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# **The Wiener-Hopf method applied to multiple angular region problems: the penetrable wedge case**

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## **Introduction**

According to the opinion of these authors, the Wiener-Hopf (W-H) technique is the most powerful method to solve field problems in presence of geometrical discontinuities .

In particular this technique has been recently applied to angular region problems, for example the impenetrable wedges with arbitrary aperture angle [1-5].

In general the W-H formulation of angular region problems yields generalized W-H equations (GWHE) that are more difficult to study with respect to the classical W-H equations (CWHE).

It is remarkable that for impenetrable wedges immersed in free space a suitable mapping reduces the GWHE to CWHE [1,2,3]. However, this property does not hold for penetrable wedges and more angular region problems [6].

Approximate techniques of factorizations are available and they can be applied to GWHE and, in particular, the GWHE can be reduced to Fredholm equations of second kind [3-6].

The aim of our work is to provide an efficient approximate evaluation of the diffraction coefficients of a dielectric wedge starting by the WH formulation.

An advantage of this approach is constituted by its capability to formulate and solve the more general wedge problems that involve anisotropic or bianisotropic media.

This method seems to extend the analysis of problems where the Sommerfeld-Malyuzhinets formulation does not exist since it is limited to media where the Helmholtz wave equation holds.

## **Angular region problems**

Without the sake of completeness let us consider an angular problem constituted by four isotropic angular regions illuminated by plane wave at skew incidence  $\beta$ . The Wiener-Hopf technique [1] for angular problems is based on the introduction of the following Laplace transforms:

$$V_{z+}(\eta, \varphi) = \int_0^\infty E_z(\rho, \varphi) e^{j\eta\rho} d\rho, \quad I_{\rho+}(\eta, \varphi) = \int_0^\infty H_\rho(\rho, \varphi) e^{j\eta\rho} d\rho \quad (1)$$

$$V_{\rho+}(\eta, \varphi) = \int_0^\infty E_\rho(\rho, \varphi) e^{j\eta\rho} d\rho, \quad I_{z+}(\eta, \varphi) = \int_0^\infty H_z(\rho, \varphi) e^{j\eta\rho} d\rho \quad (2)$$

where the subscript + indicates plus functions, *i.e.* functions having regular half-planes of convergence that are upper half-planes in the  $\eta$ -plane.

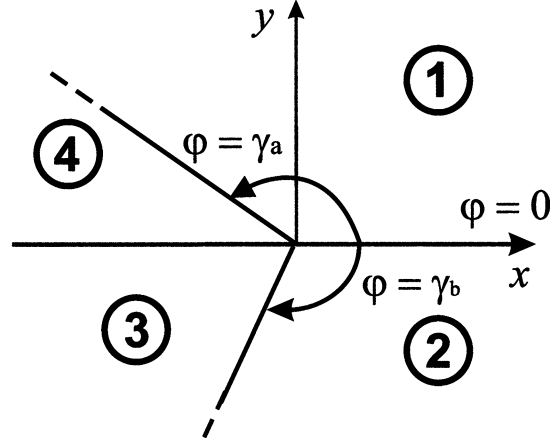


Fig. 1: four angular regions.

For region 1 we obtain the following functional equations[1-3,6]:

$$\begin{cases} \xi V_{z+}(\eta, 0) - \frac{\tau_o^2}{\omega \varepsilon} I_{\rho+}(\eta, 0) - \frac{\alpha_o \eta}{\omega \varepsilon} I_{z+}(\eta, 0) = -n V_{z+}(-m, \gamma_a) - \frac{\tau_o^2}{\omega \varepsilon} I_{\rho+}(-m, \gamma_a) + \frac{\alpha_o m}{\omega \varepsilon} I_{z+}(-m, \gamma_a) \\ \xi I_{z+}(\eta, 0) + \frac{\tau_o^2}{\omega \mu} V_{\rho+}(\eta, 0) + \frac{\alpha_o \eta}{\omega \mu} V_{z+}(\eta, 0) = -n I_{z+}(-m, \gamma_a) + \frac{\tau_o^2}{\omega \mu} V_{\rho+}(-m, \gamma_a) - \frac{\alpha_o m}{\omega \mu} V_{z+}(-m, \gamma_a) \end{cases} \quad (3)$$

where:  $\alpha_o = k \cos \beta$ ,  $\tau_o = \sqrt{k^2 - \alpha_o^2}$ ,  $\text{Im}[\tau_o] \leq 0$ ,  $\xi = \xi(\eta) = \sqrt{\tau_o^2 - \eta^2}$ .

Using symmetry and variable substitutions we obtain similar functional equations for the other regions. For example in region 3 the following equations hold:

$$\begin{cases} -\xi V_{z+}(-\eta, -\pi) + \frac{\tau_o^2}{\omega \varepsilon} I_{\rho+}(-\eta, -\pi) - \frac{\alpha_o \eta}{\omega \varepsilon} I_{z+}(-\eta, -\pi) = -n V_{z+}(-m, -\gamma_b) + \frac{\tau_o^2}{\omega \varepsilon} I_{\rho+}(-m, -\gamma_b) - \frac{\alpha_o m}{\omega \varepsilon} I_{z+}(-m, -\gamma_b) \\ -\xi I_{z+}(-\eta, -\pi) - \frac{\tau_o^2}{\omega \mu} V_{\rho+}(-\eta, -\pi) + \frac{\alpha_o \eta}{\omega \mu} V_{z+}(-\eta, -\pi) = -n I_{z+}(-m, -\gamma_b) - \frac{\tau_o^2}{\omega \mu} V_{\rho+}(-m, -\gamma_b) + \frac{\alpha_o m}{\omega \mu} V_{z+}(-m, -\gamma_b) \end{cases} \quad (4)$$

However we need to notice that the quantities involved in equations (3) and (4) depend on the constitutive parameters of the angular region (aperture angle and material), therefore:

for region 1

$$\xi = \xi_1 = \sqrt{\tau_1^2 - \eta^2}, \quad \tau = \tau_1 = \sqrt{k_1^2 - \alpha_o^2}, \quad \varepsilon = \varepsilon_1, \quad \mu = \mu_1, \quad k = k_1 = \omega \sqrt{\varepsilon_1 \mu_1}$$

$$m = m_1 = -\eta \cos \gamma_a + \xi_1 \sin \gamma_a, \quad n = n_1 = -\xi_1 \cos \gamma_a - \eta \sin \gamma_a = \sqrt{\tau_1^2 - m_1^2}$$

for region 3

$$\xi = \xi_3 = \sqrt{\tau_3^2 - \eta^2}, \quad \tau = \tau_3 = \sqrt{k_3^2 - \alpha_o^2}, \quad \varepsilon = \varepsilon_3, \quad \mu = \mu_3, \quad k = k_3 = \omega \sqrt{\varepsilon_3 \mu_3}$$

$$m = m_3 = -\eta \cos \gamma_b + \xi_3 \sin \gamma_b, \quad n = n_3 = -\xi_3 \cos \gamma_b - \eta \sin \gamma_b = \sqrt{\tau_3^2 - m_3^2}$$

### The penetrable wedge

Fig. 2 illustrates the problem of the diffraction by a plane wave at normal incidence on a dielectric wedge having relative permittivity  $\varepsilon_r$  and immersed in the free space.

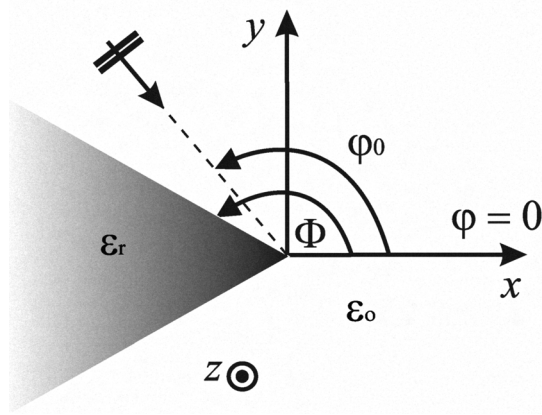


Fig. 2: the dielectric wedge

The problem can be formulated in terms of functional equations of the kind described in the previous section. Because of the symmetry we can rewrite the equations only using two angular regions and therefore by using only two kinds of constitutive parameters and functions  $m_i, n_i$ .

Let us consider the normal incidence ( $\beta = \pi/2, \alpha_o = 0$ ) with E-polarized illumination, we obtain two uncoupled system of GWHE functional equations of the following kind:

$$\begin{cases} Y_{i+}(\eta) = X_{i+}(-m_1) - \frac{\xi_{1-}}{n_{1+}} X_{j+}(-m_1) \\ Y_{j+}(\eta) = X_{i+}(-m_2) - \frac{\xi_{2-}}{n_{2+}} X_{j+}(-m_2) \end{cases} \quad (5)$$

where the unknowns are related to the physical quantities (1)-(2). Notice that the unknown are defined into three different complex planes:  $\eta, m_1, m_2$ . As reported in [1,2] we can apply a special transformation to map unknowns defined in  $\eta, m_i$  into a new unique complex plane  $\bar{\eta}_i$ , therefore we obtain CWHE from a GWHE. However, for multiple angular regions, we need to define multiple  $\bar{\eta}_i$  planes [6] (this is the main difficulty to solve in order to obtain the solution of multiple angular region problems).

### Solution

The solution of the penetrable wedge problem at normal incidence is obtained by two uncoupled systems of two functional equations whose unknown are defined into two different complex planes  $\bar{\eta}_1, \bar{\eta}_2$ :

$$\begin{cases} Y_{i+}(\bar{\eta}_1) = X_{i+}(\bar{\eta}_1) - \frac{\xi_{1-}}{n_{1+}} X_{j+}(\bar{\eta}_1) \\ Y_{j+}(\bar{\eta}_2) = X_{i+}(\bar{\eta}_2) - \frac{\xi_{2-}}{n_{2+}} X_{j+}(\bar{\eta}_2) \end{cases} \quad (6)$$

The solution of equations (6) can be obtained using the general procedure described in [3-6], the Fredholm technique, that reduce the factorization of CWHE to the solution of Fredholm integral equations of second kind.

However the penetrable wedge problem, as the more general angular region problems, involves more than one complex plane  $\bar{\eta}_i$ . The unknowns are defined into two complex planes therefore we need to relate them. This can be accomplished using the Cauchy formula [6], for instance:

$$X_{i+}(m_2) = \frac{1}{2\pi j} \oint_{\gamma_1} \frac{X_{i+}(m_1)}{m_1 - m_2} dm_1 \quad (7)$$

The use of the angular plane  $w$  and  $w_1$  and of special warping improve the convergence of the numerical discretization of the equations (6)-(7).

Further details on the procedure to get the solution and numerical results in terms of diffraction coefficients of a dielectric wedge will be discussed and presented at the conference.

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