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## Defining the crack pattern of RC beams through the golden section

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**Abstract.** Both in architecture and arts, the golden section has been exclusively taken into consideration for its geometrical properties. Specifically, among all the proportions, the golden section can inspire beauty. Indeed, it has driven the construction of buildings for centuries. Nevertheless, as discussed for the first time in the present paper, static equilibrium of structures calls the golden section into play. This is the case of reinforced concrete beam in four point bending, whose average crack spacing in the constant moment zone increases of a factor equal to the irrational number 1.61803 when the geometrical dimensions of the beam are doubled. In other words, it can be argued that the centrality of golden section in the art of construction has profound physical meanings, as it can bring together the aesthetic of nature and architecture, and the equilibrium of stress flow in solid bodies.

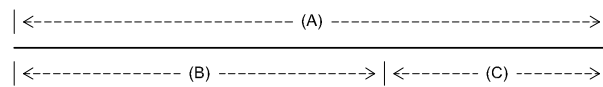
### 1. Introduction

From immemorial time, man sought the essence of existence. This research is often physically materialized in the manufactures built throughout earth by ancient civilizations. The main key areas of expression have always been astronomy and geometry, applied to art and to architecture.

The abstract rules descending from astronomy, which can be interpreted by analyzing a plenty of manufactures, find their actual realization according to the measure of the earth's diameter or radius, or to some mystic latitudes, or even according to some peculiar sites on earth which are in a privileged position during solstices or equinoxes.

In the field of geometry, even more complex and pervasive, however, is the presence of the golden ratio, or divine proportion. It is well known that the golden number has been adopted since the age of ancient Egyptians, as a geometric abstraction applied to art and architecture. Later scientists discovered that this proportion is quite often revealed in Nature, and can be considered as a privileged "rule of proportion" inducing into human perception the positive feelings of regularity and aesthetic pleasure.

A first definition of the golden section was proposed by Euclide (300 b.C.). In particular, in the VI book of the Elements, the third definition states: <<Ἀκρον καὶ μέσον λόγον εὐθεία τετμηθῆναι λέγεται, ὅταν ἡ ὥς ἡ ὅλη πρὸς τὸ μείζον τμήμα, οὕτως τὸ μείζον πρὸς τὸ ἐλάττω >> (A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less) [1].



**Figure 1.** A line segment divided in accordance with the golden section.

In other words, the golden section divides a line segment (Figure 1) at a point where the ratio of the whole line (A) to the large segment (B) is the same as the ratio of the large segment (B) to the small one (C). Only when this ratio is equal to the golden number (i.e., the irrational number  $\phi = 1.61803\dots$ ), can such a division be possible:

$$\frac{A}{B} = \frac{B}{C} = \phi = \frac{1 + \sqrt{5}}{2} = 1.61803\dots \quad (1)$$

In mathematics, this number is intimately interconnected with the Fibonacci sequence. Precisely, the limit of the ratios of successive terms of the series tends to the golden number. However it recurs in many disciplines, such as art, architecture, biology, physics, etc.

According to Livio [2], in many cases only the geometrical properties of  $\phi$  are taken into account, as they have been always considered a classical model of beauty (i.e., the golden section as equilibrium of forms). Nevertheless, it can also be interpreted as the results of static equilibrium, corresponding to a minimum of energy (i.e., the golden section as a form of equilibrium). In what follows, all these aspects are investigated.

## 2. The golden section as equilibrium of forms

Both in arts and architecture, the golden section has been exclusively taken into consideration for its geometrical properties. Specifically, among all the proportions, the golden section can inspire beauty. Thus, it has driven artists and architects for centuries.

### 2.1. Some evidences in arts

The first tangible evidence of the golden ratio in the visual arts comes from the ancient Egyptians. In particular, in the stela of King Den of Abydos (the ancient capital of Egypt), now exposed in the Louvre museum in Paris, one can notice the glyph of the king inscribed in the golden rectangle that surrounds the palace and the snake. In Greek art, a common application of the golden section occurs in amphorae dating from the IV-III cent b.C. In these amphorae, the smallest diameter is in the ratio  $1/\phi$  to the greater diameter [3].

Starting from the Renaissance, the golden section spreads mainly through the work of Luca Pacioli (1445-1517), *De Divina Proportione*, in which the author tries to convey to the artist, through the golden ratio, the secret of harmony of the visible forms. This proportion can be also evidenced in the paintings by Raffaello Sanzio (1483-1520), like the *Trasfigurazione* (Vatican Museums) and *Crocifissione* (National Gallery, London), and in the two versions of *Vergine delle Rocce* by Leonardo da Vinci, where the ratio between two sides of the canvas is the golden ratio [2].

The golden section can still be found in some paintings of the nineteenth and twentieth century, where there is a famous example - declared expressly by the author-, namely the *Ultima Cena* by Salvador Dalí. In this painting, the entire scene is set inside a dodecahedron, and the sides of its pentagonal faces are in golden proportion to the circle inscribing the polyhedron.

### 2.2. Application in architecture

The golden section can be found in many buildings of antiquity. The relationship between length and width of most Greek temples was preferably equal to  $\phi$ .

Controversial is the case of the Parthenon in Athens, which many scientists consider one of the highest expressions of the golden ratio, but where the golden section is not exactly found by usual rules of calculation. In any case, the Parthenon shows a perfectly proportioned façade and undoubtedly infuses a sense of equilibrium of forms. In the arch of Constantine in Rome, the central arch divides the total height according to golden section, while the two smaller arches play the same role between the base and the ledge above.

In the middle-age, Federico II, the Emperor of Germany, built Castel del Monte in Andria (Italy) by imposing the golden proportions even in plant (Figure 2a). Multiplying the shortest side of one of its trapezoidal halls by the golden ratio, the larger side is obtained. At the latitude of the castle ( $41^{\circ} 05' 04''$  N), the four points on the horizon where the sun rises and sets in the dates of the winter and summer solstices, when combined together, realize a golden rectangle. In this way, Castel del Monte is at the heart of this ideal golden rectangle, representing an ideal mixture of geometry and astronomy. Thus, this architectural masterpiece possesses many interesting mathematical, astronomic and esoteric implications, and can be considered as the ultimate expression of the enigmatic Medium-Age, according to Sala and Cappellato [3].



**Figure 2.** Application of golden section in architecture: a) Castel del Monte (Italy); b) Cathedral in Chartres (France).

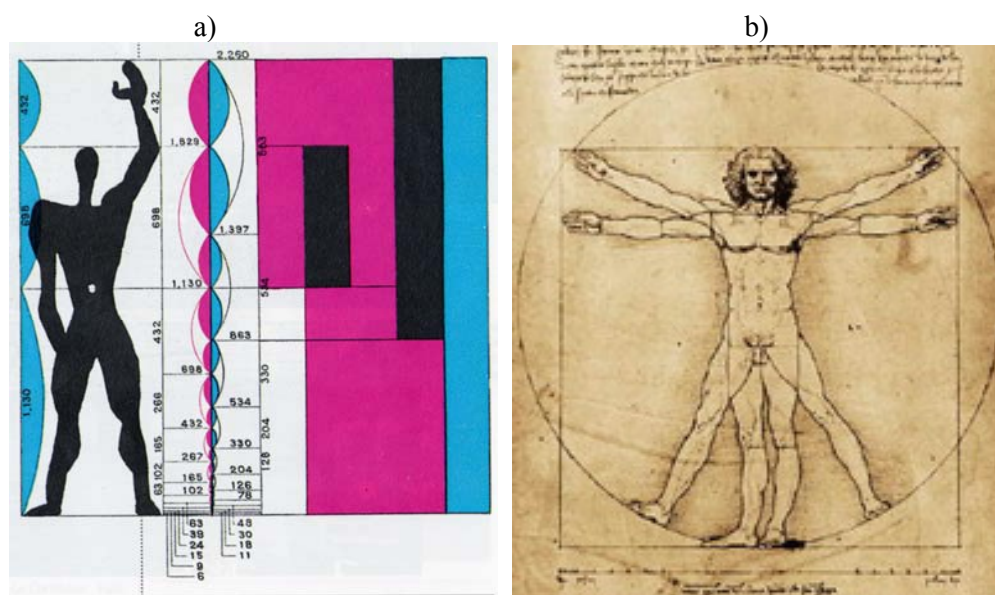
From the starry pentagon derived from the golden number, the famous ogive of the gothic cathedral in Chartres originates (Figure 2b). This is a pentagon with one star tip on the top and two on the bottom, whose construction philosophy is straightforward. If a compass is centered at one of two bottom points, e.g. at the left, using an opening to reach the upper tip of the pentagon, an arch can be described, passing through the right midpoints and reaching the two bottom points. Positioning then the compass upon the bottom right points, with the same opening as before, it describes an arch opposite to the first.

It is well known that the golden ratio in architecture is not exclusive of gothic and Romanic churches. For instance, this relationship can be already found in the megalithic complex of Stonehenge, older than 4000 years, where the ratio of the projected surfaces of the two circles of blue stone and sarsen is equal to 1.6. Eventually, the pyramid of Cheops, the Egyptian sarcophagi, and most Greek manufactures testify the ubiquity of the knowledge and of the application of the divine proportion in architecture.

Moreover, according to Catholic religion, people assert that the harmonic and aesthetic function have been magically combined with a sacred function, namely, the allegory of Jerusalem. In fact, the value 1.618 is the cotangent of the angle of  $31^{\circ} 43' N$  (i.e., the golden angle), which is the latitude of the Holy City. In other words, a church designed according to a golden rectangle (that was very

common in the past centuries), can be divided by a diagonal line into two angles, one of  $31^{\circ} 43'$ , i.e. the latitude of Jerusalem, the other of  $58^{\circ} 17' N$ , which is the value corresponding to the culmination of the Sun (zenith) during the equinoxes in the same city. This can hold a particular and intriguing significance: when a man enters the church, he puts his foot symbolically upon the Holy City, in the land of Christ. Note that in modern atlases, the latitude of Jerusalem is normally reported as  $31^{\circ} 46' N$ , with a difference of  $3'$  compared to the "golden angle", a difference corresponding to just 5 kilometers on the ground.

In 1948 Le Corbusier published *Modulor* (union of *module* and *section d'or*), an essay of studies ranging from the work by Vitruvius, through Leonardo da Vinci's Vitruvian Man and the work by Leon Battista Alberti, up to a series of attempts to find geometric and mathematical proportions in the human body (Figure 3).



**Figure 3.** Geometric and mathematical proportions in the human body: a) Graphic image of the Modulor by Le Corbusier; b) Studies about human body by Leonardo Da Vinci.

The aim was to extract some knowledge to improve the functionality of the architecture (*form, function and structure*). Actually Le Corbusier used the golden ratio in an original way to determine a set of harmonic dimensions on a human scale.

According to Figure 3a, a stylized human figure with one arm stretched above his head is close to two vertical measurements, the red series based on the height of the solar plexus (put equal to 108 cm in the original version, and to 113 cm in the revised one) and then divided into segments according to  $\phi$ , and the blue series based on the entire height of the figure, which is double with respect to the height of the solar plexus (216 cm in the original version, 226 cm in the revised one), divided into segments in the same way. A spiral, graphically developed between the red and the blue series, seems to mimic the volume of the human figure, or the DNA triple helix, many years before its discovery.

Criticism upon the Modulor immediately raised. The height of the figure seems to be arbitrary and perhaps chosen for mathematical convenience. The female body, in the words of Ostwald [4], "*was considered only later, and rejected as a source of harmony and proportion.*" The system does not exactly correspond to any current anthropometric observations.

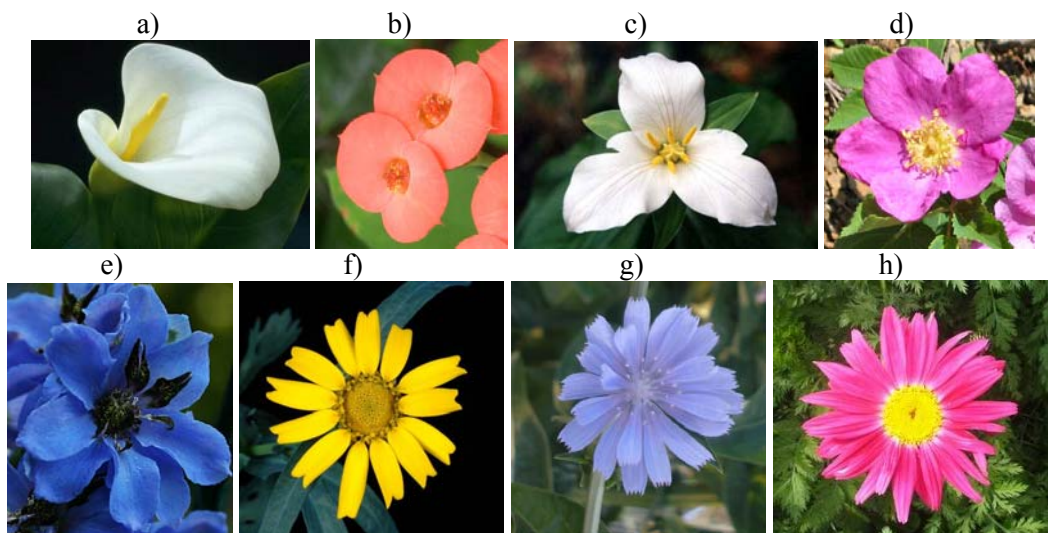
There is not an obvious and clear method to transfer these measurements to space settlements. For example, the Modulor cannot be used to calculate the optimal proportion of the stairs; indeed Modulor should be considered a dimensional exercise relating anthropometric and mathematical principles.

### 3. The golden section in nature

The golden number is the ratio that harmonizes the human body. In fact, if the distance from the navel to the ground, in an adult and well proportionate person, is multiplied by 1.618, one obtains the person's height. If the distance from elbow to hand with stretched fingers is multiplied by 1.618, the total length of the arm is obtained. The distance that goes from the hip to the knee, multiplied by the gold number, gives the length of the leg from the hip to the malleolus.

Even in the man's hand, the phalange, little phalange and middle phalange of the medium and ring finger (but only of these two fingers!), are related with one another through the golden ratio. The human face can be subdivided into a grid whose rectangular sides share the golden ratio, i.e. multiplying the shorter side of the rectangle by 1.618, the length of the longest side is obtained.

Although many natural phenomena, such as the external cardiac configuration [5] and the structure of quasi crystals, recalls the golden section, plants and flowers show the most important examples [2]. It is possible to discover the numbers of Fibonacci series in the petals of flowers, as suggested by Zeng and Wang [6].



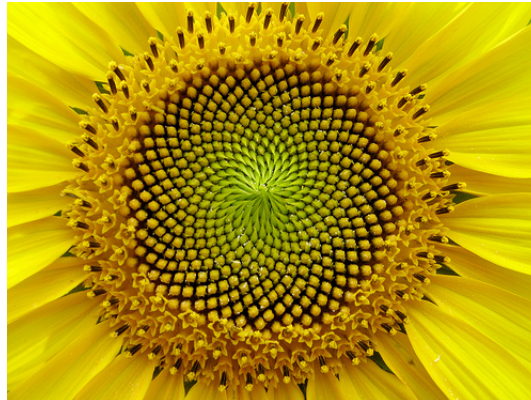
**Figure 4.** The Fibonacci numbers in the petals of flowers.

Specifically, Calle has 1 petal (Figure 4a), Euphorbia mili has 2 petals (Figure 4b), trillium has 3 petals (Figure 4c), Wild Rose has 5 petals (Figure 4d), Delphiniums has 8 petals (Figure 4e), Corn Marigold has 13 petals (Figure 4f), Chicory has 21 petals (Figure 4g), Pyrethrum has 34 petals (Figure 4h), and so on.

In general, the Phyllotaxis, a subdivision of the plant morphology that studies the arrangement of repeated units (e.g., leaves around a stem, spirals on a pine cone or on a pineapple, etc), offers several cases in which the golden section plays a fundamental role. It is sufficient to think to the seeds of a sunflower (Figure 5), which generate opposed families of spirals, respectively composed by a number of spirals equal to two consecutive terms of the Fibonacci series (and thus their ratio is equal to  $\phi$ ), such as 89/55, 144/89, 233/144 [2].

These phenomena have been investigated for centuries by botanists of all over the world [7], but only in the last decades the studies have been focused on the origin of such natural patterns. In other words, researchers are trying to answer to the following question: Why the golden section is intrinsically involved in plant growth?

Answers to this question were provided by two different schools of thought. Some researchers are mainly motivated by the functional requirements of plant structures, such as the necessity of homogeneity (i.e., it should be the same in each part) and self-similarity (i.e., it should be the same independently of the scale), without explaining the causes of the geometrical properties.



**Figure 5.** The Fibonacci numbers in the spirals generated by the seeds of a sunflower.

Conversely, by recurring to mechanical models, other botanists have shown that plant morphologies are the results of static equilibrium, corresponding to a minimum of energy. In particular, Shipman and Newell [8] demonstrate how the ribbed planforms on plants can be explained by the energy-minimizing buckling pattern of a compressed sheet (the plant's tunica) on elastic foundation.

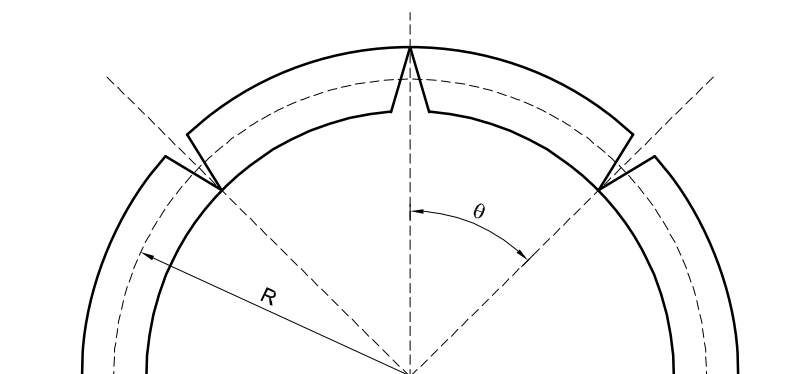
Despite further experimental and theoretical researches are needed, the mechanical approach applied to plant morphology is of great importance. In fact, the constant presence of Fibonacci numbers in Nature, and of the golden section as well, is not only due to the aesthetical reasons explained in the previous section, but is also the result of equilibrium conditions (i.e., it is a form of equilibrium).

#### 4. Crack pattern in masonry and concrete structures

Static equilibrium of structures calls the golden section into play, in particular in the crack pattern of masonry and concrete structures.

In circular hemispherical dome (Figure 6), subjected to its own weight loads and made with masonry (having no tensile strength), the position of tensile cracks can be defined by the angle  $\theta$  [9]:

$$\cos \theta = \frac{1}{\phi} \quad \rightarrow \quad \theta = 51^\circ 50' \quad (2)$$



**Figure 6.** Crack pattern in the hemispherical dome under weight load [9].

Persian architects knew such a crack pattern, which sometimes caused the failure of the dome, and frequently changed the thickness around the angle  $\theta$  [9].

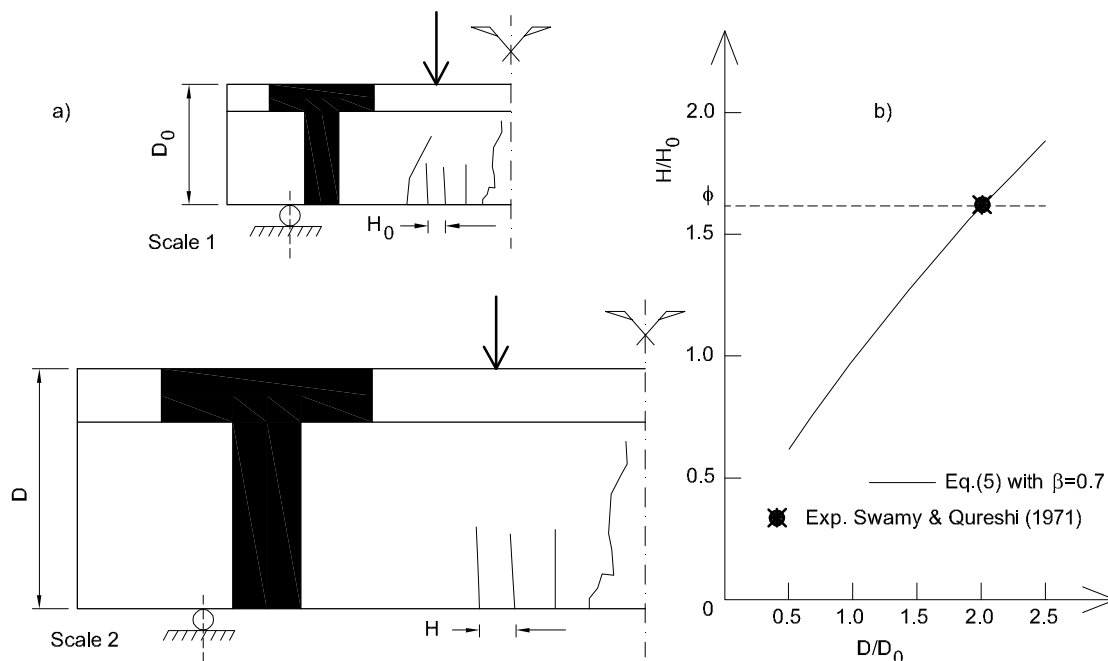
The golden section can be also detected in the crack pattern of reinforced concrete (RC) beams, when such a phenomenon is investigated at different scales. For instance, Figure 7 shows the beams tested in four point bending by Swamy and Qureshi [10]. Two types of beams, namely scale 1 and scale 2, are here taken into account (Figure 7a): the smallest beams (i.e. those of scale 1) have the geometrical dimensions equal to half those of the largest beams (i.e. those of scale 2). In all the beams the crack pattern in the constant moment zone has been observed. In these zones, crack spacing and crack widths are only affected by the bond-slip between steel and concrete in tension. Indeed, they can be predicted by solving, in a portion of beam bordered by two consecutive cracks, the classical tension stiffening problem [11]. It consists of the following equations:

$$\frac{d\sigma_s}{dz} = -\frac{p_s}{A_s} \cdot \tau \quad (3)$$

$$\frac{ds}{dz} = -\varepsilon_s(z) + \varepsilon_c(z) \quad (4)$$

where  $p_s$  and  $A_s$  = respectively, the perimeter and the cross-sectional area of reinforcing bars in tension;  $s$  = value of slip between reinforcing bars and concrete;  $\sigma_s$  = stress in the reinforcing bars;  $\varepsilon_s$  and  $\varepsilon_c$  = strains, respectively computed in the steel area in tension and in the tensile concrete at the same level of reinforcement;  $z$  = horizontal coordinate; and  $\tau$  = bond stress between steel and concrete.

However, the application of block models is not always possible, because the mechanical responses of steel and concrete, as well as the bond-slip relationship, are not known in advance. This is the case of the Swamy's and Qureshi's tests, in which the number of the observed cracks and their distances are reported in Table 1. In the same Table, the average distance between the cracks (i.e., 6.95 cm - 2.74"- for the beams of scale 1, and 11.3 cm - 4.46"- for the beams of scale 2) is also evidenced.



**Figure 7.** The crack pattern of RC beams: a) the T-section beams tested by Swamy and Qureshi [10]; b) size effect law for crack spacing in the constant moment zone.

**Table 1.** The crack pattern observed by Swamy and Qureshi [10] in the constant moment zone.

Beam	Scale	Number of cracks	Crack spacing		
			cm	inches	average
M1A	1	11	9.40	3.70	
M1B	1	19	6.10	2.40	
M1C	1	18	6.27	2.47	
M1D	1	18	6.02	2.37	6.95 cm (2.74")
M2A	2	18	11.2	4.42	
M2B	2	18	10.9	4.30	
M2C	2	17	11.8	4.66	
M2D	2	19	11.3	4.45	11.3 cm (4.46")

It can be observed how the average crack spacing in the constant moment zone increases of a factor  $\phi$  when the geometrical dimensions of the beam are doubled ( $11.3/6.95 \cong \phi$ ).

Generally, size-effect laws have the form of power functions [12]:

$$\frac{H}{H_0} = \left( \frac{D}{D_0} \right)^\beta \quad (5)$$

where,  $D_0$  = reference dimension of scale 1 beams,  $D$  = reference dimension of a generic beam (the ratio  $D/D_0$  is the scale factor);  $H_0$  = average distance between cracks in the scale 1 beams;  $H$  = average distance between cracks in a generic beam; and  $\beta$  = exponent that has to be defined by fitting some experimental results.

According to the tests performed by Swamy and Qureshi [10], when the ratio  $D/D_0 = 2$ , Eq.(5) should give  $H/H_0 = \phi$ , and therefore  $\beta = 0.7$  (see Figure 7b).

These observations are really important when the behavior of large RC structures, under complex loads, needs to be simulated. By testing prototypes made of similar materials, namely the same steel and the same concrete, but of lower dimensions, and by using the size effect laws similar to that of Eq.(5), the experimental results can be easily extended to large beams in bending. Specifically the average crack distance of RC beams, made with the same materials, but of different dimensions, seems to be ruled by the golden section. This understandable, when one recalls that the crack formation is essentially an energy-driven phenomenon.

## 5. Conclusions

In the previous sections the recurring of golden section in architecture and Nature has been widely investigated. As a result, it can be argued that the centrality of golden section in the art of construction has profound physical meanings, as it can bring together the aesthetic of nature and architecture, and the equilibrium of stress flow in solid bodies.

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