 Role of the equilibrium size of Kadanoff blocks in the loop-expansion technique

C. Degli Esposti Boschi, A. Rioli, L. Ferrari, and F. Dolcini

1Istituto Nazionale per la Fisica della Materia, Unità di Bologna, viale Berti Pichat 6/2, 40127 Bologna, Italy
2Dipartimento di Fisica dell’Università di Bologna, viale Berti Pichat 6/2, 40127 Bologna, Italy

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A method developed by the present authors in a previous paper [Phys. Rev. E 57, 2594 (1998)] leads to the introduction of the equilibrium size of the Kadanoff blocks as a useful tool to approach the critical properties of the $\phi^4$ model. The present paper aims to elucidate the role of the equilibrium size of the Kadanoff blocks in the loop-expansion technique currently used in the field-theoretic renormalization. While the standard results are readily obtained, aspects emerge that help clarify the true nature of the smallness parameter in the loop-expansion technique.

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I. INTRODUCTION

In a recent paper [1] we introduced an unconventional renormalizable approach to the $\phi^4$ model, based on the residual free energy. It is shown therein that if $\theta$ is the relevant field, there is a special value $s^*(\theta)$ of the scaling parameter $s$ having the same critical properties of divergence as the correlation length $\xi$.

Instead of calculating the second moment of the pair correlation function (the usual definition of $\xi$), $s^*(\theta)$ is obtained by minimizing the residual free energy $f_{\text{res}}(s, \theta)$, which follows from Wilson’s renormalization procedure reducing the original number of degrees of freedom by a factor $s^d$ (in $d$ dimension). Since the residual free energy $f_{\text{res}}(s)$ can be interpreted as the “formation energy” of Kadanoff blocks of linear size $s$, it turns out that $s^*$ is the equilibrium size of the Kadanoff blocks, provided they are considered as canonic systems exchanging heat with one another. This argument substantiates on a physical ground the relationship, obtained in [1] as a formal result, between the correlation length $\xi$ and $s^*$.

The equilibrium size $s^*$ of the Kadanoff blocks is obviously derived from Wilson renormalization group theory (WRGT) [2]. In the present paper we will show that $s^*$ also enters the field-theoretic approach to renormalization, in a significant way, since $1/s^*$ actually plays the role of the additional parameter to be introduced for massless field renormalization. The textbook of Amit [3] will be taken as a reference point in what follows.

The main advantage of the present method is that the correlation length (that is, $s^*$) enters the calculations with its own physical meaning, while in the standard field-theoretic renormalization (FTR), the scaling parameter $s$ is introduced ad hoc, and somehow arbitrarily, in order to remedy the infrared divergences arising in a massless theory. Since the correlation length is a physical quantity, we have no arbitrariness at all. In particular, we will show that introducing the correlation length actually produces noninfrared divergences, which are inherent to the cumulant expansion rather than to the vanishing of the mass. However, the loop-expansion technique suggests a way to shift those divergences to next-order terms, by redefining a “dressed” relevant field (renormalized mass) order by order. As a consequence, the present method makes it possible to find both the critical point and the critical exponent $\nu$ by expressing $s^*$ as a function of the dressed field and then by studying the limiting case $s^* \to \infty$. It will be shown that the failure of the loop expansion is characterized just by the impossibility for $s^*$ to diverge at any finite value of the relevant field.

In order to implement the connection between $s^*$ and FTR, we first show that the resummation of the diagrams in Fig. 1 is equivalent to the one-loop approximation in FTR (see [3], Chap. 6). In Sec. II we also show that if $r_0$ is the relevant field and $r_0=0$ is the Gaussian critical value, then the resummation of the diagrams in Fig. 1 yields a new critical point, shifted below by the quantity

$$r_c = \frac{12d}{d-2} u_0 \quad (d=3,4,5,\ldots).$$

This coincides with the results of the one-loop approximation. In addition, the values in Eq. (1) turn out to be the lowest-order approximants (in $u_0$) for the exact critical value $-\theta_c$ (Sec. IV). The new relevant field is thereby conveniently defined as

$$\theta = r_0 + \theta_c = r_0 + r_c + O(u_0^{1+a}),$$

with $a$ a positive exponent. As far as the critical exponents are concerned, we find again the standard results of the one-loop approximation, that is, the Ginzburg criterion (Sec. III). In addition, we find that the one-loop approximation maps the one-component model (discrete symmetry) onto the spherical model (continuous symmetry) in any dimension. To the authors’ knowledge, this point has never been

![FIG. 1. Vacuum bubble diagrams containing, order by order, the maximum number of tadpole subdiagrams.](image)

*Electronic address: FERRARI@GPXBOF.DF.UNIBO.IT

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stressed before, though some results pointing to the same conclusion have already been reported in the literature (see, for example, [3]). The reason is probably that the standard method of FTR does not provide the complete one-loop solution in $d=2$, while our method does. We actually find that in $d=2$ the one-loop approximation yields the same results as the Mermin-Wagner theorem [4] for the spherical model, that is, the absence of ordered phases at any finite temperature.

In Sec. V we accomplish our program by identifying those diagrams whose resummation is equivalent to the two-structure. This makes it clear that the loop expansion does not improve our knowledge of the critical phenomenon because the shrinking of the number of degrees of freedom is an important aspect to be accounted for in the calculation of the residual free energy $f_{\text{res}}(s,r_0)$ [1]. In this case, it is convenient to perform a linear change in the integral measure, leading to the definition of the partition function $Z$,

$$Z=(\pi N\gamma)^{-N/2}\int d\{\phi_{\mathbf{q}}\}e^{-\beta H},$$

with $\gamma^{\infty}\beta=1/\kappa BT$ (see details in [1]). In what follows we take $\gamma$ as a fixed quantity, but let $r_0$ change arbitrarily, since we are interested in the critical value of the relevant field. The critical value of the temperature is, of course, another matter of affairs, which should keep track of the dependence of $\gamma$ on the temperature and of the mapping between the specific physical model and the abstract model Eqs. (3) and (4).

It should be stressed that, from the very definition (3), the Gaussian critical value corresponds to $r_0=0$ and $\theta=\theta_0$ if the relevant field $\theta$ is defined by Eq. (2). On performing a single renormalization procedure on Eq. (4) and expanding in Gaussian cumulants, the residual free energy in $d$ dimensions takes the form [1]

$$\beta f_{\text{res}}(s,r_0) = \beta f_{\text{res}}^{(0)}(s,r_0) + \sum_{n=1}^{\infty} \frac{u_0^n}{n!} G_n(s,r_0),$$

where

$$\beta f_{\text{res}}^{(0)}(s,r_0) = \frac{d}{2} \int_1^{\infty} \frac{dx}{x} x^{d-1} \ln (r_0 + x^2) + \frac{1}{2} \left( 1 - \frac{1}{s^d} \right) \ln \gamma - \frac{\ln s}{s^d},$$

is the Gaussian residual free energy. In [1] the expression of $G_1$ (corresponding to the first diagram in Fig. 1) has been explicitly calculated, with the result

$$G_1(s,r_0) = 3d^2[F_d(s,r_0)]^2, \quad F_d(s,r_0) = \frac{\partial \Phi_d(s,r_0)}{\partial r_0}.$$
why we keep them under special control, when we study how an arbitrarily small \( u_0 \) removes the Gaussian singularity at \( r_0 = 0 \), possibly shifting it below. The details of the resummation can be found in Ref. [5] and the resulting expression is

\[
\beta f_{\text{res}}(s, r_0) = \beta f^{(0)}_{\text{res}}(s, r_0 + 12du_0F_d(s, r_0)) - 3d^2u_0[F_d(s, r_0)]^2 + \text{(Contributions from Other Diagrams)},
\]

(8)

showing that the residual free energy of the \( \phi^4 \) model actually contains a shifted Gaussian term resulting from the diagrams in Fig. 1. The second argument of the logarithmic part of \( \Phi_d \) [Eq. (6)] is \( r_0 + 12du_0F_d(s, r_0) \), i.e., the direct correlation function (otherwise denoted as the two-point vertex function in FTR), at first order in \( u_0 \). It is important to stress that the function \( \Phi_d(s, r_0) \), from which the diagrams in Fig. 1 originate, is singular, if \( r_0 < 0 \), at the point \( s = 1/\sqrt{|r_0|} \). It is at this stage that our method differs from the standard FTR.

We know that \( s \) will be replaced by \( s^* \) [obtained on minimizing the free energy (8)] and that \( s^* \) behaves like the correlation length. Hence \( s^* \) will unavoidably cross the singularity point \( 1/\sqrt{|r_0|} \) if the critical point corresponds to a negative value of \( r_0 \). Note that the divergence of \( F_d(s^*, r_0) \) [Eq. (7)] is not related to the vanishing of the “bare mass” \( r_0 = 0 \), but to the cumulant expansion on a Gaussian distribution, when some eigenvalues of the quadratic form become negative. Hence we need a formal prescription to extend the calculations in the region \( r_0 < 0 \) (negative bare mass), with \( s \approx 1/\sqrt{|r_0|} \). The prescription we use here follows the same line of reasoning as the one-loop correction in FTR (see [3]). We define a “dressed” relevant field through the implicit equation

\[
r_1 = r_0 + 12du_0F_d(s, r_1) \quad (r_1 > 0),
\]

(9)

which does not contain any singular integral. In addition, the difference \( r_1 - r_0 \) is formally small to first order in \( u_0 \). The procedure we adopt is simply to express \( r_0 \) in Eq. (8), in terms of \( r_1 \) [Eq. (9)], dropping all higher-order terms in \( u_0 \). This yields

\[
\beta f^{(1)}_{\text{res}}(s, r_0) = \beta f^{(1)}_{\text{res}}(s, r_1) + O(u_0^2).
\]

(10)

At this stage, the strategy to remove the singularities in the integrals is straightforward: Dressing the relevant field (renormalizing the mass) actually shifts the singular terms to the next order, taking advantage of the fact that these terms will be resummed into the renormalized quantities in the next-order approximation and so on. This will be explicitly seen at the two-loop level (Sec. V).

Suppose now that we work in a region of parameter space where the contributions to Eq. (8) from the other diagrams is negligible. The value \( s^* \) that minimizes the free energy \( f^{(1)}_{\text{res}}(s, r_1) \) is found through the equation

\[
\partial_s f^{(1)}_{\text{res}}(s, r_1) + \partial_{r_1} f^{(1)}_{\text{res}}(s, r_1) \partial_s r_1 = 0.
\]

(11)

From Eqs. (6), (7), and (9) it can be seen that, to first order in \( u_0 \), the second term in Eq. (11) is exactly canceled by the non-Gaussian part of the first term, whence Eq. (11) becomes

\[
\frac{d}{2} \ln[1 + (s^*)^2r_1] = \left(1 - \frac{d}{2} \ln \gamma \right) \Rightarrow s^* = \frac{c_d}{\sqrt{r_1}}; \quad c_d = \left(\frac{e^{2d} - 1}{\gamma}\right)^{1/2},
\]

(12)

showing that the solution is still Gaussian-like [see [1], Eqs. (13) and (14), except for replacing the Gaussian field \( r_0 \) with the dressed field \( r_1(s^*, r_0) \). On the other hand, the dressed field (9) is nothing but the renormalized mass at one-loop order in FTR [see [3], Eq. (6-35)] and \( 1/s^* \) plays the role of the so-called subtraction point in the field theory (see [3,6]).

### III. The Ginzburg Criterion Revisited

The fact that the dressed field \( r_1 \) depends on \( s^* \) itself [Eq. (9)] is the distinction between the genuine Gaussian problem and the present one-loop approximation. Indeed, we can now make use of Eq. (12) to eliminate \( r_1 \) from Eq. (9) in order to get the relationship between \( s^* \) and \( r_0 \). The resulting equations are

\[
\left(\frac{c_4}{s^*}\right)^2 = \left(\frac{r_0 + 24u_0}{c_4 + 24u_0}\right) \left(1 + \frac{24u_0c_4^2}{c_4^2 + 24u_0}\right) \left[\left(s^*\right)^2 + \frac{c_4^2}{1 + c_4^2}\right]^{-1} \quad (d = 4; \quad (13a)
\]

\[
\left(\frac{c_3}{s^*}\right)^2 = (r_0 + 36u_0) - \frac{36u_0}{s^*} \left[1 + c_3 \arctan(c_3) - \arctan\left(\frac{c_3}{s^*}\right)\right] \quad (d = 3; \quad (13b)
\]

\[
\left(\frac{c_2}{s^*}\right)^2 = r_0 + 12u_0 \ln \left[\left(s^*\right)^2 + \frac{c_2^2}{1 + c_2^2}\right] \quad (d = 2). \quad (13c)
\]

It is immediately seen that in four and three dimensions \( s^* \) diverges at the point \( r_0 = - r_c \), with \( r_c \) given by Eq. (1). Hence, in view of Eq. (2), we set \( r_0 + r_c = \theta \). In four dimensions, Eq. (13) yields an asymptotic behavior \( (s^*)^{-1} \propto \sqrt{\theta} \ln \theta \). The critical exponent is always Gaussian, according to the Ginzburg criterion, but with a logarithmic correction (that disappears for \( d > 4 \)). In three dimensions, Eq. (13b) can be transformed into a second-order equation in \( (s^*)^{-1} \) in the limit \( (s^*)^{-1} \ll 1 \). In this case, there is a crossover from a Gaussian regime \( \sqrt{\theta} \gg 36u_0(c_3^{-1} + \arctan(c_3)) \) in which \( (s^*)^{-1} \propto \sqrt{\theta} \), to a non-Gaussian re-
might suggest that the relationship

\[ s^*(\theta)|_{r_0=0} = s^*_0(\theta_0) = \sqrt{\frac{\theta^*}{\theta_0}} \left[ 1 + O(u_0^2) \right]. \]  

provided \( \theta_0 \gg u_0^2 \). The value \( \theta^* \gg u_0 \) corresponds (modulo corrections of order \( u_0^2 \)) to \( s^*_0(\theta^*) = 1 \), that is, to the value of the relevant field at which the minimum possible value of the scaling parameter is attained. It is not difficult to verify, by means of Eq. (13b), that the approximated value of \( s^* \) at \( r_0 = 0 \) in three dimensions is

\[ s^*(0) = \sqrt{\frac{r_0^*}{36u_0^*}} \left[ 1 + O(u_0^2) \right]. \]  

with \( s^*(r_0^*) = 1 \) (modulo corrections of order \( u_0^2 \)). On the other hand, we have already claimed that the expression (16) has been obtained from the diagrams in Fig. 1, which yield the largest contribution at \( r_0 = 0 \). Hence we can take the right-hand side members of Eqs. (15) and (16) to be equal to the lowest significant order in \( u_0 \), which yields

\[ \theta_c = \frac{\theta^*}{r_0^*} \left[ 1 + O(u_0^2) \right]. \]  

Note that the condition \( \theta_c \gg u_0^2 \) follows self-consistently from Eq. (17). Furthermore, one has, by definition, \( \theta^* = r_0^* + \theta_c \) and \( r_c = 36u_0^* \) for \( d = 3 \). Now it is an easy matter to obtain Eq. (14) from Eq. (17) even in three dimensions. This proves that, in three and four dimensions, the exact critical point \( -\theta_c \) and the one-loop critical point \( -r_c \) coincide to first order in \( u_0 \).

V. TWO-LOOP APPROXIMATION

In FTR, the two-loop approximation aims to iterate the mass renormalization to second order in \( u_0 \). In order to dress the relevant field at the same order, we adopt the same procedure as in the one-loop approximation (Sec. II), that is, we identify the diagrams whose resummation yields

\[ f^{(1)}_{\text{res}}(s, r_1) = f^{(1)}_{\text{res}}(s, r_2), \quad r_2 = r_1 + u_0^2 g(s, r_2) \]  

(with \( g \) a suitable function), so that the argument of the logarithmic part of \( \Phi_d \) [Eq. (6)] corresponds to the (renormalized) direct correlation function at the second order in \( u_0 \). The diagrams are those reported in Fig. 2 and give the dressed relevant field at two-loop order:

\[ r_2 = r_1 + 48 \frac{d^2 u_0^2}{\Omega_d} [r_2 \sigma_2(s, r_2) - 2 \sigma_0(s, r_2)], \]

where \( \Omega_d \) is the \( d \)-dimensional solid angle and

\[ \sigma_0(s, r) = \sigma(x = 0, s), \quad \sigma_2(s, r) = \frac{\partial^2 \sigma(x, s, r)}{\partial x^2} \bigg|_{x = 0}, \]

\[ \sigma(x, s, r) = \int_{\text{out}} \int_{\text{out}} d^d y \, d^d z \]

\[ \times \chi_{\text{out}}(x + y + z) \]

\[ \left( (x + y + z)^2 + r \right) \left( (y^2 + r)(z^2 + r) \right). \]  

work only when the resummation of the diagrams in Fig. 1 yields the exact critical exponent. However, it can be shown that Eq. (14) holds true in three dimensions too. We make use of the Ginzburg criterion (revised above) that the Gaussian behavior is recovered when the relevant field is much larger than \( u_0^2 \). In particular, if \( s^*_0(\theta) \) is the true (unknown) expression of \( s^* \) and \( \theta = r_0 + \theta_c \) is the true relevant field [Eq. (2)], then
\( x_{\text{out}} \) is the characteristic function of the hyperspherical shell:

\[
x_{\text{out}} = \{ x \in \mathbb{R}^2; 1/s < |x| \leq 1 \}.
\]

As expected, the dressed field of Eq. (19) coincides with the renormalized mass (see Sec. 6-7 of [3]) [7]. As for the residual free energy at two-loop order, we get

\[
f^{(2)}_{\text{res}}(s,r_2) = f^{(1)}_{\text{res}}(s,r_2) + u_0^2 \Delta f_2(s,r_2),
\]

with \( \Delta f_2 \) a complicated function not reported here for the sake of brevity. The calculation leading to Eqs. (19)–(21) involves a careful account of the anomalous dimension \( \eta \) since some of the diagrams in Fig. 2 also affect the coefficient of \( \chi^2 \) in the integral expressions. Details on these aspects can be found in the E-PAPS file [5], accompanying the present paper.

Let us now discuss the two-loop approximation in the three-dimensional case only. For this aim, we do not need to minimize \( f^{(2)}_{\text{res}}(s,r_2) \) with respect to \( s \) (which is indeed far from trivial) and then study the divergence of the resulting \( s^* (r_2) \). The reason is that Eq. (19) itself, in \( d = 3 \), excludes any possible divergence of \( s^* \) for \( r_2 \rightarrow 0 \). This is due to the fact that the integral \( \sigma_0 \) in Eqs. (19) and (20) behaves, in this limit, as [5]

\[
\begin{align*}
  r_2 &\rightarrow 0, \quad \frac{1}{s^* \sqrt{r_2}} \ln r_2^2 \Rightarrow \infty \Rightarrow \sigma_0 \sim 4 \pi^2 \ln 2 |\ln r_2^2|, \\
  r_2 &\rightarrow 0, \quad \frac{1}{s^* \sqrt{r_2}} \ln s^* \Rightarrow \infty \Rightarrow \sigma_0 \sim 16 \pi^2 \ln 2 |\ln r_2^2| |\ln s^*|.
\end{align*}
\]

(wheras the term \( r_2 \sigma_2 \) is convergent).

In addition, if one believes that the minimization of \( f^{(2)}_{\text{res}}(s,r_2) \) yields an inverse power law relation between \( r_2 \) and \( s^* \) (as in the one-loop case), then the two cases in Eq. (22) are seen to coincide. Therefore, Eq. (19) in the limit \( r_2 \rightarrow 0 \) cannot be satisfied by any finite \( r_c \). In this sense, the three-dimensional case at two-loop order is quite similar to the two-dimensional case at one-loop order [Eq. (13c)]. In practice, the two-loop approximation in \( d = 3 \) looks worse than the one-loop approximation discussed in Secs. II and III. Of course, the same difficulty is encountered in FTR since the equations are formally the same. The only difference is that in FTR the quantity \( s^* \) now plays the role of the “subtraction point” (denoted as \( \kappa \) in [3]). The basic reason why in \( d = 3 \) the situation gets worse, with increasing order of approximation, is fairly clear in the present framework: The loop expansion in not an expansion in powers of \( u_0 \) alone, but involves the quantity \( s^* \) as well. However, we know that \( s^* \) diverges like the correlation length at the critical point [1] and the way \( s^* \) enters the expansion depends crucially on the dimension \( d \). The results obtained so far show that in \( d = 3 \), the true “smallness parameter” of the expansion becomes divergingly large at the critical point, just at two-loop order \( (d = 3) \) or even at one-loop order \( (d = 2) \). Indeed, these results can be regarded as special cases of the following general argument. The second case in Eq. (22) turns out to be the unique case when \( s \) is considered an independent finite parameter. Let us take for granted that, at each order, the diagrams containing superfi-

VI. CONCLUSIONS

In [1], a modified version of WRGT was introduced, in which the correlation length is calculated by minimizing the residual (renormalized) free energy with respect to the scaling parameter \( s \). As stressed in [1], the method makes use of the basic ingredients of WRGT, but avoids, in principle, the necessity of iterating the procedure to approach the fixed point. This is because the value of the scaling parameter \( s^* \) is determined, with its own critical properties, by the minimization of the residual free energy.

In view of more elaborate applications, a first step is to identify which procedure is to be intended as the loop expansion for the present method. The resummation of the diagrams in Fig. 1 is shown to realize the goal at the one-loop level. The present approach yields some progress beyond the standard results. One important point is to recognize that the diagrams in Fig. 1 are dominant for large \( s \), at each order in \( u_0 \), if \( r_0 = 0 \). It is this result that makes it possible to claim that the one-loop shifting \( r_0 \) of the critical point [Eq. (1)] is actually the first-order approximation (in \( u_0 \)) of the exact critical point \( \theta_c \) (Sec. IV). A revisitation of the Ginzburg criterion (Sec. III) yields the standard results reported in textbooks (see, in particular, [3]). However, an intriguing aspect does emerge, not explicitly stressed by other authors: The one-loop approximation maps the universal critical properties of a one-component model onto those of a spherical model. This point probably deserves some attention and is left to future investigations.
The strategy used in Sec. II to identify which diagram resummation is equivalent to the one-loop approximation can be extended to next orders: One has to dress the relevant field the same way as the mass is renormalized in FTR. This means that the direct correlation function (two-point vertex function), calculated at higher and higher orders in $u_0$, must be absorbed into the logarithmic part of the residual free energy. Following this line, it is found that the diagrams to be resummed further at the two-loop level are those in Fig. 2 (Sec. V). In three dimensions, it can be seen that the second-order term in $u_0$ diverges essentially as a power of $\ln s^*$ near the critical point. This utters the failure of the loop expansion as a useful method to investigate the critical region. Indeed, the expression of the true smallness parameter $\delta_n$ of the loop expansion at the $n$th order [Eq. (24)] shows that $\delta_n$ diverges with diverging $s^*$ at any order $n > 1$ in $d\leq3$. However, $s^*$ must diverge at the critical point since we have shown that it scales like the correlation length. In practice the smallness parameter is never small in $d<3$, close enough to the critical point. This clearly explains why in $d<3$ the loop expansion is not sufficient, by itself, to improve the approximation order by order if one is interested in the critical region, unless a new perturbative parameter is introduced keeping the divergence of $s^*$ under control. This clarifies the crucial role played by the $\epsilon$ expansion.

At this stage one may wonder whether the $\epsilon$ expansion is a unique technique to approach the study of the critical properties in $d<4$. The present approach shows that the origin of the problems stems from the Gaussian cumulant expansion. The point is that a Gaussian cumulant expansion yields some problems just close to the critical region, where the "perturbation" $u_0$ becomes the dominant term. The quickest way to remove any singularity in a perturbative expansion close to the critical point should be treating the Gaussian part itself of the Hamiltonian as a perturbation. Though this approach might look discouraging at first, our next attempt will point to this direction since the physical meaning of $s^*$ seems to open some perspectives for a possible non-Gaussian (quartic) expansion in cumulants.

[5] See AIP document No. E-PAPS: E-PLEE8-58-009811 for details and a discussion about the calculations leading to some of the results reported in the main paper. E-PAPS document files may be retrieved free of charge from our FTP server (http://www.aip.org/epaps/epaps.html) or from ftp.aip.org in the directory /epaps/. For further information, e-mail: paps@aip.org or fax: 516-576-2223.
[7] It should be noticed that the identification of $r_2$ with the usual renormalized mass is not complete in as much as our approach does not need to regularize the four-point vertex function (quartic coupling constant).