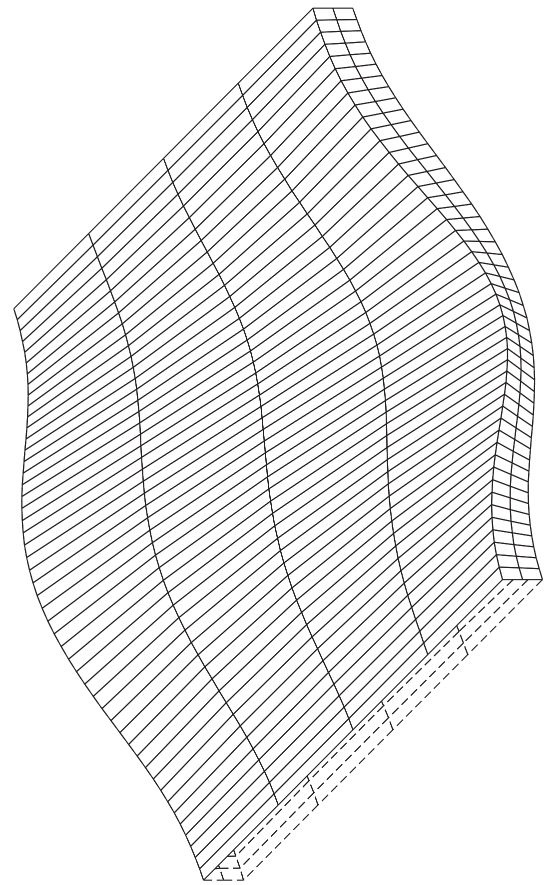

**Fig. 9.** Second mode of five layered (15°/75°/0°/75°/15°) composite panel (b) ($P = 5.46 \times 10^5 N$; $N = 6$).

The fundamental buckling load for the laminated beam evaluated using refined theories is presented in Table 2 and compared with available literature which was obtained using higher order theory [2]. It can be seen from Table 2 that the present result obtained using $N = 3$ matches very well with that of available results [2].

For further studies, the results for the three (0/90/0)-layered cross ply laminated composite beam are presented in this section. The material properties are: $[(E_1, E_2, E_3, G_{12}, G_{23}, G_{31}), \nu_{12}, \nu_{23}, \nu_{31}]$ = [(224.25,6.9,6.9, 56.58, 56.58) GPa, 1.38,0.25, 0.25, 0.25]. Geometrical properties of the beam are: length of beam ($L$) = 0.25 m, total thickness of beam ($a$) = 0.003 m, width of beam ($b$) = 0.05 m. The values of the first five buckling loads for the three layered beam obtained using different refined theories are presented in Table 3. It can be seen from Table 3 that the corresponding lower buckling loads are almost the same even for theories with higher orders whereas slight difference is observed for the higher buckling loads. This is because the length to thickness ratio is quite high ($L/a \approx 80$) and also due to the fact that the first five buckling loads are basically the bending buckling loads. Moreover, the first three torsional buckling loads are given in Table 4 and the corresponding buckling modes are plotted in Figs. 2–4. It may be noted that the torsional effects are observed in the beam considered in the present study at significantly higher buckling loads. However, the ratio of the bending mode to the corresponding torsional mode becomes larger with the subsequent higher modes; in fact it is $\approx$9, 5 and 3 for first second and third modes respectively. It is worth mentioning that these torsional modes cannot be depicted using the classical and lower order models.

### 4.2. Composite panel

In this section, buckling analysis of two different flat panels, namely panel (a) and panel (b), each having five layers is carried out. The material $[(E_1, E_2, E_3, G_{12}, G_{23}, G_{31}), \nu_{12}, \nu_{23}, \nu_{31}]$ and geometrical properties of the panel are:

Material 1 (Mat. 1): [(6.9,6.9,6.9,2.76,2.76,2.76) GPa; 0.25, 0.25,0.25]

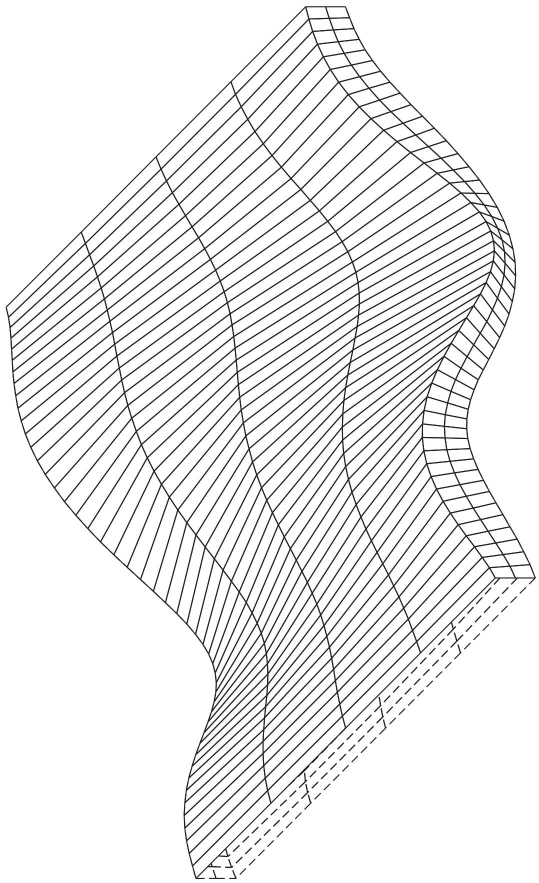

**Fig. 10.** Third mode of five layered (15°/75°/0°/75°/15°) composite panel (b) ($P = 8.77 \times 10^5 N$; $N = 6$).

Material 2 (Mat. 2): [(224.25, 6.9, 6.9, 56.58, 1.38, 56.58) GPa; 0.25, 0.25, 0.25]
Material 3 (Mat. 3): [(172.5, 6.9, 6.9, 3.45, 1.38, 3.45) GPa; 0.25, 0.25, 0.25]

Length of plate = 0.2 m; width =0.1 m; total thickness $a$ = 0.01 m; thickness of layers (first layer at bottom) = (0.1$a$/0.3$a$/0.2$a$/0.3$a$/0.1$a$);

Lamination angle and material distribution for panel (a) = (30 (Mat. 2)/−30 (Mat. 2)/0 (Mat. 2)/−30 (Mat. 2)/30 (Mat. 2)). Lamination angle and material distribution for panel (b) = (15 (Mat. 2)/75 (Mat. 3)/0 (Mat. 1)/75 (Mat. 3)/15 (Mat. 2)).

First five buckling loads of panel (a) obtained using the classical theories and higher order expansions are presented in Table 5. Buckling loads obtained using the classical theories and lower order expansion up to $N = 1$ give rather erroneous results which are significantly overestimated as compared with the actual ones. Also, only the bending buckling modes are depicted using these models. However, for expansion orders greater than 1, the modes in which the out of plane displacement is seen are also predicted. With further refinement, the error reduces and it can be seen that the converged buckling load values can be obtained using expansion order of 5 as with further refinement (expansion order 6), there is negligible improvement of the values. The modes obtained using $N = 6$ are plotted in Figs. 5–7 and it can be seen that the third mode is a torsional type mode. It can be noted that unlike the beams considered in the present study, torsional modes are depicted at significantly lower buckling load.

Another study has been carried out to a symmetric panel (b) in which the layers are made up with different materials and the first five buckling loads are presented in Table 6. Similar to the previous case of panel (a), classical theories and theory order with $N = 1$ only predicts the bending buckling modes. It is interesting to observe that by using the unified formulation up to $N = 1$, the second buckling mode ($P_2 = 10.35 \times 10^5$ for $N = 1$) predicted is of bending type. However, the modes obtained using the higher orders with buckling loads comparable to that of the second mode for $N = 1$ are all of torsional type modes. So, it can be easily said that the lower or-der models are unable to deal with torsion. Further, first three modes obtained using $N = 6$ are plotted in Figs. 8–10 and the tor-sional effects in all the modes except the first one (first mode is purely flexural mode) are easily observed.

## 5. Conclusions

In this paper, buckling analysis using the 1D formulation within the framework of CUF for typical beam sections namely, the laminated composite beam, and laminated composite flat panels have been carried out in a unified manner and compared with available literature. The lamination effects on the buckling characteristics are highlighted. Torsional modes are depicted for beams with expansion orders greater than two whereas for the case of laminated composite panels, the torsional modes can be adequately predicted by using the refined models with expansion order $N = 6$. The advantage of the higher order theory has also been highlighted for describing the torsional modes with adequate accuracy. It can be seen that the lower order models are unable to predict torsional modes.

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