Generalized Marshall-Olkin Distributions, and Related Bivariate Aging Properties

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Abstract

A class of generalized bivariate Marshall–Olkin distributions, which includes as special cases the Marshall-Olkin bivariate exponential distribution and the Marshall-Olkin type distribution due to Muliere and Scarsini (1987), are examined in this paper. Stochastic comparison results are derived, and bivariate aging properties, together with properties related to evolution of dependence along time, are investigated for this class of distributions. Extensions of results previously presented in the literature are provided as well.

Key words Positive dependence properties; aging notions; survival copulas; stochastic orders; positive dependence orders.

1 Introduction and preliminaries

Dealing with stress–strength modelling, a typical assumption is that the dependence among components arise from common environmental shocks and stress. In this case, a well-known joint distribution appropriate to describe the random lifetimes of a two–component system is the the bivariate exponential distribution proposed in Marshall and Olkin (1967), whose survival function is defined as

$$\overline{F}(x_1, x_2) = P(X_1 > x_1,\, X_2 > x_2) = \exp\big\{ -\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 \max\{x_1, x_2\}\big\}, \qquad (1.1)$$

with $x_1, x_2 \geq 0$ and $\lambda_i \geq 0$, $i = 1, 2, 3$. For example, in reliability theory this structure may describe the lifetimes of two components operating in a random environment and subjected to fatal shock governed by a poisson process, while in the theory of credit risk X_1 and X_2 may be viewed as the times to default of two counter-parties subject to three independent underlying economic or financial events.

Different generalizations of this model have been considered and applied in the literature starting from the observation that a bivariate random vector (X_1, X_2) of lifetimes has the Marshall–Olkin distribution whenever it admits the representation

$$(X_1, X_2) \overset{st}{=} (\min\{S_1, S_3\}, \min\{S_2, S_3\}), \qquad (1.2)$$

where S_1, S_2 and S_3 are independent and exponentially distributed lifetimes with parameters λ_1, λ_2 and λ_3, respectively.

On the one hand, some authors substituted in the above structure the exponential distribution by the second type Pareto distribution, or by the Weibull distribution, in order to obtain a bunch of bivariate semi–parametric models which performed well in modelling bivariate survival data (see, for example, Lu, 1989 and 1992, or Asimit et al., 2010), or, for example, in the description of occurrences of metastases at multiple sites after breast cancer (see Klein et al., 1989). Moreover, bivariate vectors defined as in (1.2) are, actually, a particular case of the family of distributions of coherent systems sharing some of their components, like the ones recently studied in Navarro et al. (2010) (see also Navarro and Balakrishnan, 2010, for dependence properties of this family of distributions).

On the other hand, those who focused on the lack-of-memory property of the Marshall–Olkin distribution devote themselves to gaining any further insight in the mechanism. For example, it was found (see Marshall and Olkin, 1967, or Galambos and Kotz, 1978) that the vector (X_1, X_2) with exponential marginal distributions has the bivariate distribution in (1.1) if and only if it achieves the lack-of-memory property

$$P(X_1 > x_1 + t,\, X_2 > x_2 + t|\; X_1 > t,\, X_2 > t) = P(X_1 > x_1,\, X_2 > x_2), \qquad (1.3)$$

for all $x_1, x_2 \geq 0$ and $t \geq 0$. Subsequently, Muliere and Scarsini (1987) further investigated the distributions satisfying the equality

$$P(X_1 > x_1 * t,\; X_2 > x_2 * t|\; X_1 > t,\; X_2 > t) = P(X_1 > x_1,\; X_2 > x_2), \tag{1.4}$$

where the binary operation $*$ is assumed to be associative (i.e., such that $x * (y * z) = (x * y) * z$) and reducible (i.e., it satisfies $x * y = x * z$ or $y * x = z * x$ if and only if $z = y$). Obviously, setting $*$ as $+$ in (1.4), one gets (1.3). Muliere and Scarsini (1987) proved that bivariate vectors (X_1, X_2) having continuous distribution possess the lack-of-memory property (1.4) and satisfy the equality

$$P(X_i > x_i * t|\; X_i > t) = P(X_i > x_i), \quad i = 1, 2,$$

for all $x_1, x_2, t \geq 0$ if and only if they have joint survival function

$$\overline{F}(x_1, x_2) = \exp\{-\lambda_1 H(x_1) - \lambda_2 H(x_2) - \lambda_3 H(\max\{x_1, x_2\})\},\; x_1, x_2 \geq 0,$$

for an increasing function H such that $H(0) = 0$ and $H(\infty) = \infty$. This kind of semi-parametric model, that they called Marshall-Olkin type survival function, is rather flexible in practice and includes several useful bivariate distributions (see, e.g., Scarsini, 1984, and Wu, 1997). Moreover, in this case $\overline{F}$ also corresponds to the survival function of a vector of lifetimes having marginal distributions satisfying a Cox proportional hazard rate model, with baseline cumulative hazard function H.

Along the line of such a kind of semi-parametric extension, in this paper we study the more general model which takes the form (1.2) where the three non–negative random variables S_1, S_2 and S_3 are assumed to be independent but not necessarily with proportional hazard rates. In other words, we consider here the class of bivariate vectors $\boldsymbol{X}$ defined as in (1.2), where the lifetimes S_i are independent and not necessarily identically distributed, thus vectors having joint survival function

$$\begin{aligned}\bar{F}_{\boldsymbol{X}}(x_1, x_2) &= P(X_1 > x_1, X_2 > x_2) \\ &= P(S_1 > x_1, S_2 > x_2, S_3 > \max\{x_1, x_2\}) \\ &= \bar{G}_1(x_1)\bar{G}_2(x_2)\bar{G}_3(\max\{x_1, x_2\}) \\ &= \exp\{-H_1(x_1) - H_2(x_2) - H_3(\max\{x_1, x_2\})\},\end{aligned} \tag{1.5}$$

where the right continuous functions H_i satisfying $H_i(0) = 0$ and $H_i(\infty) = \infty$ are the cumulative hazard functions of the lifetimes S_i (and, in particular, are the integrals of the hazard rates when the S_i are absolutely continuous). In this case, we will say that $\boldsymbol{X}$ has a *Generalized Marshall-Olkin type (GMO) distribution*, and H_1, H_2, H_3 will be called the *generating functions* of (1.5).

As already mentioned, the following are special cases of GMO distributions.

1. Bivariate exponential distribution (Marshall and Olkin, 1967)

$$H_i(x) = \lambda_i x, \qquad x \geq 0, \quad \lambda_i \geq 0,\ i = 1, 2, 3.$$

2. Bivariate Weibull distribution (Marshall and Olkin, 1967, and Lu, 1989)

$$H_i(x) = \lambda_i x^\alpha, \qquad x \geq 0, \quad \alpha > 0,\ \lambda_i \geq 0,\ i = 1, 2, 3.$$

3. Bivariate Pareto distribution (II) (Hanagal, 1996, Kotz et al., 2000, and Asimit et al., 2010)

$$H_i(x) = \alpha_i \log\left(1 + \frac{x - \mu_i}{\sigma_i}\right), \qquad x \geq \mu_i \geq 0, i = 1, 2, 3,$$

   for $\mu_1 = \mu_2 \geq 0$, $\sigma_1 = \sigma_2 \geq 0$, $\mu_3 = 0$, $\sigma_3 = 1$ and $\alpha_i \geq 0$, $i = 1, 2, 3$.

4. Marshall-Olkin type distribution (Muliere and Scarsini, 1987)

$$H_i(x) = \lambda_i H(x), \qquad \lambda_i \geq 0,\ i = 1, 2, 3,$$

   where $H(x)$ is increasing with $H(0) = 0$ and $H(\infty) = \infty$.

However, it should also be pointed out that GMO distributions defined as above have the main disadvantage that they are not absolutely continuous, having a singularity due to $P(X_1 = X_2) > 0$, thus they can not be applied in all those problems where absolute continuity is required.

The class of the generalized Marshall-Olkin type distributions does not possess the lack-of-memory property, and for this reason the aim of this paper is to investigate the aging behavior and the dependence properties of such type of random vectors.

In Section 2, we derive the copula expression for GMO distributions, and we provide the first preliminary positive dependence property satisfied by these distributions. In Section 3, we analyze stochastic comparisons among GMO distributions. Apart from the stochastic order and the increasing concave order of the random vectors themselves, the order on their copulas is built based upon the stochastic orders of the generating random variables. In Section 4, we first have a simple discussion on the aging behavior of this type of distributions due to the aging property of the three generating random variables. Then, by studying the survival copula of the residual life, we explore the evolution of the dependence as time elapses. Based on these works on dependence, a further discussion on the aging behavior of the GMO distribution is made.

Throughout this note, the terms *increasing* and *decreasing* stand for non-decreasing and non-increasing, respectively. All random variables under investigation are non–negative, with continuous distribution, and expectations are implicitly assumed to be finite once they appear.

For ease of reference, let us first briefly recall some useful notions, and stochastic orders and aging concepts which will be used in sequel.

Recall that a random vector $\boldsymbol{X} = (X_1, X_2)$ with joint survival function $\bar{F}$ and continuous marginal survival functions $\bar{F}_i$, $i = 1, 2$ has survival copula

$$\hat{C}_{\boldsymbol{X}}(u, v) = \bar{F}\big(\bar{F}_1^{-1}(u), \bar{F}_2^{-1}(v)\big), \quad 0 \leq u, v \leq 1,$$

where $\bar{F}_i^{-1}$ is the right continuous inverse of $\bar{F}_i$, $i = 1, 2$. The survival copula, which is unique under assumption of continuity of the $\overline{F}_i$, is an useful tool to describe the structure of dependence between the concerned components (see, e.g., Nelsen, 1999). For example, different positive dependence concepts have been defined by means of copulas. Among others, the well–known PQD notion: a vector $\boldsymbol{X}$ is said to be *positively quadrant dependent* (PQD) if

$$\hat{C}_{\boldsymbol{X}}(u, v) \geq uv \quad \text{for all } 0 \leq u, v \leq 1$$

(see, e.g., Denuit et al., 2005).

Definition 1.1 $\boldsymbol{X} = (X_1, X_2)$ is said to be smaller than $\boldsymbol{Y} = (Y_1, Y_2)$ in the

(i) *usual stochastic order* (denoted by $\boldsymbol{X} \leq_{st} \boldsymbol{Y}$) if $\mathrm{E}[\phi(X_1, X_2)] \leq \mathrm{E}[\phi(Y_1, Y_2)]$ for every increasing function ϕ such that expectations exist;

(ii) *increasing concave order* (denoted by $\boldsymbol{X} \leq_{icv} \boldsymbol{Y}$) if $\mathrm{E}[\phi(X_1, X_2)] \leq \mathrm{E}[\phi(Y_1, Y_2)]$ for every increasing and concave function ϕ such that expectations exist;

(iii) *upper orthant order* (denoted by $\boldsymbol{X} \leq_{uo} \boldsymbol{Y}$) if $\mathrm{E}[\phi(X_1, X_2)] \leq \mathrm{E}[\phi(Y_1, Y_2)]$ for every joint distribution function ϕ such that expectations exist, i.e., if and only if $P[X_1 > x_1, X_2 > x_2] \leq P[Y_1 > x_1, Y_2 > x_2]$ for all $x_1, x_2 \in \Re$.

See Shaked and Shanthikumar (2007) for details, properties and equivalent definitions of these stochastic orders.

The following aging notions are well–known in reliability theory. Denote with $X_t = [X - t | X > t]$ the residual life of X at time $t \geq 0$.

Definition 1.2 A non–negative random variable X is said to be

(i) of *increasing in failure rate* (IFR) if $X_s \geq_{st} X_t$ for all $t \geq s \geq 0$;

(ii) *new better than used* (NBU) if $X \geq_{st} X_t$ for all $t \geq 0$;

(iii) *new better than used in the 2nd order stochastic dominance* (NBU(2)) if $X \geq_{icv} X_t$ for all $t \geq 0$;

Let now

$$\boldsymbol{X}_t = [(X_1 - t, X_2 - t) | X_1 > t, X_2 > t]$$

be the residual life vector of $\boldsymbol{X}$ at time $t \geq 0$.

Definition 1.3 A non–negative random vector $\boldsymbol{X} = (X_1, X_2)$ is said to be

(i) of *bivariate increasing failure rate* (B-IFR) if $\boldsymbol{X}_s \geq_{st} \boldsymbol{X}_t$ for all $t \geq s \geq 0$;

(ii) *bivariate new better than used* (B-NBU) if $\boldsymbol{X} \geq_{st} \boldsymbol{X}_t$ for all $t \geq 0$;

(iii) *bivariate new better than used in the 2nd stochastic dominance* (B-NBU(2)) if $\boldsymbol{X} \geq_{icv} \boldsymbol{X}_t$ for all $t \geq 0$.

The dual notions *decreasing failure rate* (DFR), *new worse than used* (NWU) and *new worse than used in the 2nd order stochastic dominance* (NWU(2)) as well as their bivariate versions B-DFR, B-NBU, B-NBU(2) may be defined through reversing all corresponding inequalities above. It is well-known that

$$\boldsymbol{X} \leq_{st} \boldsymbol{Y} \Longrightarrow \boldsymbol{X} \leq_{uo} (\leq_{icv}) \, \boldsymbol{Y},$$

$$\text{IFR (DFR)} \Longrightarrow \text{NBU (NWU)} \Longrightarrow \text{NBU(2) (NWU(2))},$$

and

$$\text{B-IFR (B-DFR)} \Longrightarrow \text{B-NBU (B-NWU)} \Longrightarrow \text{B-NBU(2) (B-NWU(2))}.$$

For more details on stochastic orders and aging properties, readers may refer Barlow and Proschan (1981), Klefsjö (1983), Deshpande et al. (1986), Pellerey (2008), Li and Kochar (2001), Denuit et al. (2005), Shaked and Shanthikumar (2007), Lai and Xie (2006) and Mulero and Pellerey (2010).

2 Generalized Marshall-Olkin copula

Consider a bivariate vector $\boldsymbol{X} = (X_1, X_2)$ having GMO distribution, i.e.,

$$\boldsymbol{X} = (X_1, X_2) = (\min\{S_1, S_3\}, \min\{S_2, S_3\}) \tag{2.1}$$

for mutually independent random lifetimes $S_i \sim \bar{G}_i(x)$ with cumulative hazard functions H_i, thus with joint survival function

$$\begin{aligned}
\bar{F}_{\boldsymbol{X}}(x_1, x_2) &= P(X_1 > x_1, X_2 > x_2) \\
&= P(S_1 > x_1, S_2 > x_2, S_3 > \max\{x_1, x_2\}) \\
&= \bar{G}_1(x_1)\bar{G}_2(x_2)\bar{G}_3(\max\{x_1, x_2\}) \\
&= \exp\{-H_1(x_1) - H_2(x_2) - H_3(\max\{x_1, x_2\})\}, \qquad (2.2)
\end{aligned}$$

and marginal survival functions

$$\bar{F}_1(x_1) = P(X_1 > x_1) = \bar{G}_1(x_1)\bar{G}_3(x_1) = \exp\{-H_1(x_1) - H_3(x_1)\},$$

$$\bar{F}_2(x_2) = P(X_2 > x_2) = \bar{G}_2(x_2)\bar{G}_3(x_2) = \exp\{-H_2(x_2) - H_3(x_2)\}.$$

Denote

$$\tilde{H}_1(x) = H_1(x) + H_3(x), \qquad \tilde{H}_2(x) = H_2(x) + H_3(x).$$

Then,

$$\bar{F}_1^{-1}(u) = \tilde{H}_1^{-1}(-\ln u), \qquad \bar{F}_2^{-1}(v) = \tilde{H}_2^{-1}(-\ln v).$$

Let $\hat{C}_{\boldsymbol{X}}(u, v)$ be the survival copula of $\boldsymbol{X}$. Then, for (u, v) such that $\bar{F}_1^{-1}(u) > \bar{F}_2^{-1}(v)$, we have

$$\begin{aligned}
\ln \hat{C}_{\boldsymbol{X}}(u, v) &= \ln \bar{F}\big(\bar{F}_1^{-1}(u), \bar{F}_2^{-1}(v)\big) \\
&= -H_1(\bar{F}_1^{-1}(u)) - H_2(\bar{F}_2^{-1}(v)) - H_3(\bar{F}_1^{-1}(u)) \\
&= -\tilde{H}_1(\bar{F}_1^{-1}(u)) - H_2(\bar{F}_2^{-1}(v)) \\
&= \ln u - H_2(\tilde{H}_2^{-1}(-\ln v)) \\
&= \ln u + \ln v + H_3(\tilde{H}_2^{-1}(-\ln v)).
\end{aligned}$$

Likewise, for (u, v) such that $\bar{F}_1^{-1}(u) \leq \bar{F}_2^{-1}(v)$, we have

$$\ln \hat{C}_{\boldsymbol{X}}(u, v) = \ln u + \ln v + H_3(\tilde{H}_1^{-1}(-\ln u)).$$

Thus,

$$\hat{C}_{\boldsymbol{X}}(u, v) = \begin{cases} uv \exp\Big\{H_3\big(\tilde{H}_1^{-1}(-\ln u)\big)\Big\}, & \tilde{H}_1^{-1}(-\ln u) \leq \tilde{H}_2^{-1}(-\ln v), \\ uv \exp\Big\{H_3\big(\tilde{H}_2^{-1}(-\ln v)\big)\Big\}, & \tilde{H}_1^{-1}(-\ln u) > \tilde{H}_2^{-1}(-\ln v). \end{cases} \qquad (2.3)$$

To avoid ambiguity, throughout this paper any survival copula taking the form of (2.3) is called *Generalized Marshall-Olkin (GMO) survival copula*, and the functions H_1, H_2, H_3 are called as its *generating functions*.

The following corollary is an immediate consequence of (2.3)

Proposition 2.1 Every vector $\boldsymbol{X}$ having GMO distribution is always PQD.

Note that by setting $H_i(x) = \lambda_i x$ for $\lambda_i \geq 0$ and $x \geq 0$, $i = 1, 2, 3$, the GMO copula in (2.3) reduces to

$$\hat{C}_{\boldsymbol{X}}(u, v) = uv \min\left\{u^{\frac{-\lambda_3}{\lambda_1+\lambda_3}},\ v^{\frac{-\lambda_3}{\lambda_2+\lambda_3}}\right\}, \quad \text{for } 0 \leq u,\ v \leq 1, \tag{2.4}$$

which is just the survival copula for (1.1) and is known as the bivariate Marshall-Olkin survival copula. Equipped with various nonexponential margins, this copula has been utilized in a variety of applications. One may see, for example, Hutchinson and Lai (1990) for details and references.

3 Stochastic comparisons

In this section, we build some stochastic comparison results for GMO distributions, which are also useful in studying aging properties in the sequel.

Consider two sets of independent random variables $\{S_i, i = 1, 2, 3\}$ and $\{T_i, i = 1, 2, 3\}$, and let

$$\boldsymbol{X} = (\min\{S_1, S_3\}, \min\{S_2, S_3\}) \quad \text{and} \quad \boldsymbol{Y} = (\min\{T_1, T_3\}, \min\{T_2, T_3\}) \tag{3.1}$$

be the two corresponding random vectors with GMO distributions. The following result provides conditions to compare $\boldsymbol{X}$ and $\boldsymbol{Y}$ in the usual stochastic and increasing concave orders.

Theorem 3.1 Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be defined as in (3.1). If $S_i \leq_{st} \ (\leq_{icv})\, T_i$ for $i = 1, 2, 3$, then, $\boldsymbol{X} \leq_{st} \ (\leq_{icv})\, \boldsymbol{Y}$.

Proof Since $\{S_i,\ i = 1, 2, 3\}$ and $\{T_i,\ i = 1, 2, 3\}$ both are formed by independent variables, $S_i \leq_{st} T_i$ $(i = 1, 2, 3)$ imply $(S_1, S_2, S_3) \leq_{st} (T_1, T_2, T_3)$. Note that the function $\min\{x, y\}$ is increasing in both x and y, thus the comparison $\boldsymbol{X} \leq_{st} \boldsymbol{Y}$ follows immediately from Theorem 6.B.16(a) of Shaked and Shanthikumar (2007).

For the case of the increasing concave order, let us consider

$$g(\boldsymbol{s}) = g((s_1, s_2, s_3)') = (\min\{s_1, s_3\}, \min\{s_2, s_3\})'.$$

For any $0 < \alpha < 1$, it holds that

$$
\begin{aligned}
g(\alpha\boldsymbol{s} + (1-\alpha)\boldsymbol{t}) &= g\begin{pmatrix} \alpha s_1 + (1-\alpha)t_1 \\ \alpha s_2 + (1-\alpha)t_2 \\ \alpha s_3 + (1-\alpha)t_3 \end{pmatrix} \\
&= \begin{pmatrix} \min\{\alpha s_1 + (1-\alpha)t_1,\, \alpha s_3 + (1-\alpha)t_3\} \\ \min\{\alpha s_2 + (1-\alpha)t_2,\, \alpha s_3 + (1-\alpha)t_3\} \end{pmatrix}
\end{aligned}
$$

and

$$
\alpha g(\boldsymbol{s}) + (1-\alpha)g(\boldsymbol{t}) = \begin{pmatrix} \min\{\alpha s_1,\, \alpha s_3\} + \min\{(1-\alpha)t_1,\, (1-\alpha)t_3\} \\ \min\{\alpha s_2,\, \alpha s_3\} + \min\{(1-\alpha)t_2,\, (1-\alpha)t_3\} \end{pmatrix}.
$$

Since, for $i = 1, 2$,

$$
\min\{\alpha s_i,\, \alpha s_3\} + \min\{(1-\alpha)t_i,\, (1-\alpha)t_3\} \le \min\{\alpha s_i + (1-\alpha)t_i,\, \alpha s_3 + (1-\alpha)t_3\},
$$

it follows that $g(\alpha\boldsymbol{s} + (1-\alpha)\boldsymbol{t}) \ge \alpha g(\boldsymbol{s}) + (1-\alpha)g(\boldsymbol{t})$. That is, $g(\boldsymbol{s})$ is increasing and concave.

Due to the independence, $S_i \le_{icv} T_i$ for $i = 1, 2, 3$ imply $(S_1, S_2, S_3) \le_{icv} (T_1, T_2, T_3)$. By Theorem 7.A.5(a) of Shaked and Shanthikumar (2007), we get

$$
\boldsymbol{X} = g((S_1, S_2, S_3)') \le_{icv} g((T_1, T_2, T_3)') = \boldsymbol{Y}.
$$

This completes the proof. ■

The following statement, which is the main result of this section, deals on comparisons of GMO survival copulas.

Theorem 3.2 Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be defined as in (3.1). If $S_1 \le_{st} T_1$, $S_2 \le_{st} T_2$ and $S_3 \ge_{st} T_3$, then,

$$
\hat{C}_{\boldsymbol{X}}(u, v) \le \hat{C}_{\boldsymbol{Y}}(u, v), \qquad \text{for all } 0 \le u,\ v \le 1. \tag{3.2}
$$

Proof Denote $L_i(x)$ the cumulative hazard function of T_i for $i = 1, 2, 3$ and let $\tilde{L}_i = L_i + L_3$ for $i = 1, 2$. Then, $\boldsymbol{Y}$ has its survival copula

$$
\hat{C}_{\boldsymbol{Y}}(u, v) = \begin{cases} uv \exp\left\{L_3\big(\tilde{L}_1^{-1}(-\ln u)\big)\right\}, & \tilde{L}_1^{-1}(-\ln u) \le \tilde{L}_2^{-1}(-\ln v), \\ uv \exp\left\{L_3\big(\tilde{L}_2^{-1}(-\ln v)\big)\right\}, & \tilde{L}_1^{-1}(-\ln u) > \tilde{L}_2^{-1}(-\ln v). \end{cases}
$$

Let $\boldsymbol{Z} = (\min\{T_1, S_3\}, \min\{T_2, S_3\})$ and $\tilde{K}_i = L_i + H_3$ for $i = 1, 2$ so that $\boldsymbol{Z}$ has its survival copula

$$\hat{C}_{\boldsymbol{Z}}(u,v) = \begin{cases} uv \exp\left\{H_3\big(\tilde{K}_1^{-1}(-\ln u)\big)\right\}, & \tilde{K}_1^{-1}(-\ln u) \le \tilde{K}_2^{-1}(-\ln v), \\ uv \exp\left\{H_3\big(\tilde{K}_2^{-1}(-\ln v)\big)\right\}, & \tilde{K}_1^{-1}(-\ln u) > \tilde{K}_2^{-1}(-\ln v). \end{cases}$$

First we show

$$\hat{C}_{\boldsymbol{X}}(u,v) \le \hat{C}_{\boldsymbol{Z}}(u,v) \quad \text{for all } 0 \le u,\, v \le 1. \tag{3.3}$$

For $i = 1, 2$, since $S_i \le_{st} T_i$, i.e., $H_i(x) \ge L_i(x)$ for all $x \ge 0$, it holds that $H_i(x) + H_3(x) \ge L_i(x) + H_3(x)$ and hence

$$\tilde{H}_i^{-1}(x) \le \tilde{K}_i^{-1}(x) \quad \text{for all } x \ge 0. \tag{3.4}$$

Let us consider the four possible cases, one by one:

i) $\left\{(u,v) : \tilde{H}_1^{-1}(-\ln u) \le \tilde{H}_2^{-1}(-\ln v) \text{ and } \tilde{K}_1^{-1}(-\ln u) \le \tilde{K}_2^{-1}(-\ln v)\right\}$.
By (3.4), we have

$$\hat{C}_{\boldsymbol{X}}(u,v) = \exp\left\{H_3\big(\tilde{H}_1^{-1}(-\ln u)\big)\right\} \le \exp\left\{H_3\big(\tilde{K}_1^{-1}(-\ln u)\big)\right\} = \hat{C}_{\boldsymbol{Z}}(u,v).$$

ii) $\left\{(u,v) : \tilde{H}_1^{-1}(-\ln u) > \tilde{H}_2^{-1}(-\ln v) \text{ and } \tilde{K}_1^{-1}(-\ln u) > \tilde{K}_2^{-1}(-\ln v)\right\}$.
By (3.4) again, we have

$$\hat{C}_{\boldsymbol{X}}(u,v) = \exp\left\{H_3\big(\tilde{H}_2^{-1}(-\ln v)\big)\right\} \le \exp\left\{H_3\big(\tilde{K}_2^{-1}(-\ln v)\big)\right\} = \hat{C}_{\boldsymbol{Z}}(u,v).$$

iii) $\left\{(u,v) : \tilde{H}_1^{-1}(-\ln u) \le \tilde{H}_2^{-1}(-\ln v) \text{ and } \tilde{K}_1^{-1}(-\ln u) > \tilde{K}_2^{-1}(-\ln v)\right\}$.
It always holds that

$$\hat{C}_{\boldsymbol{X}}(u,v) = \exp\left\{H_3\big(\tilde{H}_1^{-1}(-\ln u)\big)\right\} \le \exp\left\{H_3\big(\tilde{H}_2^{-1}(-\ln v)\big)\right\};$$

By (3.4) again, we also have

$$\exp\left\{H_3\big(\tilde{H}_2^{-1}(-\ln v)\big)\right\} \le \exp\left\{H_3\big(\tilde{K}_2^{-1}(-\ln v)\big)\right\} = \hat{C}_{\boldsymbol{Z}}(u,v).$$

Hence, $\hat{C}_{\boldsymbol{X}}(u,v) \le \hat{C}_{\boldsymbol{Z}}(u,v)$.

iv) $\left\{(u,v) : \tilde{H}_1^{-1}(-\ln u) > \tilde{H}_2^{-1}(-\ln v) \text{ and } \tilde{K}_1^{-1}(-\ln u) \le \tilde{K}_2^{-1}(-\ln v)\right\}$.
In a similar manner to iii), we have

$$\exp\left\{H_3\big(\tilde{H}_1^{-1}(-\ln u)\big)\right\} \le \exp\left\{H_3\big(\tilde{K}_1^{-1}(-\ln u)\big)\right\} \le \exp\{H_3\big(\tilde{K}_2^{-1}(-\ln v)\big)\right\}.$$

That is, $\hat{C}_{\boldsymbol{X}}(u,v) \le \hat{C}_{\boldsymbol{Z}}(u,v)$.

Thus, (3.3) is validated.

Secondly, let us prove

$$\hat{C}_{\boldsymbol{Z}}(u,v) \le \hat{C}_{\boldsymbol{Y}}(u,v) \quad \text{for all } 0 \le u,\, v \le 1. \tag{3.5}$$

Since $S_3 \ge T_3$, i.e., $H_3(x) \le L_3(x)$ for all $x \ge 0$, it holds that

$$L_i(x) + H_3(x) \le L_i(x) + L_3(x), \quad \text{for } i = 1, 2,$$

and thus also

$$\tilde{K}_i^{-1}(x) \ge \tilde{L}_i^{-1}(x) \quad \text{for all } x \ge 0 \text{ and } i = 1, 2. \tag{3.6}$$

For $x \ge 0$ and $i = 1, 2$, denote $t = \tilde{K}_i^{-1}(x)$, $s = \tilde{L}_i^{-1}(x)$ and $b = L_i(t)$, $a = L_i(s)$. Then, it holds that $H_3(t) = x - b$ and $L_3(s) = x - a$. The inequality (3.6) implies $s < t$ and hence $a = L_i(s) < L_i(t) = b$. Thus,

$$L_3\big(\tilde{L}_i^{-1}(x)\big) = L_3(s) = x - a > x - b = H_3(t) = H_3\big(\tilde{K}_i^{-1}(x)\big). \tag{3.7}$$

Likewise, we have four possible cases.

i) $\Big\{(u,v) : \tilde{K}_1^{-1}(-\ln u) \le \tilde{K}_2^{-1}(-\ln v) \text{ and } \tilde{L}_1^{-1}(-\ln u) \le \tilde{L}_2^{-1}(-\ln v)\Big\}$.

By (3.7), it always holds that

$$\hat{C}_{\boldsymbol{Z}}(u,v) = \exp\Big\{H_3\big(\tilde{K}_1^{-1}(-\ln u)\big)\Big\} \le \exp\Big\{L_3\big(\tilde{L}_1^{-1}(-\ln u)\big)\Big\} = \hat{C}_{\boldsymbol{Y}}(u,v).$$

ii) $\Big\{(u,v) : \tilde{K}_1^{-1}(-\ln u) > \tilde{K}_2^{-1}(-\ln v) \text{ and } \tilde{L}_1^{-1}(-\ln u) > \tilde{L}_2^{-1}(-\ln v)\Big\}$.

By (3.7) again, we have

$$\hat{C}_{\boldsymbol{Z}}(u,v) = \exp\Big\{H_3\big(\tilde{K}_2^{-1}(-\ln v)\big)\Big\} \le \exp\Big\{L_3\big(\tilde{L}_2^{-1}(-\ln v)\big)\Big\} = \hat{C}_{\boldsymbol{Y}}(u,v).$$

iii) $\Big\{(u,v) : \tilde{K}_1^{-1}(-\ln u) \le \tilde{K}_2^{-1}(-\ln v) \text{ and } \tilde{L}_1^{-1}(-\ln u) > \tilde{L}_2^{-1}(-\ln v)\Big\}$.

It holds that

$$\hat{C}_{\boldsymbol{Z}}(u,v) = \exp\Big\{H_3\big(\tilde{K}_1^{-1}(-\ln u)\big)\Big\} \le \exp\Big\{H_3\big((\tilde{K}_2^{-1} - \ln v)\big)\Big\};$$

By (3.7), we also have

$$\exp\Big\{H_3\big(\tilde{K}_2^{-1}(-\ln v)\big)\Big\} \le \exp\Big\{L_3\big(\tilde{L}_2^{-1}(-\ln v)\big)\Big\} = \hat{C}_{\boldsymbol{Y}}(u,v).$$

Thus, $\hat{C}_{\boldsymbol{Z}}(u,v) \le \hat{C}_{\boldsymbol{Y}}(u,v)$.

iv) $\left\{(u,v): \tilde{K}_1^{-1}(-\ln u) > \tilde{K}_2^{-1}(-\ln v) \text{ and } \tilde{L}_1^{-1}(-\ln u) \le \tilde{L}_2^{-1}(-\ln v)\right\}$.

Similarly,

$$\exp\left\{H_3\big(\tilde{K}_2^{-1}(-\ln v)\big)\right\} \le \exp\left\{H_3\big(\tilde{K}_1^{-1}(-\ln u)\big)\right\} \le \exp\left\{L_3\big(\tilde{L}_1^{-1}(-\ln u)\big)\right\}.$$

Once again, we have $\hat{C}_{\boldsymbol{Z}}(u,v) \le \hat{C}_{\boldsymbol{Y}}(u,v)$.

Hence, (3.5) is invoked.

Now, the desired assertion in (3.2) follows immediately from (3.3) and (3.5). ■

To close this section, we present an example to illustrate the above theorem.

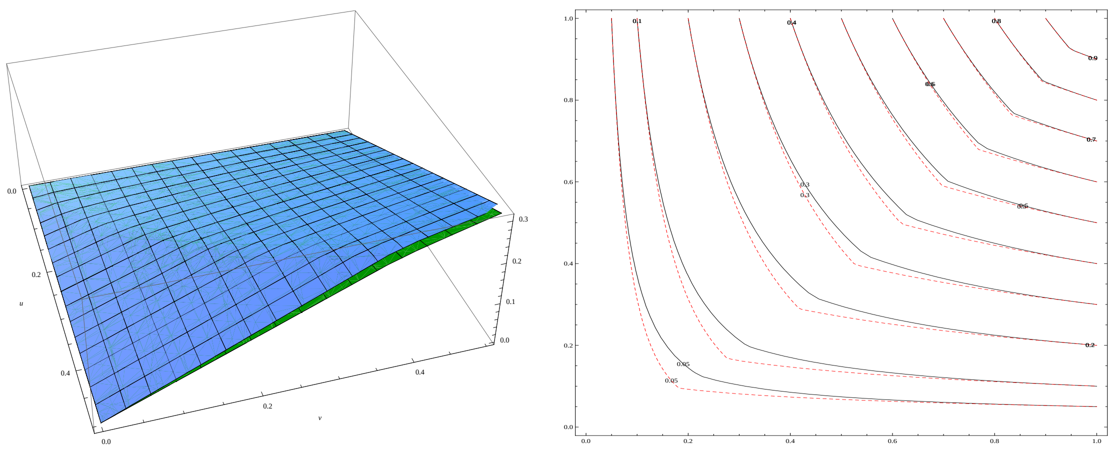
Example 3.3 Consider bivariate vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ having GMO distributions with the generating cumulative hazard functions

$$H_1(x) = x + x^2, \quad H_2(x) = 2x + x^2, \quad H_3(x) = x$$

and

$$L_1(x) = x, \quad L_2(x) = 2x, \quad L_3(x) = x + x^2.$$

It may be evaluated that



(a) Copulas: $\hat{C}_{\boldsymbol{X}}$ (lower) and $\hat{C}_{\boldsymbol{Y}}$ (upper) (b) Level curves: $\hat{C}_{\boldsymbol{X}}$ (solid) and $\hat{C}_{\boldsymbol{Y}}$ (dotted)

Figure 1: Uniform inequality $\hat{C}_{\boldsymbol{X}}(u,v) \le \hat{C}_{\boldsymbol{Y}}(u,v)$

$$\hat{C}_{\boldsymbol{X}}(u,v) = \begin{cases} uv\exp\left\{\sqrt{1-\ln u} - 1\right\}, & \sqrt{1-\ln u} \le \sqrt{\frac{9}{4} - \ln v} - \frac{1}{2}, \\ uv\exp\left\{\sqrt{\frac{9}{4} - \ln v} - \frac{1}{2}\right\}, & \sqrt{1-\ln u} > \sqrt{\frac{9}{4} - \ln v} - \frac{1}{2}, \end{cases}$$

$$\hat{C}_{\boldsymbol{Y}}(u,v) = \begin{cases} uv\exp\left\{1-\ln u-\sqrt{1-\ln u}\right\}, & \sqrt{1-\ln u} \le \sqrt{\frac{9}{4}-\ln v}-\frac{1}{2}, \\ uv\exp\left\{2-\ln v-\sqrt{\frac{9}{4}-\ln v}\right\}, & \sqrt{1-\ln u} > \sqrt{\frac{9}{4}-\ln v}-\frac{1}{2}. \end{cases}$$

Evidently, $S_1 \le_{st} T_1$, $S_2 \le_{st} T_2$ and $S_3 \ge_{st} T_3$. According to Theorem 3.2, it holds that $\hat{C}_{\boldsymbol{X}}(u,v) \le \hat{C}_{\boldsymbol{Y}}(u,v)$ for all $0 \le u,\, v \le 1$. The copulas and the corresponding level curves are displayed in Figure 1. As it can be seen, $\hat{C}_{\boldsymbol{Y}}(u,v)$ is always above $\hat{C}_{\boldsymbol{X}}(u,v)$. ■

4 Aging and dependence properties

In this section we investigate the relationships between the aging properties of the generating distributions and the dependence in the components of the vector when its distribution is of GMO type.

The first result tells that the vector of the residual lifes also has a GMO copula if the vector $\boldsymbol{X}$ does.

Theorem 4.1 If $\boldsymbol{X}$ has a GMO distribution, then, for any $t \ge 0$, $\boldsymbol{X}_t$ also has a GMO distribution. In particular, if $\boldsymbol{X}$ has a Marshall-Olkin distribution, then so does $\boldsymbol{X}_t$ for any $t \ge 0$.

Proof Since S_1, S_2 and S_3 are independent, for any $t \ge 0$ it holds

$$\begin{aligned} P[\boldsymbol{X}_t > (x_1,x_2)] &= P\big(X_1 - t > x_1, X_2 - t > x_2 |\ X_1 > t, X_2 > t\big) \\ &= P\big((\min\{S_1 - t, S_3 - t\},\, \min\{S_2 - t, S_3 - t\}) > (x_1,x_2) |\ S_i > t, i = 1,2,3\big) \\ &= \frac{P\big(S_1 > t + x_1,\, S_2 > t + x_2,\, S_3 > t + \max\{x_1,x_2\}\big)}{P(S_1 > t,\, S_2 > t,\, S_3 > t)} \\ &= \frac{\overline{G}_1(t+x_1)}{\overline{G}_1(t)}\frac{\overline{G}_2(t+x_2)}{\overline{G}_2(t)}\frac{\overline{G}_3(t+\max\{x_1,x_2\})}{\overline{G}_3(t)} \end{aligned}$$

Letting now the variables $(S_i)_t$, for $t \ge 0$ and $i = 1,2,3$, be independent and with survival functions $\overline{G}_{i,t}(x) = \frac{\overline{G}_i(t+x)}{\overline{G}_i(t)}$, i.e., letting $(S_i)_t \overset{st}{=} [S_i - t | S_i > t]$, one immediately gets that

$$\begin{aligned} P[\boldsymbol{X}_t > (x_1,x_2)] &= P((S_1)_t > x_1)\, P((S_2)_t > x_2)\, P((S_3)_t > \max\{x_1,x_2\}) \\ &= P\big(\min\{(S_1)_t,\, (S_3)_t\} > x_1,\, \min\{(S_2)_t,\, (S_3)_t\} > x_2\big). \end{aligned}$$

Thus, for any $t \ge 0$ it holds that

$$\boldsymbol{X}_t \overset{st}{=} (\min\{(S_1)_t,\, (S_3)_t\},\, \min\{(S_2)_t,\, (S_3)_t\}), \tag{4.1}$$

Recalling that $\boldsymbol{X} = (\min\{S_1, S_3\}, \min\{S_2, S_3\})$, from Theorem 6.B.16(b) of Shaked and Shanthikumar (2007) it follows immediately that $\boldsymbol{X}$ and $\boldsymbol{X}_t$ have the same type of copula, even if they have different generating functions.

The other part is trivial. ■

It should be remarked here that the stochastic equality in (4.1) is of independent interest. In fact, Li and Lu (2003) built the following univariate version

$$[\min\{S_1, S_2\} - t | \min\{S_1, S_2\} > t] \stackrel{st}{=} \min\{(S_1)_t, (S_2)_t\}, \quad \text{for all } t \geq 0,$$

and derived a preservation property under the taking of series systems for some aging properties.

Also, one may easily draw the following conclusion.

Corollary 4.2 Let $\boldsymbol{X}$ be defined as in (2.1).

(i) If S_i is NBU (NWU) for $i = 1, 2, 3$, then, $\boldsymbol{X}$ is B-NBU (B-NWU);

(ii) If S_i is NBU(2) (NWU(2)) for $i = 1, 2, 3$, then, $\boldsymbol{X}$ is B-NBU(2) (B-NWU(2));

(iii) If S_i is IFR (DFR) for $i = 1, 2, 3$, then, $\boldsymbol{X}$ is B-IFR (B-DFR).

Proof (i) NBU (NWU) property guarantees

$$S_i \geq_{st} (\leq_{st}) (S_i)_t, \quad \text{for any } t \geq 0 \text{ and } i = 1, 2, 3.$$

Recalling that the function $\min\{x, y\}$ is increasing in x and y, respectively, it follows immediately that

$$\begin{aligned} \boldsymbol{X} &= (\min\{S_1, S_3\}, \min\{S_2, S_3\}) \\ &\geq_{st} (\leq_{st}) \big(\min\{(S_1)_t, (S_3)_t\}, \min\{(S_2)_t, (S_3)_t\}\big). \end{aligned}$$

Taking (4.1) into account, we have $\boldsymbol{X} \geq_{st} (\leq_{st}) \boldsymbol{X}_t$ for any $t \geq 0$.

(ii) and (iii) may be proved in completely a similar manner. ■

Corollary 4.3 below is a direct consequence of Property 2.1 and Theorem 4.1.

Corollary 4.3 For any $t \geq 0$, the residual life $\boldsymbol{X}_t$ corresponding to a vector $\boldsymbol{X}$ having GMO distribution is always PQD.

In order to get more insight, let us take a look at the survival copula of the residual life.

Denote, for any $t \geq 0$ and $x \geq 0$,

$$W_{i,t}(x) = H_i(t+x) - H_i(t), \qquad i = 1, 2, 3.$$

Then, the residual life $\boldsymbol{X}_t$ has the survival function

$$\begin{aligned}\bar{F}_{\boldsymbol{X}_t}(x_1, x_2) &= P(X_1 - t > x_1, X_2 - t > x_2 | X_1 > t, X_2 > t) \\ &= \frac{\bar{G}_1(t+x_1)\bar{G}_2(t+x_2)\bar{G}_3(t+\max\{x_1, x_2\})}{\bar{G}_1(t)\bar{G}_2(t)\bar{G}_3(t)} \\ &= \bar{G}_{1,t}(x)\bar{G}_{2,t}(x_2)\bar{G}_{3,t}(\max\{x_1, x_2\}) \\ &= \exp\{-W_{1,t}(x) - W_{2,t}(x_2) - W_{3,t}(\max\{x_1, x_2\})\},\end{aligned}$$

and the marginal survival functions

$$\bar{F}_{1,t}(x_1) = \bar{G}_{1,t}(x_2)\bar{G}_{3,t}(x_2) = \exp\{-W_{1,t}(x_2) - W_{3,t}(x_2)\},$$

$$\bar{F}_{2,t}(x_2) = \bar{G}_{2,t}(x_2)\bar{G}_{3,t}(x_2) = \exp\{-W_{2,t}(x_2) - W_{3,t}(x_2)\}.$$

In the same manner to that in Section 2, the survival copula of the residual life $\boldsymbol{X}_t$ may be derived as follows:

$$\hat{C}_{\boldsymbol{X}_t}(u, v) = \begin{cases} uv \exp\left\{W_{3,t}\big(\tilde{W}_{1,t}^{-1}(-\ln u)\big)\right\}, & \tilde{W}_{1,t}^{-1}(-\ln u) \leq \tilde{W}_{2,t}^{-1}(-\ln v), \\ uv \exp\left\{W_{3,t}\big(\tilde{W}_{2,t}^{-1}(-\ln v)\big)\right\}, & \tilde{W}_{1,t}^{-1}(-\ln u) > \tilde{W}_{2,t}^{-1}(-\ln v), \end{cases} \tag{4.2}$$

where

$$\tilde{W}_{i,t}(x) = W_{i,t}(x) + W_{3,t}(x), \qquad i = 1, 2.$$

Since $\boldsymbol{X}$ is PQD if and only if $\bar{F}(x_1, x_2) \geq \bar{F}_1(x)\bar{F}_2(x_2)$ for all x_1, $x_2 \geq 0$, naturally,

$$D_{\boldsymbol{X}}(x_1, x_2) = \bar{F}(x_1, x_2)/\bar{F}_1(x_1)\bar{F}_2(x_2)$$

may be viewed as a measure for the degree of PQD, which permits heterogeneous margins and hence is in general more informative than the PQD order. Next proposition tells that the convexity (concavity) of H_3 dominates the evolution of the degree of PQD of the residual life.

Theorem 4.4 Let $\boldsymbol{X}$ be defined as in (2.1). Suppose H_3 is convex (concave). Then, the degree of PQD of $\boldsymbol{X}_t$ is increasing (decreasing) with respect to $t \geq 0$.

Proof For any $x_1, x_2 \geq 0$, it holds

$$
\begin{aligned}
D_{\boldsymbol{X}_t}(x_1, x_2) &= \frac{\bar{F}_t(x_1, x_2)}{\bar{F}_{1,t}(x_1)\bar{F}_{2,t}(x_2)} \\
&= \exp\big\{ - [W_{3,t}(\max\{x_1, x_2\}) - W_{3,t}(x_1) - W_{3,t}(x_2)] \big\} \\
&= \begin{cases} \exp\{H_3(x_2 + t) - H_3(t)\}, & \text{if } x_1 > x_2, \\ \exp\{H_3(x_1 + t) - H_3(t)\}, & \text{if } x_1 < x_2. \end{cases}
\end{aligned}
$$

Because the convexity (concavity) of H_3 implies that $H_3(x + t) - H_3(t)$ is increasing (decreasing) in $t \geq 0$, the desired result follows immediately. ■

By taking a comparison between (2.3) and (4.2), we reach the second main result, which asserts that the survival copula of the residual life of the Marshall-Olkin type distribution (Muliere and Scarsini, 1987) is invariant with respect to the age.

Theorem 4.5 A random vector $\boldsymbol{X}$ with GMO distribution and its residual life $\boldsymbol{X}_t$ have the same GMO copula if, and only if, H_1, H_2 and H_3 are proportional.

Proof By Theorem 1, page 34, in Aczel (1966), the functional equation $f(x + y) = f(x) + f(y)$ is satisfied for all real x, y by a function $f : \Re \to \Re^+$ continuous at a point if and only if $f = \alpha x$ for a non-negative α. Thus, for two continuous and increasing functions g and h the composition $g \circ h^{-1}$ satisfies additivity if and only if $g(h^{-1}(x)) = \alpha x$, with $\alpha \geq 0$. Letting $x = h(u)$, this is equivalent to $g(u) = \alpha h(u)$. As an immediate consequence, the composition $H_3 \circ (H_i + H_3)^{-1}$, for $i = 1, 2$, satisfies additivity if and only if $H_3(u) = \alpha(H_i(u) + H_3(u))$ for all u and an $\alpha \geq 0$, which in turns is verified if and only if H_i and H_3 are proportional, i.e., $c_1 H_1 = c_2 H_2 = H_3$ for some $c_1, c_2 \geq 0$. Thus, $H_3(H_2 + H_3)^{-1}$ and $H_3(H_1 + H_3)^{-1}$ are additive if and only if S_1, S_2 and S_3 belong to a proportional hazard family.

In view of

$$
\tilde{W}_{1,t}(x) = \tilde{H}_1(t + x) - \tilde{H}_1(t)
$$

and by the additivity of $H_3 \circ \tilde{H}_1^{-1}$, we have, for any $u \in [0, 1]$ and $t \geq 0$,

$$
\begin{aligned}
W_{3,t}\big(\tilde{W}_{1,t}^{-1}(-\ln u)\big) &= W_{3,t}\big(\tilde{H}_1^{-1}(\tilde{H}_1(t) - \ln u) - t\big) \\
&= H_3\big(\tilde{H}_1^{-1}(\tilde{H}_1(t) - \ln u)\big) - H_3(t) \\
&= H_3\big(\tilde{H}_1^{-1}(-\ln u)\big).
\end{aligned}
$$

Similarly, due to the linearity of $H_3 \circ \tilde{H}_2^{-1}$, we also have, for any $v \in [0, 1]$ and $t \geq 0$,

$$
W_{3,t}\big(\tilde{W}_{2,t}^{-1}(-\ln v)\big) = H_3\big(\tilde{H}_2^{-1}(-\ln v)\big).
$$

Thus, $\hat{C}_{\boldsymbol{X}}(u,v) = \hat{C}_{\boldsymbol{X}_t}(u,v)$ for any $t \geq 0$ and $0 \leq u, v \leq 1$. ■

As an immediate consequence of Theorem 4.5, we get the characterization of the weak lack-of-memory property of GMO distributions (see, e.g., Ghurye and Marshall (1984)).

Corollary 4.6 For a random vector $\boldsymbol{X}$ with GMO distribution, $\boldsymbol{X} \stackrel{st}{=} \boldsymbol{X}_t$ for any $t \geq 0$ if and only if $H_i(x)$ is proportional to x for $i = 1, 2, 3$.

Next corollary asserts that the condition on S_i in Corollary 4.2 may be relaxed to a similar condition on the margins when H_1, H_2 and H_3 are proportional.

Corollary 4.7 Let $\boldsymbol{X}$ be defined as in (2.1). Suppose that the generating functions H_1, H_2 and H_3 are proportional.

(i) If X_i is NBU (NWU), $i = 1, 2$, then, $\boldsymbol{X}$ is B-NBU (B-NWU);

(ii) If X_i is IFR (DFR), $i = 1, 2$, then, $\boldsymbol{X}$ is B-IFR (B-DFR).

Proof By Theorem 4.5, $\hat{C}_{\boldsymbol{X}}(u,v) = \hat{C}_{\boldsymbol{X}_t}(u,v)$ for any $t \geq 0$. Due to the similarity, we only prove the assertion (ii).

(ii) Let (U_1, U_2) be the vector having distribution $\hat{C}_{\boldsymbol{X}}(u,v)$. Then, for any $t \geq 0$,

$$\boldsymbol{X} \stackrel{st}{=} \big(\bar{F}_1^{-1}(U_1), \bar{F}_2^{-1}(U_2)\big), \qquad \boldsymbol{X}_t \stackrel{st}{=} \big(\bar{F}_{1,t}^{-1}(U_1), \bar{F}_{2,t}^{-1}(U_2)\big),$$

where $\bar{F}_{i,t}(x) = \frac{\bar{F}_i(x+t)}{\bar{F}_i(t)}$, $i = 1, 2$. By the IFR (DFR) property, we have

$$\bar{F}_{i,t}(x) \leq (\geq)\ \bar{F}_{i,s}(x), \qquad \text{for all } x \geq 0 \text{ and } \ i = 1, 2,$$

and hence

$$\boldsymbol{X}_s \stackrel{st}{=} \big(\bar{F}_{1,s}^{-1}(U_1), \bar{F}_{2,s}^{-1}(U_2)\big) \geq_{a.s.} (\leq_{a.s.}) \big(\bar{F}_{1,t}^{-1}(U_1), \bar{F}_{2,t}^{-1}(U_2)\big) \stackrel{st}{=} \boldsymbol{X}_t,$$

for $t \geq s \geq 0$ and $x \geq 0$. ■

Now, let us analyze the behavior of dependence due to aging. We will address conditions to compare in dependence the entire bivariate life and the bivariate residual life.

Suppose S_1, S_2 are NBU and S_3 is NWU. Then, for any $t \geq 0$, $S_3 \leq_{st} (S_3)_t$ and $S_i \geq_{st} (S_i)_t$, $i = 1, 2$. In view of (4.1), it stems from Theorem 3.2 that $\hat{C}_{\boldsymbol{X}}(u,v) \geq \hat{C}_{\boldsymbol{X}_t}(u,v)$ for any $0 \leq u, v \leq 1$ and $t \geq 0$. That is, the bivariate residual life becomes less dependent as the age of time elapsed. Actually, we have a more general conclusion as below. Recall that a real valued function h is said to be superadditive if $h(x+y) \geq h(x) + h(y)$ for all $x, y \geq 0$, while it is said to be subadditive if the inequality is reversed. Also, observe that an increasing function h is superadditive if, and only if, h^{-1} is subadditive.

Theorem 4.8 Let $\boldsymbol{X}$ be defined as in (2.1), and suppose $H_i \circ H_3^{-1}$ is superadditive (subadditive) for $i = 1, 2$. Then,

$$\hat{C}_{\boldsymbol{X}}(u, v) \geq \ (\leq)\, \hat{C}_{\boldsymbol{X}_t}(u, v), \qquad \text{for any } 0 \leq u, v \leq 1 \text{ and } t \geq 0.$$

Proof Let $i = 1, 2$. Since

$$(H_i + H_3) \circ H_3^{-1}(x) = H_i \circ H_3^{-1}(x) + x,$$

from superadditivity of $H_i \circ H_3^{-1}$ follows the superadditivity of $(H_i + H_3) \circ H_3^{-1}$. Observing now that $(H_i + H_3) \circ H_3^{-1}$ is increasing, being the composition of two increasing functions, it follows that its inverse $H_3 \circ \tilde{H}_i^{-1}$ is subadditive. Then we have, for $u \in [0, 1]$, $t \geq 0$ and $i = 1, 2$,

$$\begin{aligned} W_{3,t}\big(\tilde{W}_{i,t}^{-1}(-\ln u)\big) &= W_{3,t}\big(\tilde{H}_i^{-1}(\tilde{H}_i(t) - \ln u) - t\big) \\ &= H_3\big(\tilde{H}_i^{-1}(\tilde{H}_i(t) - \ln u)\big) - H_3(t) \\ &\leq H_3\big(\tilde{H}_i^{-1}(-\ln u)\big). \end{aligned}$$

So, from (2.3) and (4.2), it follows that $\hat{C}_{\boldsymbol{X}}(u, v) \geq \hat{C}_{\boldsymbol{X}_t}(u, v)$ for $t \geq 0$ and $0 \leq u, v \leq 1$.

The assertion for subadditivity may be proved by reversing all inequalities above. ■

Replacing the superadditivity (subadditivity) assumption for the composition $H_i \circ H_3^{-1}$ with the stronger property of convexity (concavity), then the monotonicity in dependence of the residual life can be asserted, as described in the following result.

Theorem 4.9 Let $\boldsymbol{X}$ be defined as in (2.1). Suppose $H_i \circ H_3^{-1}$ is convex (concave) for $i = 1, 2$. Then,

$$\hat{C}_{\boldsymbol{X}_s}(u, v) \geq \ (\leq)\, \hat{C}_{\boldsymbol{X}_t}(u, v), \quad \text{for any } 0 \leq u, v \leq 1 \text{ and } t \geq s \geq 0.$$

Proof Let $i = 1, 2$. The convexity (concavity) of $H_i \circ H_3^{-1}$ clearly implies convexity (concavity) of $(H_i + H_3) \circ H_3^{-1} = \tilde{H}_1 \circ H_3^{-1}$. Since $\tilde{H}_1^{-1} \circ H_3^{-1}$ is increasing, then its inverse $H_3 \circ \tilde{H}_1^{-1}$ is increasing and concave, i.e, $H_3\big(\tilde{H}_1^{-1}(x + y)\big) - H_3(\big(\tilde{H}_1^{-1}(x)\big))$ is decreasing in x, for all $y \geq 0$. Letting $x = \tilde{H}_1(t)$ and $y = \ln(u)$, it follows that

$$H_3\big(\tilde{H}_1^{-1}(\tilde{H}_1(t) - \ln u)\big) - H_3(t), \quad \text{and} \quad H_3\big(\tilde{H}_2^{-1}(\tilde{H}_2(t) - \ln v)\big) - H_3(t)$$

are increasing (decreasing) with respect to $t \geq 0$ for any $u, v \in [0, 1]$. The desired results follow immediately. ■

As a direct application of Theorem 4.8 and Theorem 4.9, we may build a condition for the upper-orthant comparison of the bivariate residual lifes of two GMO distributions, which are supplements to what stated in Corollary 4.2.

Corollary 4.10 Let $\boldsymbol{X}$ be defined as in (2.1). Suppose S_3 is NWU (NBU) and S_i is NBU (NWU) for $i = 1, 2$. Then,

$$\hat{C}_{\boldsymbol{X}}(u, v) \geq (\leq)\ \hat{C}_{\boldsymbol{X}_t}(u, v), \quad \text{for any } u, v \in [0, 1] \text{ and } t \geq 0.$$

Further, if $\min\{S_i, S_3\}$ is NBU (NWU) for $i = 1, 2$, then,

$$\boldsymbol{X} \geq_{uo}\ (\leq_{uo})\, \boldsymbol{X}_t \quad \text{for any } t \geq 0.$$

Proof Observe that the NWU property of S_3 is equivalent to subadditivity of H_3, and similarly the NBU property of S_i implies superadditivity of H_i, $i = 1, 2$. As a result, $H_i \circ H_3^{-1}$ is superadditive, $i = 1, 2$. By Theorem 4.8, we have

$$\hat{C}_{\boldsymbol{X}}(u, v) \geq (\leq)\ \hat{C}_{\boldsymbol{X}_t}(u, v), \quad \text{for any } 0 \leq u, v \leq 1 \text{ and } t \geq 0.$$

For $i = 1, 2$, since $\min\{S_i, S_3\}$ is NBU,

$$\bar{F}_i(x_i) \geq \bar{F}_{i,t}(x_i), \quad \text{for any } x_i \geq 0.$$

Thus, for any $t \geq 0$ and x_1, $x_2 \geq 0$,

$$\hat{C}_{\boldsymbol{X}}\big(\bar{F}_1(x_1), \bar{F}_2(x_2)\big) \geq \hat{C}_{\boldsymbol{X}}\big(\bar{F}_{1,t}(x_1), \bar{F}_{2,t}(x_2)\big) \geq \hat{C}_{\boldsymbol{X}_t}\big(\bar{F}_{1,t}(x_1), \bar{F}_{2,t}(x_2)\big).$$

Thus we get

$$\bar{F}_{\boldsymbol{X}}(x_1, x_2) \geq \bar{F}_{\boldsymbol{X}_t}(x_1, x_2) \quad \text{for any } t \geq 0 \text{ and } x_i \geq 0,\ i = 1, 2.$$

That is, $\boldsymbol{X} \geq_{uo} \boldsymbol{X}_t$ for any $t \geq 0$.

The other case may be proved in a similar manner. ■

The last corollary confirms Theorem 4.4, and the proof is omitted due to similarity.

Corollary 4.11 Let $\boldsymbol{X}$ be defined as in (2.1). Suppose that S_3 is DFR (IFR) and that S_i is IFR (DFR), for $i = 1, 2$. Then,

$$\hat{C}_{\boldsymbol{X}_s}(u, v) \geq (\leq)\ \hat{C}_{\boldsymbol{X}_t}(u, v), \quad \text{for any } 0 \leq u, v \leq 1 \text{ and } t \geq s \geq 0.$$

Further, if $\min\{S_i, S_3\}$ is IFR (DFR) for $i = 1, 2$, then,

$$\boldsymbol{X}_s \geq_{uo}\ (\leq_{uo})\, \boldsymbol{X}_t \quad \text{for any } t \geq s \geq 0.$$

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