

A geometric proof of Karpelevich-Mostow's theorem

Original

A geometric proof of Karpelevich-Mostow's theorem / DI SCALA, ANTONIO JOSE'; Olmos, C.. - In: BULLETIN OF THE LONDON MATHEMATICAL SOCIETY. - ISSN 0024-6093. - 41:(2009), pp. 634-638.

Availability:

This version is available at: 11583/1927380 since:

Publisher:

Published

DOI:

Terms of use:

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

Nature --> vedi Generico

[DA NON USARE] ex default_article_draft

(Article begins on next page)

A geometric proof of the Karpelevich-Mostow's Theorem

Antonio J. Di Scala and Carlos Olmos

Post print (i.e. final draft post-refereeing) version of an article published on *Bull. London Math. Soc.* (2009) 41 (4): 634-638. doi: 10.1112/blms/bdp036.

Beyond the journal formatting, please note that there could be minor changes from this document to the final published version.

The final published version is accessible from here:

<http://blms.oxfordjournals.org/content/41/4/634>

This document has made accessible through PORTO, the Open Access Repository of Politecnico di Torino (<http://porto.polito.it>), in compliance with the Publisher's copyright policy as reported in the SHERPA-ROMEO website: <http://www.sherpa.ac.uk/romeo/issn/0024-6093/>

ABSTRACT

In this paper we give a geometric proof of the Karpelevich's theorem that asserts that a semisimple Lie subgroup of isometries, of a symmetric space of non compact type, has a totally geodesic orbit. In fact, this equivalent to a well-known result of Mostow about existence of compatible Cartan decompositions.

1. Introduction.

In this paper we address the problem of giving a geometric proof of the following theorem of Karpelevich.

THEOREM 1.1. (*Karpelevich [7]*) *Let M be a Riemannian symmetric space of non positive curvature without flat factor. Then any connected and semisimple subgroup $G \subset \text{Iso}(M)$ has a totally geodesic orbit $G.p \subset M$.*

It is well-known that Karpelevich's theorem is equivalent to the following algebraic theorem.

THEOREM 1.2. (*Mostow [8, Theorem 6]*) *Let \mathfrak{g}' be a real semisimple Lie algebra of non compact type and let $\mathfrak{g} \subset \mathfrak{g}'$ be a semisimple Lie subalgebra. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition for \mathfrak{g} . Then there exists a Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ for \mathfrak{g}' such that $\mathfrak{k} \subset \mathfrak{k}'$ and $\mathfrak{p} \subset \mathfrak{p}'$.*

The proof of the above theorems is very algebraic in nature and uses delicate arguments related to automorphisms of semisimple Lie algebras.

For the real hyperbolic spaces, i.e. when $\mathfrak{g}' = \mathfrak{so}(n, 1)$, there are two geometric proofs of Karpelevich's theorem [4], [2]. The proof in [4] is based on the study of minimal orbits of isometries subgroups, i.e. orbits with zero mean curvature. The approach in [2] is based on hyperbolic dynamics. It is interesting to note that both proofs are strongly based on the fact that the boundary at infinity of real hyperbolic spaces has a simple structure.

The only non-trivial algebraic tool that we will use is the existence of a Cartan decomposition of a non compact semisimple Lie algebra. But this can also be proved geometrically as was explained by S.K. Donaldson in [5].

Here is a brief explanation of our proof of Theorem 1.1. We first show that a simple subgroup $G \subset \text{Iso}(M)$ has a minimal orbit $G.p \subset M$. Then, by using a standard totally geodesic embedding $M \hookrightarrow \mathcal{P}$, where $\mathcal{P} = SL(n, \mathbb{R})/SO(n)$, we will show that $G.p$ is, actually, a totally geodesic submanifold of M .

2. Preliminaries.

The results in this section are well known and are included to orient the non-specialist reader.

The equivalence between Theorems 1.1 and 1.2 is a consequence of the following Elie Cartan's famous and remarkable theorem.

THEOREM 2.1. *(Elie Cartan) Let M be a Riemannian symmetric space of non positive curvature without flat factor. Then the Lie group $\text{Iso}(M)$ is semisimple of non compact type. Conversely, if \mathfrak{g} is a semisimple Lie algebra of non compact type then there exist a Riemannian symmetric space M of non positive curvature without flat factor such that \mathfrak{g} is the Lie algebra of $\text{Iso}(M)$.*

The difficult part of the proof of the above theorem is the second part. Namely, the construction of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where \mathfrak{k} is maximal compact subalgebra of \mathfrak{g} and the Killing form B of \mathfrak{g} is positive definite on \mathfrak{p} . The standard and well-known proof of the existence of a Cartan decomposition is long and via the classification theory of complex semisimple Lie algebras, i.e. the existence of a real compact form (see e.g. [6]). There is also a direct and geometric proof of the existence of a Cartan decomposition [5].

On the other hand, when $\mathfrak{g} = \text{Lie}(\text{Iso}(M))$, where M is a Riemannian symmetric space of non positive curvature without flat factor, a Cartan decomposition of \mathfrak{g} can be constructed geometrically. Namely, $\mathfrak{g} = \text{Lie}(\text{Iso}(M)) = \mathfrak{k} \oplus \mathfrak{p}$ where \mathfrak{k} is the Lie algebra of the isotropy group $K_p \subset \text{Iso}(M)$ and $\mathfrak{p} := \{X \in \text{Lie}(\text{Iso}(M)) : (\nabla X)_p = 0\}$.

It is well-known that the Riemannian symmetric spaces $\mathcal{P} = SL(n, \mathbb{R})/SO(n)$ are the *universal* Riemannian symmetric space of non positive curvature. Namely, any Riemannian symmetric space of non compact type $M = G/K$ can be totally geodesically embedded in some \mathcal{P} (up to rescaling the metric in the irreducible De Rham factors). A proof of this fact follows from the following well-known result (c.f. Theorem 1 in [5]).

PROPOSITION 2.2. *Let $\mathfrak{g} \subset \mathfrak{sl}(n, \mathbb{R})$ be a semisimple Lie subalgebra and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. Then there exists a Cartan decomposition $\mathfrak{sl}(n, \mathbb{R}) = \mathcal{A} \oplus \mathcal{S}$ such that $\mathfrak{k} \subset \mathcal{A}$ and $\mathfrak{p} \subset \mathcal{S}$. Thus, if $G \subset SL(n)$ is semisimple, G has a totally geodesic orbit in $\mathcal{P} = SL(n)/SO(n)$. Indeed, any Riemannian symmetric space of non positive curvature M , without flat factor, can be totally geodesically embedded in some $\mathcal{P} = SL(n)/SO(n)$.*

Proof. Notice that any Cartan decomposition of $\mathfrak{sl}(n, \mathbb{R})$ is given by the anti-symmetric \mathcal{A} and symmetric matrices \mathcal{S} w.r.t. a positive definite inner product on \mathbb{R}^n . Since $\mathfrak{g}^* := \mathfrak{k} \oplus i\mathfrak{p}$ is a compact Lie subalgebra of $\mathfrak{sl}(n, \mathbb{C})$, there exists a positive definite Hermitian form $(\cdot | \cdot)$ of \mathbb{C}^n

invariant by g^* . By defining $\langle , \rangle := \text{Real}(\ |)$ it follows that $\mathfrak{k} \subset \mathcal{A}$ and $\mathfrak{p} \subset \mathcal{S}$. \square

Let $S_\infty(M)$ be the *sphere or boundary at infinity* of M , i.e. $S_\infty(M)$ is the set of equivalence classes of asymptotic geodesics rays (see [5] or [3, Chapter II.8] for details).

Here is another corollary of the existence of the totally geodesic embedding $M \hookrightarrow \mathcal{P}$.

COROLLARY 2.3. *Let M be a Riemannian symmetric space of non positive curvature without flat factor. Then a connected and semisimple Lie subgroup $G \subset \text{Iso}(M)$ of non compact type has no fixed points in $S_\infty(M)$.*

This corollary is false see Corrigendum in <http://arxiv.org/abs/1104.0892>

We include the following proposition.

PROPOSITION 2.4. *Let M be a Riemannian symmetric space of non positive curvature without flat factor. Let $S = \mathbb{R}^N \times M$ be a symmetric space of non positive curvature with flat factor \mathbb{R}^N . If $G \subset \text{Iso}(S)$ is a connected non compact simple Lie group then $G \subset \text{Iso}(M)$.*

Proof. Let \mathfrak{g} be the Lie algebra of G . Then the projection $\pi : \mathfrak{g} \mapsto \text{Lie}(\text{Iso}(\mathbb{R}^N))$ is injective or trivial i.e. $\pi \equiv 0$. If π is injective then a further composition with the projection to $\mathfrak{so}(N)$ gives that \mathfrak{g} must carry a bi-invariant metric. So, \mathfrak{g} can not be simple and non compact. \square

Let G be a semisimple Lie group and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. A subspace $T \subset \mathfrak{p}$ is called a *Lie triple system* if $[T, [T, T]] \subset T$. It is well-known that there is a 1-1 correspondence between Lie triple systems T of \mathfrak{p} and totally geodesic submanifolds through the base point $[K] \in G/K$ (see [6]).

3. Minimal and totally geodesic orbits.

We will need the following proposition (see Lemma 3.1. in [4] or Proposition 5.5. in [1]).

PROPOSITION 3.1. *Let M be a Riemannian symmetric space of non positive curvature without flat factor and let $G \subset \text{Iso}(M)$ be a connected group of isometries. Assume that G has a totally geodesic orbit $G.p$. Then any other minimal orbit $G.q$ is also a totally geodesic submanifold of M . Moreover, if G is semisimple then the union of totally geodesic G -orbits T_G is a totally geodesic submanifold of M which is a Riemannian product $T_G = (G.p) \times A$ where A is a totally geodesic submanifold of M .*

Proof. Let $G.p$ be the totally geodesic orbit and let $G.q \neq \{q\}$ be another orbit. Let γ be a geodesic in M that minimizes the distance between q and $G.p$ (such geodesic do exists since totally geodesic submanifolds of M are closed and embedded). Eventually by changing the base point p by another in the orbit we may assume that $\gamma(0) = p$ and $\gamma(1) = q$. A simple computation using the Killing equation shows that $\dot{\gamma}(t)$ is perpendicular to $T_{\gamma(t)}(G.\gamma(t))$, for all t .

Let X be a Killing field in the Lie algebra of G such that $X.q \neq 0$ and let ϕ_s^X be the one-parameter group of isometries generated by X . Define $h : I \times \mathbb{R} \rightarrow M$ by $h_s(t) := \phi_s^X.\gamma(t)$. Note that $X.h_s(t) = \frac{\partial h}{\partial s}$ and that, for a fixed s , $h_s(t)$ is a geodesic.

Let $A_{\dot{\gamma}(t)}$ be the shape operator, in the direction of $\dot{\gamma}(t)$ of the orbit $G \cdot \gamma(t)$. Define $f(t) := -\langle A_{\dot{\gamma}(t)}(X \cdot \gamma(t)), X \cdot \gamma(t) \rangle = \langle \frac{D}{ds} \frac{\partial h}{\partial t}, X \cdot h_s(t) \rangle |_{s=0}$. Now a computation as in Lemma 3.1. in [4] or Proposition 5.5. in [1] implies that $\frac{d}{dt} f(t) \geq 0$. Since $f(0) = 0$, due to the fact that $G \cdot p$ is totally geodesic, we obtain that $f(1) = -\langle A_{\dot{\gamma}(1)}(X \cdot q), X \cdot q \rangle \geq 0$. Hence $A_{\dot{\gamma}(1)}$ is negative semidefinite. Since $G \cdot q$ is minimal, $\text{trace}(A_{\dot{\gamma}(1)}) = 0$, we get that $f(t) \equiv 0$. Thus, $\langle R(\dot{\gamma}(t), X \cdot \gamma(t))\dot{\gamma}(t), X \cdot \gamma(t) \rangle \equiv 0$ and $\nabla_{\dot{\gamma}(t)}(X \cdot \gamma(t)) \equiv 0$. Notice that the tangent spaces $T_{\gamma(t)}G \cdot \gamma(t)$ are parallel along $\gamma(t)$ in M . So the normal spaces $\nu_{\gamma(t)}G \cdot \gamma(t)$ are also parallel along $\gamma(t)$ in M . Since M is a symmetric space of non positive curvature the condition $\langle R(\dot{\gamma}(t), X \cdot \gamma(t))\dot{\gamma}(t), X \cdot \gamma(t) \rangle \equiv 0$ implies $R(\dot{\gamma}(t), X \cdot \gamma(t))(\cdot) \equiv 0$. Let $\eta(t) \in \nu_{\gamma(t)}G \cdot \gamma(t)$ be a parallel vector along $\gamma(t)$ and let X, Y two Killing vector fields in the Lie algebra of G . Then $\frac{d}{dt} \langle \nabla_X Y, \eta(t) \rangle = \langle \nabla_{\dot{\gamma}(t)} \nabla_X Y, \eta(t) \rangle = \langle \nabla_X \nabla_{\dot{\gamma}(t)} Y, \eta(t) \rangle + \langle R(\dot{\gamma}(t), X \cdot \gamma(t))(Y), \eta(t) \rangle \equiv 0$. Since $\langle \nabla_X Y, \eta(0) \rangle = 0$ we get that the G -orbits $G \cdot \gamma(t)$ are totally geodesic submanifolds of M . This show the first part.

For the second part let $K' := \text{Iso}(M)_p$ be the isotropy subgroup at $p \in M$ and let \mathfrak{k}' its Lie algebra. Let $\mathfrak{p}' \subset \text{Lie}(\text{Iso}(M))$ be such that $X \in \mathfrak{p}'$ iff $(\nabla X)_p = 0$. Thus, $\text{Lie}(\text{Iso}(M)) = \mathfrak{k}' \oplus \mathfrak{p}'$ is a Cartan decomposition of $\text{Lie}(\text{Iso}(M))$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$. Since $G \cdot p$ is totally geodesic in M we get that $\mathfrak{k} \subset \mathfrak{k}'$ and $\mathfrak{p} \subset \mathfrak{p}'$. Let $\alpha := \{Y \in \mathfrak{p}' : Y \perp \mathfrak{p} \text{ and } [Y, \mathfrak{p}] = 0\}$ which is a Lie triple system of \mathfrak{p}' . Moreover, $\mathfrak{n} := \mathfrak{p} \oplus \alpha$ is also a Lie triple system of \mathfrak{p}' . So, $N := \exp_p(\mathfrak{n}) = \exp_p(\mathfrak{p}) \times \exp_p(\alpha)$ is a G -invariant totally geodesic submanifold of M . Notice that (by construction) $N \subset T_G$.

Let $G \cdot q$ any other totally geodesic G -orbit. From the computation in the first part we get $R(\dot{\gamma}(t), X \cdot \gamma(t))(\cdot) \equiv 0$ which implies $\gamma'(0) \in \alpha$. This shows $T_G \subset N$. Then $N = T_G = (G \cdot p) \times A$ where $A := \exp_p(\alpha)$ is a totally geodesic submanifold of M associated to the Lie triple system α . \square

4. Karpelevich's Theorem for G a simple Lie group.

Here is the first step to prove Theorem 1.1.

THEOREM 4.1. *Let M be a Riemannian symmetric space of non positive curvature without flat factor. Then any connected, simple and non compact Lie subgroup $G \subset \text{Iso}(M)$ has a minimal orbit $G \cdot p \subset M$.*

Proof. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra $\mathfrak{g} := \text{Lie}(G)$ and let $K \subset G$ be the maximal compact subgroup associated to \mathfrak{k} . Let Σ be the set of fixed points of K . Notice that $\Sigma \neq \emptyset$ by Cartan's fixed point theorem. Since G is simple all G -orbits $G \cdot x$ through points in $x \in \Sigma$ are homothetic i.e. the Riemannian metric induced on $G \cdot x$ and $G \cdot y$ differ from a constant multiple for $x, y \in \Sigma$. Let $x_0 \in \Sigma$ be a point in Σ and let g_0 be the Riemannian metric on $G \cdot x_0 = G/K$ induced by the Riemannian metric $g = \langle \cdot, \cdot \rangle$ of M . So if $y \in \Sigma$ the Riemannian metric g_y on $G \cdot y$ is given by $g = \lambda(y) \cdot g_0$. Notice that if $X \in \mathfrak{p}$ is unitary at x_0 (i.e. $g_0(X(x_0), X(x_0)) = 1$) then $\lambda(y) = g(X(y), X(y)) = \|X(y)\|^2$. We claim that $\lambda(y)$ has a minimum in Σ . Indeed, if $y_n \rightarrow \infty \in S_\infty(\Sigma) \subset S_\infty(M)$ ($y_n \in \Sigma$) and $\lambda(y_n) \leq \text{const}$ then the monoparametric Lie group $\psi_t^X \subset G$ associated to any unitary $X \in \mathfrak{p}$ at $x_0 \in \Sigma$ must fix $\infty \in S_\infty(\Sigma) \subset S_\infty(M)$. Thus, since $X \in \mathfrak{p}$ is arbitrary and \mathfrak{p} generate \mathfrak{g} we get that $\infty \in S_\infty(\Sigma) \subset S_\infty(M)$ is a fixed point of G . This contradicts Corollary 2.3. So there exist $y_0 \in \Sigma$ such that λ has a minimum. Notice that the volume element Vol_y of an orbit $G \cdot y$ is given by $\lambda^{\frac{n}{2}} \text{Vol}_{x_0}$, where $n = \dim(G/K)$. Now a simple computation shows that the mean curvature vector of $G \cdot y_0$ vanish and we are done. \square

Now we are ready to prove Karpelevich's Theorem 1.1 for G a simple non compact Lie subgroup of $Iso(M)$.

THEOREM 4.2. *Let M be a Riemannian symmetric space of non positive curvature. Then any connected, simple and non compact Lie subgroup $G \subset Iso(M)$ has a totally geodesic orbit $G.p \subset M$.*

Proof. According to Proposition 2.4 we can assume that M has no flat factor. Let $i : (M, g) \hookrightarrow (\mathcal{P}, h)$ be a totally geodesic embedding as in Proposition 2.2. Notice that the pull-back metric i^*h can eventually differ (up to constant factors) from g on each irreducible De Rham factor of M . Anyway, totally geodesic submanifolds of (M, g) and (M, i^*h) are the same since totally geodesic submanifolds are defined in terms of the same Levi-Civita connection $\nabla^g = \nabla^{i^*h}$. Notice that G also acts by isometries on (M, i^*h) . Indeed, G can be also regarded as a subgroup of $Iso(\mathcal{P})$. Now Proposition 2.2 implies that G has a totally geodesic orbit $G.p$ in \mathcal{P} . The above proposition shows that G has a minimal orbit $G.y_0$ in (M, i^*h) . Since the embedding $M \hookrightarrow \mathcal{P}$ is totally geodesic we get that the G -orbit $G.y_0$ is also a minimal submanifold of \mathcal{P} . Then Proposition 3.1 implies that $G.y_0$ is a totally geodesic submanifold of \mathcal{P} . Thus, $G.y_0$ is a totally geodesic submanifold of (M, i^*h) and so $G.y_0$ is also a totally geodesic submanifold of (M, g) . \square

5. Karpelevich's Theorem.

Let $G \subset Iso(M)$ be a semisimple, connected Lie group. Then the Lie algebra $\mathfrak{g} = Lie(G) = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a sum of a simple Lie algebra \mathfrak{g}_1 and a semisimple Lie algebra \mathfrak{g}_2 . Due to Cartan's fixed point theorem we can assume that each simple factor of \mathfrak{g} is non compact. We are going to make induction on the number of simple factors of the semisimple Lie algebra \mathfrak{g} . Let G_1 (resp. G_2) be the simple Lie group associated to \mathfrak{g}_1 (resp. the semisimple Lie subgroup associated to \mathfrak{g}_2). Let $T_{G_1} \subset M$ be the union of the totally geodesic orbits of the simple subgroup G_1 acting on M . Notice that Theorem 4.2 implies that $T_{G_1} \neq \emptyset$ and Proposition 3.1 implies that $T_{G_1} = (G_1 \cdot p) \times A$ is a totally geodesic submanifold of M , where $G_1 \cdot p$ is a totally geodesic G_1 -orbit. Notice that G_2 acts on $T_{G_1} = (G_1 \cdot p) \times A$. Then \mathfrak{g}_2 (or eventually a quotient \mathfrak{g}_2 / \sim of it) acts on A . Since A is symmetric space of non positive curvature we get (by induction) that the semisimple subgroup G_2 (or eventually a quotient G_2 / \sim of it) has a totally geodesic orbit $S \subset A$. Then $(G_1 \cdot p) \times S$ is a totally geodesic orbit of G and this finish our proof of Karpelevich's Theorem 1.1.

References

1. DMITRI V. ALEKSEEVSKY and ANTONIO J. DI SCALA, 'Minimal homogeneous submanifolds of symmetric spaces', *Lie Groups and Symmetric Spaces: In Memory of F. I. Karpelevich*, Edited by: S. G. Gindikin, Rutgers University, New Brunswick, NJ. Series: American Mathematical Society Translations-Series 2, Volume: 210. (2003), 11-25.
2. CHARLES BOUBEL and ABDELGHANI ZEGHIB, 'Isometric actions of Lie subgroups of the Moebius group', *Nonlinearity*, 17(5):1677-1688, (2004).
3. MARTIN R. BRIDSON and ANDRÉ HAEFLIGER, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften 319. Springer-Verlag, Berlin, (1999).
4. ANTONIO J. DI SCALA and CARLOS OLMOS, 'The geometry of homogeneous submanifolds in hyperbolic space', *Mathematische Zeitschrift*, 237(1):199-209, (2001).
5. SIMON K. DONALDSON, 'Lie algebra theory without algebra', ArXiv:, math.DG/0702016v1, (2007).
6. SIGURDUR HELGASON, *Differential Geometry, Lie Groups, and Symmetric Spaces*. Academic Press, (1978).

7. F.I. KARPELEVICH, 'Surfaces of transitivity of semisimple group of motions of a symmetric space', *Soviet Math. Dokl.*, 93:401–404, (1953).
8. G. MOSTOW, 'Some new decomposition theorems for semi-simple groups', *Mem. Amer. Math. Soc.*, 14:31–54, (1955).
9. A. ONISHICK, *Lectures on real semisimple Lie algebras and their representations*, ESI Lectures in Mathematics and Physics, European Mathematical Society, (2004).

Antonio J. Di Scala
Dipartimento di Matematica,
Politecnico di Torino,
Corso Duca degli Abruzzi 24,
10129 Torino Italy
antonio.discal@polito.it

Carlos Olmos
Fa.M.A.F.,
Universidad Nacional de Córdoba,
Ciudad Universitaria,
5000 Cordoba, Argentina
olmos@mate.uncor.edu