A branch of kinetic theory - the lattice Boltzmann (LB) method - has recently met with a remarkable success as a powerful alternative for solving the hydrodynamic Navier–Stokes equations, with applications ranging from large Reynolds number flows to flows at a micron scale, porous media, and multiphase flows, see, e.g., [1–3] and references therein. The LB method solves a fully discrete kinetic equation for populations \( f_\alpha(x,t) \), designed in a way that it reproduces the Navier–Stokes equations in the hydrodynamic limit in \( D \) dimensions. Populations correspond to discrete velocities \( v_\alpha \) for \( \alpha = 0, 1, ..., Q-1 \), which fit into a regular spatial lattice with the nodes \( x \). This enables a simple and highly efficient algorithm based on (a) nodal relaxation and (b) streaming along the links of the regular spatial lattice. On the other hand, numerical stability of the LB method remains a critical issue [2]. Recalling the role played by the Boltzmann's \( H \)-theorem in enforcing macroscopic evolutionary constraints (the second law of thermodynamics), pertinent entropy functions have been proposed [4–8]. The full connection of LB to kinetic theory was established by the discrete-velocity analog of the Maxwellian (see Eq. (2) below).

Admittedly, however, that other heuristic methods were proposed recently to enhance stability of LB. The rationale behind one of them, the multiple–relaxation–time (MRT) [9–11], is sound: Since the incompressible flow is the only concern, the bulk viscosity arising in the quasi-compressible LB scheme can be viewed as a free parameter and tuned in order to enhance stability. However, in spite of popularity of the MRT method, to date, it cannot be considered as a consistent kinetic theory but rather a numerical trick where tuning of parameters is based on experience rather than on physics.

In this paper, we present a new consideration of the LB models, and derive a crucial result: the closed-form generalized equilibrium (see Eq. (3) below). The generalized equilibrium is the analog of the anisotropic Gaussian, and is a long-needed relevant distribution in the LB method. This finding further allows us to introduce an innovative class of entropy-based MRT LB models which enjoy both the \( H \)-theorem and the additional free–tunable parameter for controlling the bulk viscosity, where the range is dictated by the entropy.

For the sake of presentation and without any loss of generality, we consider the popular nine-velocity model, the so-called D2Q9 lattice, of which the discrete velocities are: \( v_0 = (0, 0) \), \( v_i = (\pm c, 0) \) and \( (0, \pm c) \) for \( i = 1-4 \), and \( v_i = (\pm c, \pm c) \), for \( i = 5-8 \) [12] where \( c \) is the lattice spacing. Recall that the D2Q9 lattice derives from the three-point Gauss–Hermite formula [13], with the following weights of the quadrature \( w(-1) = 1/6 \), \( w(0) = 2/3 \) and \( w(+1) = 1/6 \). Let us arrange in the list \( v_x \) all the components of the lattice velocities along the \( x \)-axis and similarly in the list \( v_y \). Analogously let us arrange in the list \( f_\alpha \) all the populations \( f_\alpha \). Algebraic operations for the lists are always assumed component-wise. The sum of all the elements of the list \( p \) is denoted by \( \langle p \rangle = \sum_{\alpha=0}^{Q-1} p_\alpha \).

The dimensionless density \( \rho \), the flow velocity \( \mathbf{u} \) and the second–order moment (pressure tensor) \( \Pi \) are defined by \( \rho = \langle f \rangle \), \( \rho \mathbf{u}_i = \langle v_i f \rangle \) and \( \rho \Pi_{ij} = \langle v_i v_j f \rangle \) respectively.

On the lattice under consideration, the convex entropy function (\( H \)-function) is defined as [5]

\[
H(f) = \langle f \ln (f/W) \rangle, \tag{1}
\]

where \( W = w(v_x) w(v_y) \). The \( H \)-function minimization problem is considered in the sequel. It is well known [5] that the equilibrium population list \( f_M \) is defined as the solution of the minimization problem \( f_M = \min_{f \in \mathcal{P}_M} H(f) \), where \( \mathcal{P}_M \) is the set of functions such that \( \mathcal{P}_M = \{ f > 0 : \langle f \rangle = \rho, \langle vf \rangle = \rho \mathbf{u} \} \). In other words, minimization of the \( H \)-function (1) under the constraints of mass and momentum conservation yields [6]

\[
f_M = \rho \prod_{\alpha=1,2} w(v_\alpha) (2 - \varphi(u_\alpha/c)) \left( \frac{2(u_\alpha/c) + \varphi(u_\alpha/c)}{1 - (u_\alpha/c)} \right)^{v_\alpha/c}, \tag{2}
\]

where \( \varphi(z) = \sqrt{3z^2+1} \). A remarkable feature of the equilibrium (2) which it shares with the ordinary Maxwellian is that it is a product of one-dimensional equilibria. In order to ensure the positivity of \( f_M \), the low Mach number limit must be considered, i.e. \( |u_\alpha| < c \).

In this paper, we derive a novel constrained equilibrium, or quasi–equilibrium [14], by requiring, in addition, that the diagonal components of the pressure tensor \( \Pi \)}
have some prescribed values. Hence let us introduce a
different minimization problem. The quasi–equilibrium
population list \( f_G \) is defined as the solution of the mini-
mization problem \( f_G = \min_{f \in P_G} H(f) \), where \( P_G \subset P_M \)
is the set of functions such that \( P_G = \{ f > 0 : \langle f \rangle = \rho, \langle u f \rangle = \rho u, \langle v^2 f \rangle = \rho \Pi_{\alpha \alpha} \} \). In other words, mini-
mization of the \( H \)–function (1) under the constraints of
mass and momentum conservation and prescribed diag-
"onal components of the pressure tensor yields

\[
f_G = \rho \prod_{\alpha=x,y} w(v_{\alpha}) \frac{3 (c^2 - \Pi_{\alpha \alpha})}{2 c^2} \left( \frac{\Pi_{\alpha \alpha} + c u_{\alpha}}{\Pi_{\alpha \alpha} - c u_{\alpha}} \right)^{v_{\alpha}/c} \left( 2 \frac{\Pi_{\alpha \alpha}^2 - c^2 u_{\alpha}^2}{c^2 - \Pi_{\alpha \alpha}} \right)^{v_{\alpha}^2/c^2}.
\]

To ease notation, we use \( \Pi = (\Pi_{xx}, \Pi_{yy}) \) for a generic
point on the two-dimensional plane of parameters. In
order to ensure the positivity of \( f_G \), it is required that
\( \Pi \in \Omega \) where \( \Omega = \{ \Pi : c |u_x| < \Pi_{xx} < c^2, c |u_y| < \Pi_{yy} < c^2 \} \) is a convex rectangular in the plane of parameters for
each velocity \( u \) (see Fig. 1).

The generalized Maxwellian (3) is the central result of
this paper, and is the key to the derivations below. It
is interesting to note that, while the equilibrium (2) is
analogous to the ordinary Maxwellian (spherically sym-
metric Gaussian \( f_M \sim \exp\{-m(v-u)^2/2kBT_0\} \), shifted
from the origin by the amount of mean flow velocity \( u \),
and with the width proportional to the fixed tempera-
ture \( T_0 = c^2/3 \), the quasi-equilibrium (3) resembles the
anisotropic Gaussian, \( f_G \sim \exp\{-1/2(v-u) \cdot \Pi^{-1} \cdot
(v-u)\} \). The latter generalized Maxwellian corresponds
to the ellipsoidal symmetry, and is among the only few
analytic results on the relevant distribution functions in
the classical kinetic theory [15]. It is revealing that also in
the LB realm the analog of the generalized Maxwellian
has a nice closed form (3). The physical sense of (3) is
that it distinguishes the relaxation of the diagonal
components of the pressure tensor (and hence also of the
trace of this tensor) among other non-conserved
moments, and hence one expects a control over the dynam-
ics of the trace which is responsible for the bulk viscosity
(see below). Moreover, it is possible to evaluate explicitly
the \( H \)–function in the generalized Maxwell states (3),
\( H_G = H(f_G) \), the result is elegantly written

\[
H_G = \rho \ln \rho + \rho \sum_{\alpha=x,y} \sum_{k=-u,+} w_k a_k(\Pi_{\alpha \alpha}) \ln (a_k(\Pi_{\alpha \alpha})),
\]

where \( w_\pm = w(\pm 1), w_0 = w(0), \quad a_\pm(\Pi_{\alpha \alpha}) = 3(\Pi_{\alpha \alpha} \pm
c u_{\alpha})/c^2 \) and \( a_0(\Pi_{\alpha \alpha}) = 3(c^2 - \Pi_{\alpha \alpha})/(2 c^2) \) (see Fig. 1).

Finally, with the help of \( f_G (3) \), let us derive a con-
strained equilibrium \( f_C \) which brings the \( H \)–function to
a minimum among all the population lists with a fixed
trace of the pressure tensor \( T(\Pi) = \Pi_{xx} + \Pi_{yy} \). In
terms of the parameter set \( \Omega \), this is equivalent to re-
quire that the point \( C = (\Pi_{xx}' \Pi_{yy}') \) belongs to a line
segment \( L_T = \{ \Pi \in \Omega : \Pi_{xx} + \Pi_{yy} = T \} \), and
the constrained equilibrium \( C \) is that minimizing the
function \( H_G \) (4) on \( L_T \) (see Fig. 1). Since the re-
striction of a convex function to a line is also convex,
the solution to the latter problem exists and is found by
\( (\partial H_G/\partial \Pi_{xx}) - (\partial H_G/\partial \Pi_{yy}) ) (\Pi_{xx}, \Pi_{xx}) L_T = 0 \), which
yields a cubic equation in terms of the normal stress dif-

\[
\begin{align*}
\text{FIG. 1: (Color online) Contour plot of the entropy } H_G \text{ (4) at } \\
\rho = 1, \quad u_x = -0.2 \text{ and } u_y = 0.1 \quad (c = 1). \quad \text{Rectangular} \\
\text{domain is the positivity domain } \Omega. \quad \text{M is the image of the} \\
\text{Maxwellian (2). } O \text{ is the image of a generic non-equilibrium} \\
\text{state while } C \text{ is the image of the constrained equilibrium (7)} \\
\text{(minimum of } H_G \text{ on the line } L_T). \quad \text{C'} \text{ is the low Mach number} \\
\text{approximation of } C, \text{ while the line segment connecting } C' \\
\text{and } C'' \text{ represents admissible generalized equilibria } E(\omega) \text{ (16)} \\
\text{with } E(1) = C' \text{ and } E(\omega^*) = C'' \text{ at } \omega^* \approx -1.
\end{align*}
\]
ference \( N = \Pi_{xx}^C - \Pi_{yy}^C \),
\[
N^3 + a N^2 + b N + d = 0,
\]
\[
a = -\frac{1}{2} (u_x^2 - u_y^2), \quad b = (2c^2 - T) (T - u^2),
\]
\[
d = -\frac{1}{2} (u_x^2 - u_y^2) (2c^2 - T)^2.
\]

Let us define \( p = -a^2/3 + b, \quad q = 2a^3/27 - ab/3 + d \) and
\( \Delta = (q/2) + (p/3)^3 \). As long as \( \Delta \geq 0 \), which is
well satisfied in the low Mach number limit, the Cardano
formula implies
\[
\Pi_{xx}^C = \frac{T}{2} + \frac{1}{2} \left( r - \frac{p}{3r} - \frac{a}{3} \right), \quad r = \sqrt[3]{\frac{q}{2}} + \sqrt{\frac{q}{2} + \sqrt{\Delta}},
\]
while \( \Pi_{yy}^C = T - \Pi_{xx}^C \). Thus, substituting (6) into (3), we
find the constrained equilibrium
\[
f_C = f_C(\rho, u, \Pi_{xx}^C(u, T), \Pi_{yy}^C(u, T)).
\]

Before proceeding any further, we mention that the
generalized Maxwellian (3) is consistent with and extends
the previously known results:
(i) The point of global minimum of the function \( H_G(4) \)
on \( \Omega \) is found from \( \partial H_G/\partial \Pi_{\alpha \alpha} = 0 \). The corre-
sponding solution \( M = (\Pi_{xx}^M, \Pi_{yy}^M) \), where \( \Pi_{\alpha \alpha}^M =
-c^2/3 + (2c^2/3) \sqrt{1 + 3(u_\alpha/c)^2} \), recovers the equilib-
rium \( f_M \) (2) upon substitution into (3):
\[
f_M = f_G(\rho, u, \Pi_{xx}^M(u), \Pi_{yy}^M(u)).
\]
(ii) In Ref. [7], a different LB equilibrium \( f_\Theta \) was intro-
duced as the entropy minimization problem under fixed
density, momentum and energy. That equilibrium was
evaluated exactly only for vanishing velocity in [7] while
a series expansion was used for \( u \neq 0 \). The new result
reported above solves the problem of Ref. [7] exactly
for any velocity: Substituting \( T = 2\Theta + u^2 \)
two-dimensional ideal gas equation of state, with \( \Theta \)
into (7), it is simply \( f_\Theta(\rho, u, \Theta) = f_G(\rho, u, \Pi_{xx}^C(u, 2\Theta +
u^2), \Pi_{yy}^C(u, 2\Theta + \nu^2)) \). Expanding the exact solution \( \Pi_{xx}^C \)
(6) in terms of the velocity \( u \) yields the approximate solu-
tion consistent with Ref. [7], namely
\[
\Pi_{xx}^C = \Theta + \left( \frac{\Theta + 1}{4\Theta} \right) u_x^2 + \left( \frac{3\Theta - 1}{4\Theta} \right) u_y^2 + O(u^4).
\]
(iii) In Ref. [16], a so-called guided equilibrium \( \tilde{f}_G \) was
introduced in order to derive LB method for compressible
flows. That equilibrium is recovered by simply assum-
ing the Maxwell-Boltzmann form of the diagonal com-
ponents, \( \Pi_{xx} = \Theta + u_x^2 \) and \( \Pi_{yy} = \Theta + u_y^2 \), in (3):
\[
f_\Theta(\rho, u, \Theta) = f_G(\rho, u, \Theta + u_x^2, \Theta + u_y^2).
\]
Thus, the generalized Maxwellian (3) and its impli-
cation, the constrained equilibrium (7), unifies all the
equilibria introduced previously on the D2Q9 lattice.

Armed with the constrained equilibrium, we now pro-
cceed with the derivation of the kinetic equation. By
means of the usual equilibrium \( M \) and the newly found
constrained equilibrium \( C \), let us define the generalized
equilibrium \( E(\omega) = (\Pi_{xx}^E(\omega), \Pi_{yy}^E(\omega)) \) as a linear inter-
polation between the points \( M \) and \( C \) on the \( \Pi \)-plane
\[
E(\omega) = (1 - \omega) M + \omega C,
\]
where \( \omega \) is a free parameter, and its admissible range will
be defined next. Thus, the generalized equilibrium list is
defined as
\[
f_{GE}(\omega) = f_G(\rho, u, \Pi_{xx}^E(\omega), \Pi_{yy}^E(\omega)).
\]

Considering kinetic equation of the form, \( \partial_t f + v \cdot \partial_x f =
J(f) \), let us define the following collision operator
\[
J(f) = \lambda \left[ f_{GE}(\omega) - f \right],
\]
where \( \lambda > 0 \) is a parameter, ruling the relaxation to-
ward the generalized equilibrium. In the continuum
limit, \( \lambda \) is related to the kinematic viscosity. While
Eq. (11) reminds the popular Bhatnagar-Gross-Krook
(BGK) model [17], the collision integral (11) depends
on two parameters, \( \lambda \) and \( \omega \). In view of the analogy
of \( f_G \) with the anisotropic Gaussian, this is somewhat
similar to the so-called ellipsoidal statistical model [17].
However, in our case, the leading order of the macro-
scopic equations recovered in the continuum limit does
not depend on \( \omega \), which is a tunable parameter for en-
hancing the stability of the corresponding LB scheme.

Collision operator (11) conserves mass and momentum,
i.e., \( \langle J(f) \rangle = 0 \) and \( \langle v J(f) \rangle = 0 \). Note that, at \( \omega = 0 \),
(11) reduces to the BGK LB model of Ref. [5], while at
\( \omega = 1 \) it becomes the so-called consistent LB model with
energy conservation [7] (see Remark (ii) above).

The key for proving the \( H \)-theorem for the kinetic
equation is to establish the non-positivity of the entropy
production \( \sigma \) due to the relaxation term (11), where
\[
\sigma = \langle \ln(\rho/W) J(f) \rangle.
\]

Clearly, if \( f = f_M \), then \( C = M \) and \( \Pi^E(\omega) = \Pi^M \) for
any \( \omega \). From Remark (i), it follows that entropy produc-
tion annihilates at the equilibrium, \( \sigma(f_M) = 0 \). In the
general case, we have
\[
\frac{\sigma}{\lambda} \leq H_G(\omega) - H(f) \leq H_G(\omega) - H_G(\Pi),
\]
where \( H_G(\omega) = H_G(\Pi_{xx}^E(\omega), \Pi_{yy}^E(\omega)) \). The first
inequality is due to the convexity of the \( H \)-function, while
the second holds because \( f_G(\Pi_{xx}, \Pi_{yy}) \), by definition,
minimizes \( H \) among all the population lists with the
moments \( (\Pi_{xx}, \Pi_{yy}) \). Recalling that \( \Pi(f_{GE}(1)) \) and
\( \Pi(f_G(\Pi_{xx}, \Pi_{yy})) \) have the same trace and taking into
account the definition of the point \( C \), inequality (13) can
be rewritten
\[
\frac{\sigma}{\lambda} \leq H_G(\omega) - H_G(1) + H_G(1) - H_G(\Pi)
\leq H_G(\omega) - H_G(1).
\]

What remains is to estimate the range of \( \omega \) such that
\( H_G(\omega) \leq H_G(1) \). Clearly, since \( M = E(0) \) is the ab-

\vspace{1cm}
function (a restriction of a convex function to a line), the right hand side of Eq. (14) is non-positive if $0 \leq \omega < 1$. This proves non-positivity of the entropy production in the interval $0 \leq \omega < 1$. In order to extend the proof to $\omega < 0$, let us consider the entropy estimate [5] (see also [18]):

$$H_{GE} (\omega^*) = H_{GE} (1).$$

(15)

Thanks to the convexity of $H_{GE} (\omega)$, the non-trivial solution $\omega^* < 0$ to this equation is unique when it exists. In the opposite case, we take $\omega^* < 0$ from the condition, $E(\omega^*) \in \partial \Omega$, where $\partial \Omega$ is the boundary of the positivity domain $\Omega$. In both cases, for $\omega^* \leq \omega < 1$, the entropy production is non-positive, $\sigma \leq 0$, which proves the existence of the $H$–theorem for the proposed model. Note that, from the entropy estimate, it follows that $\omega^*$, in general, depends on the state $f$. However, Eq. (15) drastically simplifies at low Mach numbers which we consider next.

In the case of diffusion scaling [19, 20], i.e. the flow regime with $Kn \sim Ma \sim u/c \ll 1$, where $Kn$ is the Knudsen number and $Ma$ is the Mach number, equation (8) simplifies to

$$\Pi_{\alpha \alpha}^E (\omega) = \Pi_{\alpha \alpha}^M + \omega \frac{T-T_M}{2} + O(u^4).$$

(16)

Using (16) in the definition of the collision operator (11) and neglecting all the terms in the higher moments which are two order of magnitude (with regards to $u$) smaller than the corresponding terms required to recover incompressible Navier–Stokes equations [20], the collision operator can be simplified to

$$J'(f) = A (f_M - f),$$

(17)

where $A = \lambda B^{-1} \Lambda B$ and $9 \times 9$ matrices $B$ and $\Lambda$ are

$$A = \text{diag} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1, v_x, v_y, v_x^2, v_y^2, v_x v_y, v_x^2 v_y, v_x v_y^2, v_x^2 v_y^2 \end{bmatrix}^T,$$

with $r_\pm = (r \pm 1)/2$ and $r = 1-\omega$. Operator $J'$ is a MRT collisional operator with collision matrix $A$ (characterized by two relaxation frequencies $\lambda$ and $\delta = r\lambda$). It is possible to prove by means of the asymptotic analysis [21] that, in the continuum limit, $J'$ leads to the kinematic (shear) viscosity and the second (bulk) viscosity coefficients given respectively by

$$\nu = \frac{c^2}{3\lambda}, \quad \xi = \frac{c^2}{3\delta}.$$  

(19)

Finally, for low Mach numbers, the entropy $H_{GE}$ can be estimated as follows:

$$H_{GE} = \rho \ln \rho - \frac{3}{2} \rho u^2 + \frac{9}{8} \rho (T-T_M) \omega^2 + O(u^6).$$

(20)

Using (20) in the entropy condition (15), we find $\omega^* \approx -1$ (see Fig. 1). Consequently, the stability region of the relaxation frequency $\delta$ controlling the bulk viscosity is estimated $0 < \delta < 2\lambda$ or, taking into account Eq. (19), equivalently $0 < \nu/\xi < 2$. In particular, for high Reynolds number flows, the ratio $\nu/\xi$ tends to the lowest limit, i.e. large bulk viscosity is required to make more stable the numerical computations.

Since the bulk viscosity controls the attenuation of acoustic waves, which are fictitious when searching for the incompressible problem, increasing this tunable parameter allows one to mitigate the effects of fictitious compressibility and hence it increases the stability region of the scheme. In order to check the accuracy of the scheme, let us consider the Taylor-Green vortex flow test. Let us consider a square domain $(x, y) \in [0, 2\pi/k] \times [0, 2\pi/k]$. The Taylor-Green vortex flow has the following analytical solution [22]:

$$u_x = -u_0 \cos (kx) \sin (ky) \exp (-2k^2t),$$

(21)

$$u_y = u_0 \cos (ky) \sin (kx) \exp (-2k^2t),$$

(22)

$$p = -\frac{u_0^2}{4} \left( \cos (2kx) \sin (2ky) - \sin (2kx) \cos (2ky) \right) \exp (-4k^2t).$$

(23)

where the pressure $p = (c^2\rho)/(3\rho_0)$. It is immediate to prove that

$$\Phi(t) = \frac{1}{2} \int_0^{2\pi/k} \int_0^{2\pi/k} (u_x^2 + u_y^2) k^2 dx dy = \frac{u_0^2}{4} \exp (-4k^2t).$$

(24)

The previous formula suggests a simple way to measure the actual kinematic viscosity. Introducing the simulation time $t \in [0, t_0]$, the measured kinematic viscosity is given by

$$\nu_s = \frac{\ln (\Phi(t_0)/\Phi(0))}{4k^2 t_0}$$

(25)

In the following numerical results, we set $k = 1, u_0 = 1, \rho_0 = 1$ and $t_0 = 5$. Consequently the Reynolds number becomes $Re = 2\pi/\nu$. Let us consider a homogeneous mesh made of $160 \times 160$ nodes, which implies Knudsen number $Kn = 1/160$. Let us select the Mach number as $Ma = 1/16$. Some numerical tests are reported for different kinematic viscosity $\nu$ and bulk viscosity $\xi$. The numerical results are reported in Table I and compared with the standard lattice BGK (LBGK) model. First of all, this test shows that the model recovers the right kinematic viscosity. Moreover, the relaxation frequency $\delta$, controlling the bulk viscosity, does not affect the leading part of the solution. According to the previous test, even large bulk viscosities may be adopted without affecting significantly the numerical results.

To conclude, the generalized Maxwellian (3) opens a new perspective on the LB method. Various LB equilibrium introduced in the past are special cases of (3). Important application of (3), considered in this paper, is a novel
TABLE I: Taylor-Green vortex flow test. Some numerical tests are reported for different kinematic viscosity $\nu$ and bulk viscosity $\xi$. The mesh is made of $160 \times 160$ nodes. The Knudsen number is $Kn = 1/160$, the Mach number $Ma = 1/16$ and finally the Reynolds number $Re = 2\pi/\nu$. The actual kinematic viscosity $\nu^*$ is measured by means of Eq. (25) and the relative error $(\nu - \nu^*)/\nu$ is reported as well.

<table>
<thead>
<tr>
<th>$\nu/\xi$</th>
<th>$\nu$</th>
<th>Measured $\nu^*$</th>
<th>Error [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>LBGK</td>
<td>0.001</td>
<td>0.00102065</td>
<td>2.0648</td>
</tr>
<tr>
<td>present</td>
<td>0.1 0.001</td>
<td>0.00102071</td>
<td>2.0713</td>
</tr>
<tr>
<td>present</td>
<td>0.01 0.001</td>
<td>0.00102106</td>
<td>2.1058</td>
</tr>
<tr>
<td>LBGK</td>
<td>0.010 0.0998599</td>
<td>-0.1491</td>
<td></td>
</tr>
<tr>
<td>present</td>
<td>0.1 0.010 0.0998555</td>
<td>-0.1445</td>
<td></td>
</tr>
<tr>
<td>present</td>
<td>0.01 0.010 0.0998654</td>
<td>-0.1346</td>
<td></td>
</tr>
<tr>
<td>LBGK</td>
<td>0.100 0.09977323</td>
<td>-0.2268</td>
<td></td>
</tr>
<tr>
<td>present</td>
<td>0.1 0.100 0.09977355</td>
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</tr>
<tr>
<td>present</td>
<td>0.01 0.100 0.09977230</td>
<td>-0.2277</td>
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</tr>
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</table>

class of entropic multiple–relaxation–time (E–MRT) LB models. They enjoy both the $H$–theorem and the additional free–tunable parameter for controlling the bulk viscosity. Hence, they combine the two most successful strategies for enhancing stability of LB for high Reynolds number simulations. Because all the results above are derived in a closed form, numerical implementation of the E–MRT LB models is straightforward. Preliminary numerical results demonstrated that efficient stabilization of the LB simulation without loss of accuracy is indeed achieved with the suggested scheme. Moreover, the implementation is not much different from the familiar LBGK scheme, unlike the standard MRT model. These results show that the present model can be used for enhancing stability instead of the most popular LBGK method. Details of the implementation and numerical results will be reported in a separate publication. I.V.K. acknowledges support of CCEM-CH.