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***Abstract*—We discuss recent results leading to a full statistical characterization of the noise spectrum of a free running oscillator perturbed by white Gaussian noise sources, including the effect of orbital fluctuations and of their correlation with phase noise. This extends a previous theory based on the Floquet decomposition of the linearized oscillator equations originally applied to phase fluctuations only.**

I. Introduction

The analysis of fluctuations in autonomous systems, i.e. in systems described by differential equations not explicitly dependent on time, has been widely studied in the physical, mathematical and engineering community for decades [1]–[10]. From the application standpoint, the study of noise in oscillators gained momentum when electrical oscillators came into play in the development of telecommunication systems, where oscillator fluctuations may severely limit the performance (see [10], [11] and references therein).

Oscillator noise manifests itself as two, in general correlated, variations: *phase noise*, often represented in time domain by the equivalent concept of *time jitter,* which describes the variations along the oscillator noiseless working point caused by the noise sources present in the circuit, and *amplitude fluctuations* (or *orbital noise*) corresponding to the variations in the amplitude of the instantaneous working point. This decomposition is universally recognized in the literature, although a less firmly established issue is the mathematical definition of the two components, and therefore their statistical characterization as a function of the circuit noise sources. In general, experimental results show that phase noise is the dominant component. This can be traced back to the very nature of autonomous systems, and in particular to their lack of a fixed time reference which makes time jitter virtually unbounded [10]. Orbital noise, on the other hand, is naturally quenched by the same mechanism (indispensable in practice for a working oscillator) which makes the oscillator noiseless working point a stable solution of the autonomous system. As a consequence, orbital noise is often neglected altogether [10] or has received a comparatively lower attention [8]. Nevertheless, its importance may raise in specific applications (such as telecommunication systems in presence of strong adjacent channels [11]) or far from the harmonics of the fundamental oscillator frequency, where orbital noise represents the dominant contribution and may also give rise to specific effects such as spectrum asymmetry around the noiseless oscillator harmonics [12].

In order to fix the ideas, we consider the oscillator modeled by an autonomous Ordinary Differential Equation (ODE) [10]

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} - \mathbf{f}(\mathbf{x}) = \mathbf{0}, \tag{1}$$

where $\mathbf{x}(t) \in \ ^n$ is the state vector, and $\mathbf{f}(\cdot) : \ ^n \to \ ^n$ is a nonlinear function smooth enough to allow for a unique solution. The choice of an ODE allows for a simpler formulation of the problem and of the resulting equations, but represents a simplification since in practical circuits the state equations are in general written in terms of a Differential Algebraic Equation (DAE) system: the generalization to the DAE case, however, can be carried out at the cost of more complex algebra. We assume that (1) admits of a nontrivial solution $\mathbf{x}_\mathrm{S}(t)$ (time-periodical of period T), representing the oscillator noiseless working point or limit cycle.

In presence of p white Gaussian noise sources $\boldsymbol{\xi}(t)$ (the more general case of non-white noise sources, such as low-frequency noise generators, requires proper extension, see [13] for phase noise only), (1) is modified into

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} - \mathbf{f}(\mathbf{z}) = \mathbf{B}(\mathbf{z})\boldsymbol{\xi}(t), \tag{2}$$

where $\mathbf{B}(\mathbf{z}(t))$ is a $n \times p$ solution-dependent matrix which takes into account the possible modulation of the circuit noise sources (i.e., cyclostationary noise sources). The effect of the Langevin noise sources is of course to transform the ODE (1) into the stochastic ODE (2), whose unknown $\mathbf{z}(t)$ is a random process. Since we are looking for the statistical properties of $\mathbf{z}(t)$, (2) should be tackled through the conversion into the corresponding Fokker-Planck equation [14], i.e. a deterministic partial differential equation having the $\mathbf{z}(t)$ probability density as an unknown. This approach, albeit rigorous, is in general quite hard to pursue from both mathematical [15] and numerical standpoints, at least if general (circuit-independent) results are sought for. Therefore, a different procedure is typically used, which is based on the assumption that the noise sources are of small amplitude.

The most widely used approach is the so-called Linear Time Varying (LTV) description [10], [11], [16], [17], wherein noise determines a purely additive term

$$\mathbf{z}(t) = \mathbf{x}_{\mathrm{S}}(t) + \mathbf{y}_{\mathrm{lin}}(t), \tag{3}$$

where $\mathbf{y}_{\mathrm{lin}}(t)$ is small enough to linearly perturb the solution of the noiseless circuit. The LTV approximation is of course rather appealing from an application standpoint, also because of the comparatively easy implementation in circuit simulators, but has been proven to fail very close to the oscillator output harmonic frequencies [10], where it yields a diverging phase noise spectrum.

More recent results were derived recognizing that noise induces fluctuations both in the time reference and in the amplitude [6]–[8], [10]

$$\mathbf{z}(t) = \mathbf{x}_{\mathrm{S}}(t + \alpha(t)) + \mathbf{y}(t). \tag{4}$$

In (4), $\alpha(t)$ and $\mathbf{y}(t)$ are stochastic processes representing time and orbital fluctuations, respectively. For a stable oscillator, $\mathbf{y}(t)$ should be small (at least assuming that noise sources provide perturbations of the noiseless limit cycle not large enough to push $\mathbf{z}(t)$ out of the attraction basin of $\mathbf{x}_{\mathrm{S}}(t)$), therefore the common assumption is that it can be studied as a linear perturbation of the system around the orbit. Time fluctuations, on the other hand, might become large, and therefore their characterization should be carried out avoiding the linearity assumption. For this reason, these analyses are denoted as *nonlinear perturbative* approaches.

Defining the auto-correlation function of $\mathbf{z}(t)$ as $\mathbf{R}_{\mathbf{z},\mathbf{z}}(t,\tau) = \mathrm{E}\left\{\mathbf{z}(t)\mathbf{z}^{\dagger}(t+\tau)\right\}$, where $\mathrm{E}\left\{\cdot\right\}$ is the ensamble average and † denotes hermitian conjugation, (4) yields

$$\begin{aligned}\mathbf{R}_{\mathbf{z},\mathbf{z}}(t,\tau) = \mathbf{R}_{\mathbf{x}_{\mathrm{S}},\mathbf{x}_{\mathrm{S}}}(t,\tau) + \mathbf{R}_{\mathbf{y},\mathbf{y}}(t,\tau) \\ + \mathbf{R}_{\mathbf{x}_{\mathrm{S}},\mathbf{y}}(t,\tau) + \mathbf{R}_{\mathbf{y},\mathbf{x}_{\mathrm{S}}}(t,\tau).\end{aligned} \tag{5}$$

The first two terms correspond to phase and orbital noise, while phase-orbital correlation is represented by the last two contributions.

The basic difference among the theories presented in [6]–[8], [10] is the definition of the governing equations for the phase and orbital fluctuations. To provide a common framework, we make use of the Floquet theory for the LTV system obtained linearizing (1) around $\mathbf{x}_{\mathrm{S}}(t)$, briefly recalled in Section II. Our treatment of orbital noise is summarized in Section III, while examples are discussed in Section IV.

II. A brief introduction to Floquet theory

When (1) is perturbed by a small term $\mathbf{b}(t)$, the solution can be decomposed as $\mathbf{x}(t) = \mathbf{x}_{\mathrm{S}}(t) + \mathbf{w}(t)$. Thus the linearization of the perturbed systems, neglecting high order terms, reads

$$\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{w} + \mathbf{b}(t) \tag{6}$$

where $\mathbf{A}(t)$ is the Jacobian matrix evaluated at the limit cycle of the unperturbed system (1)

$$\mathbf{A}(t) = \left.\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_{\mathrm{S}}(t)}. \tag{7}$$

By definition, $\mathbf{A}(t)$ is a T-periodic matrix, hence the linear system (6) is characterized by a state transition matrix of the form [18]

$$\mathbf{\Phi}(t,s) = \mathbf{U}(t)\exp\left(\mathbf{D}(t-s)\right)\mathbf{V}(s) \tag{8}$$

where $\mathbf{U}$ and $\mathbf{V}$ are square T-periodic matrices and $\mathbf{D}$ is a square diagonal constant matrix. The general solution of (6) can be finally written as

$$\mathbf{w}(t) = \mathbf{\Phi}(t,0)\mathbf{w}(0) + \int_0^t \mathbf{\Phi}(t,s)\mathbf{b}(s)\,\mathrm{d}s. \tag{9}$$

The columns $\mathbf{u}_k(t)$ of $\mathbf{U}$ and the rows $\mathbf{v}_j^{\mathrm{T}}(t)$ ($^{\mathrm{T}}$ denotes the transpose) of $\mathbf{V}$ are the *direct* and *adjoint Floquet eigenvectors,* respectively, satisfying the orthonormality condition ($\delta_{j,k}$ is Kronecker's symbol)

$$\mathbf{v}_j^{\mathrm{T}}(t)\mathbf{u}_k(t) = \delta_{j,k}. \tag{10}$$

The diagonal elements μ_j of $\mathbf{D}$ are the so called *Floquet exponents*. It can be shown that, for autonomous systems, one of them is always zero [18] and it can be fixed as μ_1 without loss of generality. The remaining μ_j determine the stability of the orbit solution of (1): when all of them have strictly negative real part the orbit is asymptotically stable, conversely if at least one has positive real part the orbit is unstable [18].

The n direct Floquet eigenvector $\mathbf{u}_k(t)$ have the following meaning: $\exp(\mu_k t)\mathbf{u}_k(t)$ form a linearly independent (l.i.) solution set of the homogeneous system associated to (6) (i.e., the equation obtained setting $\mathbf{b} = \mathbf{0}$). On the other hand, $\exp(-\mu_j t)\mathbf{v}_j(t)$ form a complete l.i. solution set of the adjoint system associated to the homogeneous part of (6), i.e.

$$\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} = -\mathbf{A}^{\mathrm{T}}(t)\mathbf{w}. \tag{11}$$

III. Statistical characterization of phase and orbital noise

The noise decomposition in phase and orbital components, as outlined in equation (4), defines $n+1$ stochastic variables satisfying the $n+1$ equations of the Langevin system

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}t} = \mathbf{F}\left[\mathbf{Y}(t),t\right]\mathbf{Y}(t) + \mathbf{G}\left[\mathbf{Y}(t),t\right]\mathbf{b}(t), \tag{12}$$

where $\mathbf{Y}^{\mathrm{T}}(t) = \left[\alpha(t), \mathbf{y}^{\mathrm{T}}(t)\right]^{\mathrm{T}}$, and matrices $\mathbf{F}$ and $\mathbf{G}$, of size $(n+1)\times(n+1)$ and $(n+1)\times p$, respectively, are defined as

$$\mathbf{F}\left[\mathbf{Y}(t),t\right] = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}(t+\alpha(t)) \end{bmatrix} \tag{13}$$

$$\mathbf{G}\left[\mathbf{Y}(t),t\right] = \begin{bmatrix} \mathbf{v}_1^{\mathrm{T}}(t+\alpha(t))\mathbf{B}(t+\alpha(t)) \\ \displaystyle\sum_{k=2}^{n} \mathbf{u}_k(t+\alpha(t))\mathbf{v}_k^{\mathrm{T}}(t+\alpha(t))\mathbf{B}(t+\alpha(t)) \end{bmatrix}. \tag{14}$$

In (14), $\mathbf{B}(t+\alpha(t)) = \mathbf{B}\left[\mathbf{x}_{\mathrm{S}}(t+\alpha(t))\right]$.

A thorough study of (12) carried out by estimating the characteristic function associated to process $\mathbf{Y}(t)$ by means of the associated Fokker-Plank equation, allows to prove that the fluctuations at the same time along the limit cycle $\mathbf{x}_S(t+\alpha(t))$ (i.e., the phase noise contribution) and the orbital deviation $\mathbf{y}(t)$ asymptotically become statistically independent [19]. This implies that for the phase noise correlation function the same results as in [10] hold.

On the other hand, asuming that orbital noise can be estimated as a first order perturbation, we were able to exactly estimate all the terms in (5), besides phase noise, using the statistical properties of the correlation between $\alpha(t_1)$ and $\boldsymbol{\xi}(t_2)$ and the representation in Fourier series of $\mathbf{x}_S(t)$ and of the direct and adjoint associated Floquet eigenvectors [20].

Calculations in [20] show that, asymptotically with time t, processes $\mathbf{x}_S(t+\alpha(t))$ and $\mathbf{y}(t)$ become stationary. Therefore the terms in (5) become asymptotically dependent on τ only, and can be characterized by the corresponding Fourier transforms defining the spectra

$$\mathbf{S}_{\mathbf{x}_S,\mathbf{x}_S}(\omega) = \sum_h \tilde{\mathbf{X}}_h \tilde{\mathbf{X}}_h^\dagger \frac{h^2\omega_0^2 c}{\Xi_h^2(\omega)} \tag{15}$$

$$\begin{aligned}\mathbf{S}_{\mathbf{y},\mathbf{y}}(\omega) = \sum_{l=2}^{n}\sum_{h,j} & \left\{ \frac{\left(\mathbf{C}_{lhj}^\dagger + \mathbf{C}_{lhj}\right)\left[\frac{1}{2}h^2\omega_0^2 c - \mathrm{Re}\{\mu_l\}\right]}{\Delta_{lhj}^2(\omega)} \right. \\ & \left. + \frac{\mathrm{i}\left(\mathbf{C}_{lhj}^\dagger - \mathbf{C}_{lhj}\right)(\omega + j\omega_0 + \mathrm{Im}\{\mu_l\})}{\Delta_{lhj}^2(\omega)} \right\},\end{aligned} \tag{16}$$

$$\begin{aligned}\mathbf{S}_{\mathrm{corr}}(\omega) = \sum_{l=2}^{n}\sum_{h,j} & \left\{ \frac{\left(\mathbf{D}_{lhj}^\dagger + \mathbf{D}_{lhj}\right)\left[\frac{1}{2}h^2\omega_0^2 c - \mathrm{Re}\{\mu_l\}\right]}{\Delta_{lhj}^2(\omega)} \right. \\ & + \frac{\mathrm{i}\left(\mathbf{D}_{lhj}^\dagger - \mathbf{D}_{lhj}\right)[\omega + j\omega_0 + \mathrm{Im}\{\mu_l\}]}{\Delta_{lhj}^2(\omega)} \\ & - \frac{\left(\mathbf{D}_{lhj}^\dagger + \mathbf{D}_{lhj}\right)\left[\frac{1}{2}h^2\omega_0^2 c\right]}{\Xi_h^2(\omega)} \\ & \left. - \frac{\mathrm{i}\left(\mathbf{D}_{lhj}^\dagger - \mathbf{D}_{lhj}\right)[\omega + h\omega_0]}{\Xi_h^2(\omega)} \right\}\end{aligned} \tag{17}$$

where ($\mathrm{i} = \sqrt{-1}$):

$$\Delta_{lhj}^2(\omega) = \left[\frac{1}{2}h^2\omega_0^2 c - \mathrm{Re}\{\mu_l\}\right]^2 + [\omega + j\omega_0 + \mathrm{Im}\{\mu_l\}]^2 \tag{18}$$

$$\Xi_h^2(\omega) = \left[\frac{1}{2}h^2\omega_0^2 c\right]^2 + [\omega + h\omega_0]^2 \tag{19}$$

$$c = \frac{1}{T}\int_0^T \mathbf{v}_1^{\mathrm{T}}\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{v}_1 \; \mathrm{d}t \tag{20}$$

$$\mathbf{C}_{lhj} = \sum_{l'=2}^{n}\sum_{j'} \frac{1}{\mathrm{i}(j-j')\omega_0 - \mu_{l'} - \mu_l^*} \tilde{\mathbf{U}}_{l'_{j'}} \tilde{\mathbf{\Lambda}}_{l'_{h-j'}}^{\mathrm{T}} \tilde{\mathbf{\Lambda}}_{l_{h-j}}^{*} \tilde{\mathbf{U}}_{l_j}^\dagger \tag{21}$$

$$\mathbf{D}_{lhj} = \tilde{\mathbf{X}}_h \tilde{\mathbf{V}}_{1_0}^{\mathrm{T}} \tilde{\mathbf{\Lambda}}_{l_{h-j}}^{*} \tilde{\mathbf{U}}_{l_j}^\dagger \frac{\mathrm{i}h\omega_0}{-\mu_l^* - \mathrm{i}(h-j)\omega_0}. \tag{22}$$

Coefficients $\tilde{\mathbf{X}}_h$ are the harmonic components of $\mathbf{x}_S(t)$, while $\tilde{\mathbf{U}}_{l_j}$ and $\tilde{\mathbf{\Lambda}}_{l_k}^{\mathrm{T}}$ are the Fourier coefficients of $\mathbf{u}_l(t)$ and $\mathbf{v}_l^{\mathrm{T}}(t)\mathbf{B}(t)$, respectively.

IV. Examples

A. Coram oscillator

The first example is the autonomous system proposed in [21] to assess the various approaches to phase and amplitude noise decomposition. The noisy oscillator is represented in polar coordinates by

$$\dot{\rho} = \rho - \rho^2 + \beta\xi_\rho(t) \tag{23a}$$

$$\dot{\theta} = 1 + \rho + \beta\xi_\theta(t) \tag{23b}$$

where ρ and θ are, respectively, the radial and angular coordinates, β is a parameter tuning the noise source amplitude, and $\xi(t)$ are unit white Gaussian noise sources.

Turning to cartesian coordiantes $\mathbf{x}(t) = [x_1(t), x_2(t)]^{\mathrm{T}}$, (23) becomes

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\boldsymbol{\xi}(t) \tag{24}$$

where

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} x_1 - x_2 - (x_1+x_2)\sqrt{x_1^2+x_2^2} \\ x_1 + x_2 + (x_1-x_2)\sqrt{x_1^2+x_2^2} \end{bmatrix} \tag{25}$$

$$\mathbf{B}(\mathbf{x}) = \beta \begin{bmatrix} x_1/\sqrt{x_1^2+x_2^2} & -x_2 \\ x_2/\sqrt{x_1^2+x_2^2} & x_1 \end{bmatrix} \tag{26}$$

$$\boldsymbol{\xi}(t) = \begin{bmatrix} \xi_\rho(t) \\ \xi_\theta(t) \end{bmatrix}. \tag{27}$$

The noiseless oscillator limit cycle reads simply

$$x_{S,1}(t) = \cos(2t) \qquad x_{S,2}(t) = \sin(2t), \tag{28}$$

therefore $T = \pi$ and $\omega_0 = 2$.

This simple system allows for a completely analytical determination of all of the Floquet quantities defined in Section II. The two Floquet exponents are $\mu_1 = 0$ and $\mu_2 = -1$, and the corresponding direct and adjoint Floquet eigenvectors contain only harmonic components at the fundamental frequency

$\omega_0 = 2$:

$$\mathbf{u}_1(t) = 2\begin{bmatrix} -\sin(2t) \\ \cos(2t) \end{bmatrix} \tag{29}$$

$$\mathbf{u}_2(t) = \begin{bmatrix} -\cos(2t) - \sin(2t) \\ \cos(2t) - \sin(2t) \end{bmatrix} \tag{30}$$

$$\mathbf{v}_1(t) = \frac{1}{2}\begin{bmatrix} \cos(2t) - \sin(2t) \\ \cos(2t) + \sin(2t) \end{bmatrix} \tag{31}$$

$$\mathbf{v}_2(t) = \begin{bmatrix} -\cos(2t) \\ -\sin(2t) \end{bmatrix}. \tag{32}$$

This oscillator was studied in [22] as a function of the noise source intensity β, exploiting the Fokker-Planck equation associated to (23). The results in [22] make use of a specific form of the asymptotic correlation function, and therefore of the corresponding noise spectrum. A scalar (asymptotic) correlation function $R(\tau)$ is defined as the sum of the 4 components of the matrix correlation function normalized to the same sum calculated for $\tau = 0$ (in other words, this guarantees that $R(0) = 1$). The corresponding spectrum $S(\omega)$ simply is the Fourier transform of $R(\tau)$.

Since all the Floquet quantities are available in analytical form, we can explicitly calculate the spectral components of the total noise process, i.e. including phase noise, orbital noise and phase orbital correlation, according to Section III. One of the basic ingredients is the c coefficient (20):

$$c = \frac{\beta^2}{2} \tag{33}$$

that fully characterizes the phase noise component.

Applying (15)-(16) and summing the 4 components of the 2×2 matrices, the results are

$$S_{\mathbf{x_S},\mathbf{x_S}}(\omega) = \frac{\frac{1}{4}\omega_0^2\beta^2}{\frac{1}{16}\omega_0^4\beta^4 + (\omega+\omega_0)^2} + \frac{\frac{1}{4}\omega_0^2\beta^2}{\frac{1}{16}\omega_0^4\beta^4 + (\omega-\omega_0)^2} \tag{34a}$$

$$\begin{aligned} S_{\mathbf{y},\mathbf{y}}(\omega) = {} & \frac{\beta^2\left(1+\frac{1}{4}\omega_0^2\beta^2\right)}{\left(1+\frac{1}{4}\omega_0^2\beta^2\right)^2 + (\omega+\omega_0)^2} \\ & + \frac{\beta^2\left(1+\frac{1}{4}\omega_0^2\beta^2\right)}{\left(1+\frac{1}{4}\omega_0^2\beta^2\right)^2 + (\omega-\omega_0)^2} \end{aligned} \tag{34b}$$

$$\begin{aligned} S_{\mathrm{corr}}(\omega) = {} & -\frac{\beta^2\left(1+\frac{1}{4}\omega_0^2\beta^2\right)}{\left(1+\frac{1}{4}\omega_0^2\beta^2\right)^2 + (\omega+\omega_0)^2} \\ & - \frac{\beta^2\left(1+\frac{1}{4}\omega_0^2\beta^2\right)}{\left(1+\frac{1}{4}\omega_0^2\beta^2\right)^2 + (\omega-\omega_0)^2} \\ & + \frac{\beta^2(\omega+\omega_0)}{\left(1+\frac{1}{4}\omega_0^2\beta^2\right)^2 + (\omega+\omega_0)^2} \\ & - \frac{\beta^2(\omega-\omega_0)}{\left(1+\frac{1}{4}\omega_0^2\beta^2\right)^2 + (\omega-\omega_0)^2} \\ & + \frac{\frac{1}{4}\omega_0^2\beta^4}{\frac{1}{16}\omega_0^4\beta^4 + (\omega+\omega_0)^2} + \frac{\frac{1}{4}\omega_0^2\beta^4}{\frac{1}{16}\omega_0^4\beta^4 + (\omega-\omega_0)^2} \\ & - \frac{\beta^2(\omega+\omega_0)}{\frac{1}{16}\omega_0^4\beta^4 + (\omega+\omega_0)^2} + \frac{\beta^2(\omega-\omega_0)}{\frac{1}{16}\omega_0^4\beta^4 + (\omega-\omega_0)^2} \end{aligned} \tag{34c}$$

Finally, summing these components and applying the normalization we find the spectrum calculated numerically in [22]

$$\begin{aligned} S(\omega) = \frac{\beta^2}{1+\beta^2}\Bigg[& \frac{\frac{1}{4}\omega_0^2(1+\beta^2)}{\frac{1}{16}\omega_0^4\beta^4 + (\omega+\omega_0)^2} \\ & + \frac{\frac{1}{4}\omega_0^2(1+\beta^2)}{\frac{1}{16}\omega_0^4\beta^4 + (\omega-\omega_0)^2} \\ & + \frac{\omega+\omega_0}{\left(1+\frac{1}{4}\omega_0^2\beta^2\right)^2 + (\omega+\omega_0)^2} \\ & - \frac{\omega-\omega_0}{\left(1+\frac{1}{4}\omega_0^2\beta^2\right)^2 + (\omega-\omega_0)^2} \\ & - \frac{\omega+\omega_0}{\frac{1}{16}\omega_0^4\beta^4 + (\omega+\omega_0)^2} \\ & + \frac{\omega-\omega_0}{\frac{1}{16}\omega_0^4\beta^4 + (\omega-\omega_0)^2}\Bigg]. \end{aligned} \tag{35}$$

We compare the results of this nonlinear perturbative theory to the numerical solution of the full Fokker-Planck equation reported in [22]. We consider two values of the β parameter, i.e. the amplitude of the noise sources (see (23)). Since we assume that orbital noise is a first order perturbation of the noiseless limit cycle, the precision of our results is expected to degrade as β grows.

Fig. 1 shows the caculated noise spectrum for $\beta^2 = 0.1$. For this low value of noise source, phase noise dominates the

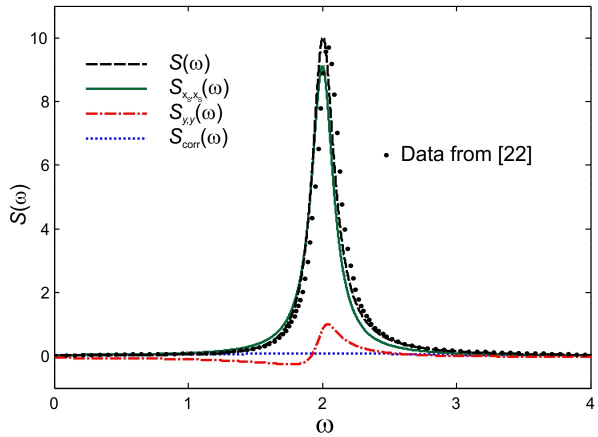

Fig. 1: Normalized noise spectrum of the Coram oscillator for $\beta^2 = 0.1$.

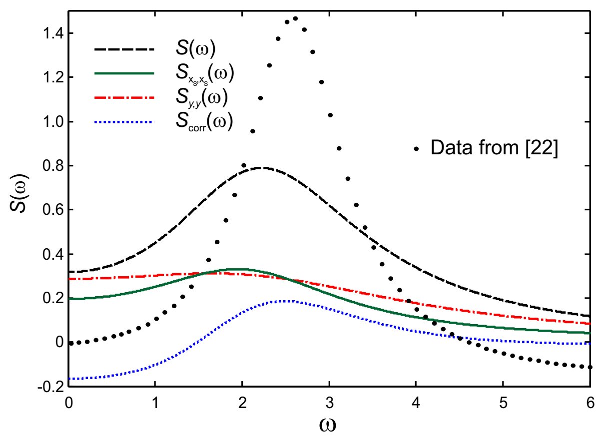

Fig. 2: Normalized noise spectrum of the Coram oscillator for $\beta^2 = 1.4$.

total noise spectrum, while the orbital component provides a correction which enhances the peak value of $S(\omega)$ and breaks the symmetry of the spectrum around the fundamental frequency. Phase-orbital correlation is in this case negligible. The agreement with the full nonlinear calculations is good, and the effect of the orbital contribution helps in shifting the noise maximum towards higher frequencies, in agreement with the discussion in [22].

Increasing the value of β the nonlinear effect becomes more important (see Fig. 2), and the perturbative approach is no longer accurate. In fact, neither the value nor the position of the peak are well approximated by the analytical result. Notice however that both the orbital noise and phase-orbital correlation contributions provide a shift of the peak noise frequency towards the position calculated in [22].

B. A Colpitts oscillator

The second example we discuss is the Colpitts oscillator presented in Fig. 3, where the bipolar transistor is represented by a memoryless, simplified model as follows:

$$i_{\mathrm{B}} = I_{\mathrm{S}} \left[\mathrm{e}^{v_{\mathrm{B}}/V_T} - 1 \right] \qquad i_{\mathrm{C}} = \beta_{\mathrm{F}} i_{\mathrm{B}}, \tag{36}$$

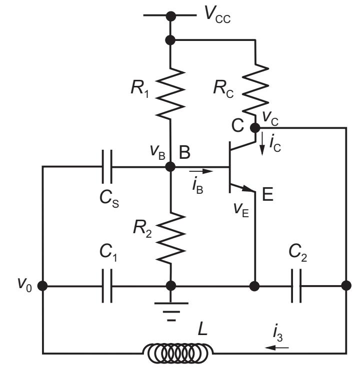

Fig. 3: Circuit of the Colpitts oscillator.

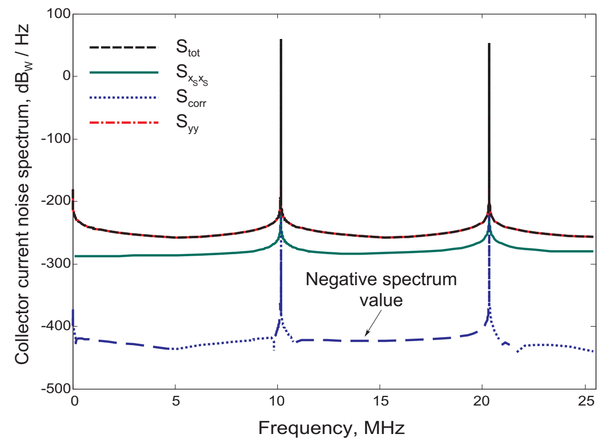

Fig. 4: Collector current noise spectrum of the Colpitts oscillator as a function of the absolute frequency.

where $I_{\mathrm{S}} = 10^{-18}$ A, $V_T = 26$ mV and $\beta_{\mathrm{F}} = 100$. The circuit parameters are: $V_{\mathrm{CC}} = 15$ V, $R_1 = 400$ kΩ, $R_2 = 71.429$ kΩ, $R_{\mathrm{C}} = 4.9$ kΩ, $C_1 = 300$ nF, $C_2 = 9.09$ nF, $C_{\mathrm{S}} = 1$ μF and $L = 27.78$ nH. The circuit has been analyzed with the harmonic balance technique including 60 harmonics, while the Floquet quantities have been determined with the method in [23], [24]. The parameter choice yields a limit cycle in the phase space not very distorted (i.e., the first harmonic is dominant with respect to the others) and is characterized by the frequency $f_0 = 10.166$ MHz. The four Floquet exponents of the limit cycle are

$$\mu_1 = -1.22622749728103 \times 10^{-8} \approx 0 \tag{37}$$
$$\mu_2 = -1.12929350360548 \tag{38}$$
$$\mu_3 = -14946.3250189117 - 73525.956036186i \tag{39}$$
$$\mu_4 = -14946.3250189117 + 73525.956036186i. \tag{40}$$

Considering as an output variable the collector current i_{C}, the calculation of the c costant yields $c = 4.192 \times 10^{-26}$ s^2 Hz assuming for simplicity that only the transistor is noisy, and affected by white shot noise. The collector current noise spectrum, in dB$_{\mathrm{W}}$/Hz, is reported in Fig. 4 as a function of frequency, showing that, at least far from the limit cycle harmonics, the spectrum is dominated by the orbital deviation

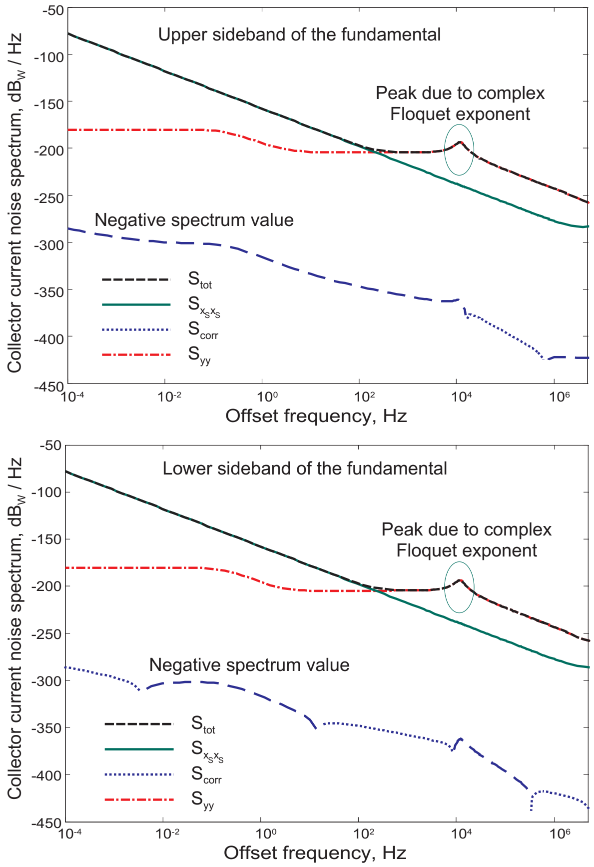

Fig. 5: Upper (above) and lower (below) sideband frequency dependence of the collector current noise spectrum of the Colpitts oscillator around the fundamental frequency f_0.

contribution (16). The correlation between phase and orbital noise (17), on the other hand, is negligible.

A better insight is obtained by considering the upper (i.e., $\omega > \omega_0$) and lower (i.e., $\omega < \omega_0$) sidebands of the fundamental frequency. The two spectra as a function of the sideband frequency (i.e., $|\omega - \omega_0|$) are shown in Fig. 5, highlighting the effect of the complex Floquet exponents, which yield the small resonance-like peak 11.7 kHz away from the fundamental (see also [11]). Notice also that the phase-orbital contribution, albeit negligible in this example in comparison with the phase and orbital spectra, is nor symmetric with respect to the central frequency f_0, neither Lorentzian in shape.

V. Conclusion

We have presented a review of recent results in the evaluation of the full noise spectrum of autonomous systems, including in the calculations the effect of orbital noise and of the correlation between phase and orbital fluctuations. The theory is based on the application of Floquet theory to the linearization of the oscillator equations around the noiseless working point, and makes use of a description of the noisy system as a superposition of time-reference fluctuations along the limit cycle and of orbital noise taking place as an orbital noise contribution. This approach extends previous results [10] originally applied to phase noise only, i.e. neglecting completely the orbital contribution. Exploiting stochastic analysis and a Fourier series description of the periodic functions, closed form expressions for the relevant noise spectra are derived and applied to two relevant oscillator examples.

References

[1] P.I. Kuznetsov, R.L. Stratonovich, V.I. Tikhonov, *Non-linear transformations of stochastic processes,* Pergamon Press, 1965.
[2] W.A. Edson, "Noise in oscillators," *Proc. of the IRE,* vol. 48, no. 8, pp. 1454-1466, Aug. 1960.
[3] E. Hafner, "The effects of noise in oscillators," *Proc. IEEE,* vol. 54, no. 2, p. 179, Feb. 1966.
[4] M. Lax, "Classical noise. V. Noise in self-sustained oscillators," *Physical Review,* vol. 160, no. 2, pp. 290–306, Aug. 1967.
[5] K. Kurokawa, "Noise in synchronized oscillators," *IEEE Trans. Microwave Theory Tech.,* vol. MTT-16, pp. 234–240, Apr. 1968.
[6] F. X. Kaertner, "Determination of the correlation spectrum of oscillators with low noise", *IEEE Trans. Microwave Theory and Techniques*, vol. 37, no. 1, pp.90-101, Jan. 1989.
[7] F. X. Kaertner, "Analysis of white and $f^{-\alpha}$ noise in oscillators", *Int. J. Circ. Theory App.*, vol. 18, pp. 485–519, 1990.
[8] A. Hajimiri, T.H. Lee, "A general theory of phase noise in electrical oscillators," *IEEE J. Solid State Circ.*, vol. 32, no. 2, pp. 179–194, Feb. 1998.
[9] T.H. Lee, A. Hajimiri, "Oscillator phase noise: a tutorial," *IEEE J. Solid State Circ.*, vol. 35, no. 3, pp. 326–336, Mar. 2000.
[10] A. Demir, A. Mehrotra, J. Roychowdhury, "Phase noise in oscillators: A unifying theory and numerical methods for characterization," *IEEE Trans. Circ. Syst. I: Fund. Theory and App.*, vol. 47, no. 5, pp. 655–674, May 2000.
[11] A. Hajimiri, T.H. Lee, *The design of low noise oscillators,* Dordrect: Kluwer Academic Publishers, 1999.
[12] A. Brambilla, G. Gruosso, M.A. Redaelli, G. Storti Gajani, D.D. Caviglia, "Improved small signal analysis for circuits working in periodic steady state," *IEEE Trans. Circ. Syst. I: Regular Papers*, vol. 57, no. 2, pp. 427–437, Feb. 2010.
[13] A. Demir, "Phase noise and timing jitter in oscillators with colored-noise sources," *IEEE Trans. Circ. Syst. I: Fund. Theory and App.*, vol. 49, no. 12, pp. 1782–1791, Dec. 2000.
[14] H. Risken, *The Fokker-Planck equation*, Second Ed., Heidelberg: Springer Verlag, 1989.
[15] S. Shankar Sastry, "The effects of small noise on implicitly defined nonlinear dynamical systems", *IEEE Trans. Circ. Syst. I: Fund. Theory and App.*, vol. 30, no. 9, pp. 651–663, Sept. 1983.
[16] A. Brambilla, G. Gruosso, M.A. Redaelli, G. Storti Gajani, D.D. Caviglia, "Improved small signal analysis for circuits working in periodic steady state," *IEEE Trans. Circ. Syst. I: Regular Papers*, vol. 57, no. 2, pp. 427–437, Feb. 2010.
[17] D. Murphy, J.J. Rael, A.A. Abidi, "Phase noise in LC oscillators: a phasor-based analysis of a general result and of loaded Q," *IEEE Trans. Circ. Syst. I: Regular Papers*, vol. 57, no. 6, pp. 1187–1203, June 2010.
[18] M. Farkas, *Periodic motions*, New York: Springer-Verlag, 1994.
[19] F.L. Traversa, F. Bonani, "Asymptotic stochastic characterization of phase and amplitude noise in free-running oscillators," accepted for publication in *Fluctuations and noise letters,* 2011.
[20] F.L. Traversa, F. Bonani, "Oscillator noise: a nonlinear perturbative theory including orbital fluctuations and phase-orbital correlation," accepted for publication in *IEEE Trans. Circ. Syst. I: Regular Papers,* 2011.
[21] G.J. Coram, "A simple 2-D oscillator to determine the correct decomposition of perturbations into amplitude and phase noise," *IEEE Trans. Circuits and Systems I*, Vol. 48, pp. 896–898, July 2001.
[22] R.S. Swain, J.P. Gleeson, M.P. Kennedy, "Influence of noise intensity on the spectrum of an oscillator," *IEEE Trans. Circ. Syst. II: Express Briefs*, vol. 52, no. 11, pp. 789–793, Nov. 2005.
[23] F.L. Traversa, F. Bonani, S. Donati Guerrieri, "A frequency-domain approach to the analysis of stability and bifurcations in nonlinear systems described by differential-algebraic equations", *Int. J. Circuit Theory & App.*, Vol. 36, No. 4, pp. 421–439, June 2008.
[24] F.L. Traversa, F. Bonani, "Frequency-domain evaluation of the adjoint Floquet eigenvectors for oscillator noise characterisation", *IET Circuits Devices Syst.*, Vol. 5, No. 1, pp. 46–51, 2011.