EMBEDDING ALMOST-COMPLEX MANIFOLDS IN ALMOST-COMPLEX EUCLIDEAN SPACES

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ABSTRACT. We show that any compact almost-complex manifold (M, J) of complex dimension m can be pseudo-holomorphically embedded in $\mathbb{R}^{6m}$ equipped with a suitable almost-complex structure $\widetilde{J}$.

KEYWORDS: embedding, almost-complex structure, manifold, pseudo-holomorphic embedding.

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1. INTRODUCTION

An almost-complex structure on a $2n$-dimensional smooth manifold M is a tensor $J \in \mathrm{End}(TM)$ such that $J^2 = -\mathrm{id}$. If M is oriented we say that J is *positive* if the orientation induced by J on M agrees with the given one. An almost-complex structure is called *integrable* if it is induced by a holomorphic atlas. In dimension two any almost-complex structure is integrable, while in higher dimension this is far from true. A smooth map $f\colon N \to M$ between two almost-complex manifolds (N, J'), (M, J) is called *pseudo-holomorphic* if $J \circ Tf = Tf \circ J'$, where $Tf : TN \to TM$ is the tangent map of f. When the map f is an embedding, (N, J') is said to be an *almost-complex submanifold* of (M, J). In this case we can identify N with its image $f(N) \subset M$ and the almost-complex structure J' with the restriction of J to $TN \cong T(f(N)) \subset TM$.

If we equip $\mathbb{R}^{2n}$ with the canonical complex structure, that is to say $\mathbb{R}^{2n} \cong \mathbb{C}^n$, then it does not admit any compact complex submanifold (by the maximum principle). Thus, it is a very natural problem to ascertain if it is possible to find compact complex manifolds pseudo-holomorphically embedded in $\mathbb{R}^{2n}$ equipped with an integrable or non-integrable almost-complex structure.

In [2] Calabi and Eckmann constructed the first examples of compact, simply connected complex manifolds $M_{p,q}$ which are not algebraic. Topologically $M_{p,q}$ is the product $S^{2p+1} \times S^{2q+1}$. Then by deleting a point on each factor one obtains a complex structure J on $\mathbb{R}^{2p+2q+2}$. In section 5 of [2] it was shown that when $p, q > 1$ there exists a complex torus as a complex submanifold of $(\mathbb{R}^{2p+2q+2}, J)$ [2, p. 499]. It follows that the Calabi-Eckmann complex structure

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J on $\mathbb{R}^{2n}$ cannot be tamed by any symplectic form and in particular cannot be Kähler. Calabi and Eckmann also observed that the only holomorphic functions on $(\mathbb{R}^{2p+2q+2}, J)$ are the constants answering negatively to a question raised by Bochner about the uniformization of complex structures on $\mathbb{R}^{2n}$. In [1] Bryant constructed pseudo-holomorphic non-constant maps $\varphi : M^2 \to S^6$ for any compact Riemann surface M^2, where S^6 is equipped with the almost-complex structure induced by the octonion multiplication. These maps realize compact Riemann surfaces as pseudo-holomorphic singular curves in S^6.

In [3] was showed that any almost-complex torus $\mathbb{T}^n = \mathbb{R}^{2n}/\Lambda$ can be pseudo-holomorphically embedded into $(\mathbb{R}^{4n}, J_\Lambda)$ for a suitable almost-complex structure J_Λ. It follows that any compact Riemann surface can be realized as a pseudo-holomorphic curve of some $(\mathbb{R}^{2n}, J)$, where J is a suitable almost-complex structure.

In this paper we prove the following general theorem.

Theorem 1. *Any compact almost-complex manifold (M, J) of real dimension $2m$ can be pseudo-holomorphically embedded in $(\mathbb{R}^{6m}, \widetilde{J})$ for a suitable positive almost-complex structure $\widetilde{J}$.*

In particular, any compact Riemann surface can be realized as a pseudo-holomorphic curve in $(\mathbb{R}^6, \widetilde{J})$. In [3] was shown that the torus is the only compact Riemann surface that can be pseudo-holomorphically embedded in $(\mathbb{R}^4, \widetilde{J})$ for some $\widetilde{J}$.

2. Preliminaries

The space of positive linear complex structures on $\mathbb{R}^{2n}$ is diffeomorphic to the homogeneus space $\widetilde{\mathfrak{J}}(n) = GL^+(2n, \mathbb{R})/GL(n, \mathbb{C})$ and is homotopy equivalent to $\mathfrak{J}(n) = SO(2n)/U(n)$. So, an almost-complex structure J on $\mathbb{R}^{2n}$ can be regarded as a smooth map $J : \mathbb{R}^{2n} \to \widetilde{\mathfrak{J}}(n)$.

Lemma 2. *Let $M \subset \mathbb{R}^{2n}$ be a closed submanifold and let $J : M \to \widetilde{\mathfrak{J}}(n)$ be a smooth map. Then there exists a smooth extension $\widetilde{J} : \mathbb{R}^{2n} \to \widetilde{\mathfrak{J}}(n)$ if and only if J is homotopic to a constant.*

Proof. The 'only if' part follows immediately from the fact that $\mathbb{R}^{2n}$ is contractible.

Let us prove the 'if' part. Consider a smooth homotopy $H : M \times [0, 1] \to \widetilde{\mathfrak{J}}(2n)$ such that $H_0(x) = J_0$ for all $x \in M$, and $H_1 = J$ where $H_t(x) = H(x, t)$ and $J_0 \in \widetilde{\mathfrak{J}}(n)$. We can extend H to $\mathbb{R}^{2n} \times \{0\} \subset \mathbb{R}^{2n} \times [0, 1]$ by setting $H(x, 0) = J_0$ for any $x \in \mathbb{R}^{2n}$. By the homotopy extension property [4, Chapter 0] there exists $\widetilde{H} : \mathbb{R}^{2n} \times [0, 1] \to \widetilde{\mathfrak{J}}(n)$ which extends H. We conclude the proof by setting $\widetilde{J} = \widetilde{H}_1$. □

Let (M, J) be an almost-complex manifold. The strategy to prove Theorem 1 will be to choose an arbitrary embedding $f : M \hookrightarrow \mathbb{R}^{6m}$, which exists for the weak Whitney embedding theorem, and to show that J extends to the pullback $f^*(T\mathbb{R}^{6m})$ and this extension is null-homotopic.

Consider the standard filtration $SO(1) \subset SO(2) \subset \cdots$. Since $SO(n-1)$ contains the $(n-2)$-skeleton of $SO(n)$ (because the standard fibration $SO(n) \to S^{n-1}$) it follows that the k-skeleton of $SO(n)$ is contained on $SO(k+1)$ for $0 \le k \le n-2$.

Since $SO(n) \subset U(n)$ it follows that $U(n)$ contains the $(n-1)$-skeleton of $SO(2n)$ for $n \ge 1$. Then the homomorphism induced by the inclusion $i_* : \pi_j(U(n)) \to \pi_j(SO(2n))$ is an isomorphism for $j \le n-2$ and is an epimorphism for $j = n-1$.

From the homotopy exact sequence of the fibre bundle $SO(2n) \to \mathfrak{J}(n)$ given by the projection map it follows that $\pi_j(\widetilde{\mathfrak{J}}(n)) \cong \pi_j(\mathfrak{J}(n)) \cong 0$ for $j \le n-1$.

Definition 3. *A space X is said to be n-connected if $\pi_j(X) \cong 0$ for all $j \le n$.*

In particular, 0-connected means path-connected.

From the above considerations we have that $\widetilde{\mathfrak{J}}(n)$ is $(n-1)$-connected. The following proposition is well-known in the theory of CW-complexes.

Proposition 4. *If X is n-connected then any map $Y \to X$ defined on a CW-complex Y of dimension $\leq n$ is homotopic to a constant.*

Also the following proposition is standard, and we give only the idea of the proof.

Proposition 5. *Let $\xi : E \to M$ be an oriented real vector bundle of rank $2k$ over an m-manifold M. If $k \geq m$ then ξ admits a positive complex structure.*

Proof. Consider the bundle $\xi^{\mathfrak{J}} : \widetilde{\mathfrak{J}}(E) \to M$ with fibre $\widetilde{\mathfrak{J}}(k)$ induced by ξ. Namely, for any $p \in M$ the fibre of $\xi^{\mathfrak{J}}$ over p is the space of positive linear complex structures on $\xi^{-1}(p)$. Since $\widetilde{\mathfrak{J}}(k)$ is $(k-1)$-connected, it follows that $\xi^{\mathfrak{J}}$ admits a section if $k \geq m$, see [7, Part III]. This section is a positive complex structure on ξ. □

Let $f : M \to \mathbb{R}^N$ be an immersion. The normal bundle $\nu_f(M)$ is, as usual, the orthogonal complement of TM in $f^*(T\mathbb{R}^N)$, that is to say:

$$f^*(T\mathbb{R}^N) = TM \oplus \nu_f(M).$$

If M is oriented then the normal bundle can be equipped with a canonical orientation, namely that which makes the splitting of $f^*(T\mathbb{R}^N)$ into a Whitney sum of oriented fibre bundles, where $\mathbb{R}^N$ is considered with the standard orientation.

3. Proof of the main results

Theorem 6. *Let $M \subset \mathbb{R}^{2n}$ be a submanifold of even dimension endowed with an almost-complex structure J. If the normal bundle of M in $\mathbb{R}^{2n}$ admits a positive complex structure with respect to the canonical orientation, then for any $k \geq \max(0, \dim_{\mathbb{R}} M - n + 1)$ there exists an almost-complex structure $\tilde{J}$ on $\mathbb{R}^{2n} \times \mathbb{R}^{2k}$ such that $M \times \{0\} \subset \mathbb{R}^{2n} \times \mathbb{R}^{2k}$ is an almost-complex submanifold.*

Proof. Let us choose a positive complex structure on the normal bundle of M. Then by taking the Whitney sum with the almost-complex structure on M we get a complex structure on $(T\mathbb{R}^{2n})_{|M}$. So we obtain a smooth map $J : M \to \widetilde{\mathfrak{J}}(n)$.

In view of Lemma 2 our target is to get a J null-homotopic. This is so if $\dim_{\mathbb{R}} M \leq n-1$ because $\widetilde{\mathfrak{J}}(n)$ is $(n-1)$-connected and Proposition 4.

If $\dim_{\mathbb{R}} M > n-1$ we take the product $\mathbb{R}^{2n} \times \mathbb{R}^{2k}$, where $\mathbb{R}^{2k}$ is endowed with the standard complex structure, and we embed M as $M \times \{0\}$. We get a complex structure on the normal bundle of M in $\mathbb{R}^{2n} \times \mathbb{R}^{2k}$ in the obvious way. So we obtain a map $J_k : M \to \widetilde{\mathfrak{J}}(n+k)$. It follows that J_k is homotopic to a constant if $k \geq \dim_{\mathbb{R}} M - n + 1$. In this case J_k extends on $\mathbb{R}^{2n} \times \mathbb{R}^{2k}$ by Lemma 2. □

It follows that if (M, J) is contained in $\mathbb{C}^n$ with a complex normal bundle and if $n \geq 2 \dim_{\mathbb{C}} M + 1$, then there is a positive almost-complex structure $\widetilde{J}$ on $\mathbb{C}^n$ which makes (M, J) an almost-complex submanifold of $(\mathbb{C}^n, \widetilde{J})$.

Proof of Theorem 1. Let $f : M \hookrightarrow \mathbb{R}^{6m}$ be any embedding. The normal bundle $\nu_f(M)$ has rank $4m$ and is orientable. By Proposition 5 there is a complex structure on the normal bundle and then we conclude by an application of Theorem 6 with $k = 0$. □

In some cases we can construct an embedding in an euclidean space of lower dimension. Recall that an *s-inverse* of the tangent bundle TM is a vector bundle ξ such that $TM \oplus \xi$ is a trivial vector bundle. Observe that if $f : M \to \mathbb{R}^N$ is an immersion then the normal bundle $\nu_f(M)$ is a real s-inverse of the tangent bundle TM. The converse also holds and is a Theorem of Hirsch [5]. Namely,

Theorem 7. *(Hirsch [5]) Any s-inverse of TM is the normal bundle of some immersion $f : M \to \mathbb{R}^N$.*

Let ξ be a complex s-inverse of (TM, J) of complex rank k, namely $TM \oplus \xi$ is trivial as a complex vector bundle. Now Hirsch's Theorem 7 imply that there exists an immersion $f : M \to \mathbb{R}^{2(m+k)}$ such that ξ is isomorphic to $\nu_f(M)$ as real vector bundles. So $\nu_f(M)$ carries a complex structure.

Up to a product with some $\mathbb{R}^{2h}$, we can assume that $k \geq m+1$, and then f is regularly homotopic, namely homotopic through immersions, to an embedding $f_1 : M \to \mathbb{R}^{2(m+k)}$. It follows that $\nu_{f_1}(M) \cong \nu_f(M)$ carries a complex structure. Now apply Theorem 6 to get $\widetilde{J}$.

If the rank of ξ satisfies $m+1 \leq k \leq 2m-1$ we get a pseudo-holomorphic embedding in an euclidean space of complex dimension $m+k < 3m$.

Let (S^6, J) be the six dimensional sphere equipped with the standard almost-complex structure J obtained from the octonion multiplication. Theorem 1 imply that (S^6, J) can be pseudo-holomorphically embedded in $(\mathbb{R}^{18}, \widetilde{J})$ for a suitable positive almost-complex structure $\widetilde{J}$. Using the existence of a low dimensional s-inverse of (TS^6, J) we have the following result:

Corollary 8. *The almost-complex sphere (S^6, J) can be pseudo-holomorphically embedded in $(\mathbb{R}^{14}, \widetilde{J})$ for a suitable positive almost-complex structure $\widetilde{J}$.*

Proof. Since S^6 is embedded in $\mathbb{R}^8$ with trivial normal bundle we conclude by an application of Theorem 6 with $k = 3$. □

Notice that (S^6, J) can not be pseudo-holomorphically embedded in $(\mathbb{R}^{12}, \widetilde{J})$. In fact, the Euler class of the normal bundle of any embedding of S^6 in $\mathbb{R}^{12}$ is zero by a theorem of Whitney, see [6, p. 138]. On the other hand, if S^6 is contained pseudo-holomorphically in $(\mathbb{R}^{12}, \widetilde{J})$, by a straightforward computation with the Chern class, we obtain for the Euler class $e(\nu(S^6)) = c_3(\nu(S^6)) = -2\lambda \neq 0$, which is a contradiction, where $\lambda \in H^6(S^6)$ is the standard generator.

We conclude with a question. Since our construction is essentially homotopy-theoretic, we are unable to control the integrability of the almost-complex structure $\widetilde{J}$ of Theorem 1. So the following question is very natural.

Question 9. *Let (M, J) be an integrable complex manifold. Is there an embedding of (M, J) into an integrable $(\mathbb{R}^{2n}, \widetilde{J})$?*

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