Parallel submanifolds of complex projective space and their normal holonomy*

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Abstract

The object of this article is to compute the holonomy group of the normal connection of complex parallel submanifolds of the complex projective space. We also give a new proof of the classification of complex parallel submanifolds by using a normal holonomy approach. Indeed, we explain how these submanifolds can be regarded as the unique complex orbits of the (projectivized) isotropy representation of an irreducible Hermitian symmetric space. Moreover, we show how these important submanifolds are related to other areas of mathematics and theoretical physics. Finally, we state a conjecture about the normal holonomy group of a complete and full complex submanifold of the complex projective space.

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Key Words: normal holonomy group, symmetric submanifolds, parallel second fundamental form, normal bundle.

1 Introduction.

The study of the normal holonomy group, started by C. Olmos in [Ol1] (see also [BCO] for more details and applications), turned out to be a powerful tool for the study of submanifolds with simple geometric invariants, e.g. homogeneous submanifolds, isoparametric submanifolds and their generalizations [Ol4, DiOl]. In

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particular, normal holonomy methods were used by C. Olmos to give simpler and geometric proofs of Berger-Simons' theorems on holonomy [Ol2, Ol3].

The restricted normal holonomy group $\Phi_p^\perp$ of a complex submanifold $M \subset \mathbb{C}P^N$ at a point $p \in M$ acts on the normal space $N_p(M)$. Under suitable and very general conditions (see [AlDi] and Section 2) this action agrees with the isotropy action of an irreducible Hermitian symmetric space; i.e., the pair $(\Phi_p^\perp, N_p(M))$ is given by $(K, T_{[K]}G/K) = (K, \mathfrak{p})$, (with $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition) where G/K is an irreducible Hermitian symmetric space and the action coincides with the isotropy representation of G/K on $\mathfrak{p}$.

Symmetric submanifolds play a distinguished role among submanifolds with simple geometric invariants. They are analogous to symmetric spaces for submanifold theory. Indeed, they always come equipped with a symmetry at each point, namely the geodesic reflection in the corresponding normal submanifold. This implies that the second fundamental form is parallel. Actually, for a complex submanifold of $\mathbb{C}P^N$ being extrinsically symmetric is the same as having parallel second fundamental form (see [BCO, Proposition 9.3.1, page 256] for more details). For real space forms, symmetric submanifolds turn out to be orbits of the isotropy representation, the so called R-spaces. This is a classical result of D. Ferus [Fer] (see also [BCO, Chapter 3, Section 7]). Notice that these submanifolds are also related to the theory of Jordan algebras [Ber, p. 239].

In algebraic geometry, complex symmetric submanifolds of $\mathbb{C}P^N$ are called *characteristic projective subvarieties*. They are related with (a part of) the celebrated Borel-Weil theorem since they are the unique complex orbit of the action of a compact Lie group (see [BoWe], [GuSt, p. 166] and [Mok, p. 103, Remark]). Furthermore they are the main ingredient of a polarization-type argument used by N. Mok to prove his well-known rigidity theorems for higher rank Hermitian symmetric spaces [Mok, p. 111, Prop. 3]. In the theory of Jordan algebras, complex symmetric submanifolds of $\mathbb{C}P^N$ are described by means of *minimal tripotents,* see [Kaup, p. 579]. These submanifolds are also important in physics and chemistry. Namely, they are related to the so-called approximation of Hartree-Fock (e.g. Slater determinants, for details see [GuSt, p. 165]).

Complex symmetric submanifolds of $\mathbb{C}P^N$ were classified by Nakagawa-Takagi [NaTa]. Besides the standard techniques from representation theory (cf. [Take]), their classification depends upon the work of Calabi-Vesentini and Borel [CaVe, Borel]. Namely, they managed to give a link between the norm of the covariant derivatives of the second fundamental form of the canonical embeddings of Hermitian symmetric spaces and the eigenvalues of the curvature operator introduced by Nakano [Na].

The first aim of this paper is to compute the pairs $(\Phi_p^\perp, N_p(M))$ when M is a complex parallel submanifold of the projective space $\mathbb{C}P^N$. We are going to collect our results in the third column of Table 1 below. Besides the classification of the possible pairs $(\Phi_p^\perp, N_p(M))$ given in [AlDi], our method is based on Proposition 2.4 and the classification of complex parallel submanifolds.

Hermitian symmetric space G/K	M as complex K-orbit	Normal holonomy	Remarks
$\frac{E_7}{T^1 \cdot E_6}$	$\frac{E_6}{T^1 \cdot Spin_{10}}$	$\frac{SO(12)}{T^1 \cdot SO(10)}$	
$\frac{E_6}{T^1 \cdot Spin_{10}}$	$\frac{SO(10)}{U(5)}$	$\frac{U(6)}{U(5)}$	
$\frac{Sp(n+1)}{U(n+1)}$	$\mathbb{C}P^n$	$\frac{Sp(n)}{U(n)}$	Veronese
$Gr_2^+(\mathbb{R}^{n+2}) := \frac{SO(n+2)}{T^1 \cdot SO(n)}$	$Gr_2^+(\mathbb{R}^n)$	$\frac{U(2)}{U(1)}$	Quadrics
$\frac{SO(2n)}{U(n)}$	$Gr_2(\mathbb{C}^n)$	$\frac{SO(2(n-2))}{U(n-2)}$	Plücker
$Gr_a(\mathbb{C}^{a+b}) := \frac{SU(a+b)}{S(U(a) \times U(b))}$	$\mathbb{C}P^{a-1} \times \mathbb{C}P^{b-1}$	$\frac{SU(a+b-2)}{S(U(a-1) \times U(b-1))}$	Segre

Table 1: Symmetric complex submanifolds $M \subset \mathbb{P}(T_{[K]}G/K)$. The space in the third column is the Hermitian symmetric space whose isotropy representation gives the normal holonomy action.

The second goal of this paper is to obtain the classification of complex parallel submanifolds of the complex projective space without making use of the work of Calabi-Vesentini and Borel [CaVe, Borel], as in the classical work of Nakagawa and Takagi [NaTa]. Indeed, such a classification is contained in the second column of Table 1. It turns out that complex parallel submanifolds of $\mathbb{C}P^n$ consist of the first canonical embeddings of rank two Hermitian symmetric spaces, the Veronese and Segre embeddings. Our approach (presented in Section 4) is based on holonomy techniques and the knowledge of the codimension of the canonical embeddings only. We will also explain how complex parallel submanifolds of $\mathbb{C}P^N$ can be regarded as the unique complex orbits of the (projectivized) isotropy representation of an irreducible Hermitian symmetric space. Notice the analogy with the above mentioned result of Ferus in the real setting.

Our main result is encoded in Table 1 and can be read as follows.

Let G/K be an irreducible Hermitian symmetric space from the complete list in the first column of Table 1. Then, the second column contains the unique complex *orbit of the K-action on the projective space $\mathbb{P}(T_{[K]}G/K)$. This K-orbit has parallel second fundamental form and all complex parallel submanifolds arise in this way. Moreover, the third column of the table contains the Hermitian symmetric space whose isotropy representation gives the normal holonomy action $(\Phi_p^\perp, N_p(M))$.*

In particular we prove the following

Theorem 1.1 *Let $M \subset \mathbb{C}P^N$ be a full (connected) complex submanifold with parallel second fundamental form. Then M is an open subset of the unique complex orbit of the (projectivized) isotropy representation of an irreducible Hermitian symmetric space G/K. Moreover, their normal holonomy actions $(\Phi_p^\perp, N_p(M))$ agree with the isotropy representations of the Hermitian symmetric spaces listed in the third column of Table 1.*

For higher canonical embeddings of Hermitian symmetric spaces we get

Theorem 1.2 *Let $f_d : G/K \hookrightarrow \mathbb{C}P^{N_d}$ be the $d-th$ canonical embedding of an irreducible Hermitian symmetric space. If $d > 2$ then the normal holonomy group is the full unitary group of the normal space.*

Motivated by the above theorem we propose the following extrinsic analog of Berger's theorem as a conjecture

Conjecture 1.3 *Let $M \hookrightarrow \mathbb{C}P^N$ be a complete (connected) and full (i.e. not contained in a proper hyperplane) complex submanifold. If the normal holonomy group is not the full unitary group, then M has parallel second fundamental form.*

Notice that if the above conjecture is true, then the realization problem of normal holonomy group of complex submanifolds of $\mathbb{C}P^N$ is solved. Namely, up to the isotropy representation of the exceptional $\dfrac{E_7}{T^1 \cdot E_6}$ any other isotropy representation of an irreducible Hermitian symmetric space can be obtained as a normal holonomy action. Recall that the realization problem of the normal holonomy group of submanifolds of the sphere was solved in [HeOl], up to eleven exceptions. Finally, Conjecture 1.3 can be regarded as the complex version of the conjecture posed in [Ol5]. Namely, *an irreducible and full homogeneous submanifold of the sphere, different from a curve, whose normal holonomy group is not transitive, must be an orbit of an s-representation.*

2 Preliminaries.

Throughout this paper by *complex submanifold M of $\mathbb{C}P^N$* we mean a holomorphic and isometric embedding $M \hookrightarrow \mathbb{C}P^N$, where $\mathbb{C}P^N$ carries the standard Fubini-

Study Kähler form of constant holomorphic curvature 1. We will always assume that M is connected.

We refer to [BCO] for the definitions of the normal bundle $N(M) \to M$, its normal connection and its holonomy group $\Phi_p^\perp$ at a point $p \in M$ (the so called *normal holonomy group*).

Recall that the shape operator of a complex submanifold anticommutes with the complex structure J, i.e., $A_\xi J = JA_\xi$, for any normal vector ξ. Moreover, the equation of Gauss yields the following expression for the holomorphic sectional curvature of M

$$(*) \qquad \frac{1}{2}\left(\langle X,Y\rangle^2 + \langle X,JY\rangle^2 + \|X\|^2\|Y\|^2\right) = \langle R_{X,JX}JY,Y\rangle + 2\|\alpha(X,Y)\|^2\,.$$

We say that $M \subset \mathbb{C}P^N$ is *full* if it is not contained in a hyperplane of $\mathbb{C}P^N$. Recall that the first normal space $N^1(M)$ is defined by $N^1(M) := \text{span}\,\{\alpha(X,Y)\}$.

The following result gives a sufficient condition for a submanifold of $\mathbb{C}P^N$ not to be full.

Theorem 2.1 [Ce], [ChO1] *Let M be a Kähler submanifold of $\mathbb{C}P^N$. If there exists a complex $\nabla^\perp$-parallel subbundle $V \neq 0$ of the normal bundle $N(M)$ such that $V \perp N^1(M)$, then M is non-full.*

Calabi rigidity theorem of complex submanifolds $M \hookrightarrow \mathbb{C}P^N$ [Cal] implies that isometric and holomorphic immersions are equivariant (see Subsection 2.3) Namely, any intrinsic isometry can be extended to the ambient space, i.e., to the projective space $\mathbb{C}P^N$. For a detailed explanation see [NaTa, p. 655, Theorem 4.3] or [Take].

Finally, let us recall the following well-known proposition (see e.g. [NaTa]).

Proposition 2.2 *Let M be a Kähler manifold not necessarily complete. Let $f : M \to \mathbb{C}P^N$ be a holomorphic and isometric immersion with parallel second fundamental form. Then M is a locally Hermitian symmetric space of compact type. Moreover, there exists a complete Hermitian symmetric space of compact type $\widetilde{M}$ and a holomorphic and isometric embedding $\widetilde{M} \overset{\tilde{f}}{\hookrightarrow} \mathbb{C}P^N$ with parallel second fundamental form such that $f = \tilde{f} \circ i$, where $i : M \to \widetilde{M}$ is the canonical inclusion.*

2.1 Normal holonomy.

The link between isotropy and holonomy is well-known for Riemannian symmetric spaces. For submanifolds with parallel second fundamental form there is a similar relationship between isotropy and normal holonomy group.

As we quoted in the introduction, it was proved in [AlDi] that, under suitable and very general conditions, the normal holonomy group acts on the normal space as the isotropy representation of an irreducible Hermitian symmetric space. All known examples (complete submanifolds, Kähler-Einstein submanifolds and manifolds with zero index of relative nullity) satisfy these conditions. Moreover, it is not known whether there exists a full complex submanifold whose normal holonomy group acts in reducible way. For completeness let us state here the following special case of the main result in [AlDi].

Theorem 2.3 [AlDi] *Let $M \subset \mathbb{C}P^N$ be a complete and full submanifold of $\mathbb{C}P^N$. Let $\Phi_p^\perp$ be the normal holonomy group at $p \in M$. Then, there exists an irreducible Hermitian symmetric space H/S such that $\Phi_p^\perp = S$. Indeed, $\dim_{\mathbb{C}}(N_p(M)) = \dim_{\mathbb{C}}(H/S)$ and the $\Phi_p^\perp$-action on $N_p(M)$ agrees with the isotropy representation of S on $T_{[S]}(H/S)$.*

The following proposition gives a nice application of the above theorem.

Proposition 2.4 *Let $M = G/K \hookrightarrow \mathbb{C}P^N$ be a full parallel submanifold where $M = G/K$ is a Hermitian symmetric space. Then the normal holonomy group $\Phi_p^\perp$ is homomorphic with K i.e. $\Phi_p^\perp = K/I$, where I is normal in K.*

Proof. Let us denote by $K^\perp$ the image of the restriction of the isotropy representation of K to the normal space $N_p(M)$, the so called *slice representation* . Thus, we are going to show that $K^\perp = \Phi_p^\perp$. First of all, notice that, since isometries preserve parallel transport, $K^\perp \subset \mathrm{Nor}\,(\Phi_p^\perp)$, where $\mathrm{Nor}\,(\Phi_p^\perp) \subset U(N_p(M))$ is the normalizer of $\Phi_p^\perp$ in the full unitary group $U(N_p(M))$. By the above Theorem 2.3 $\Phi_p^\perp$ is isomorphic to the isotropy S of an irreducible Hermitian symmetric space. Thus, $\mathrm{Nor}\,(\Phi_p^\perp) = \Phi_p^\perp$ and we get the inclusion $K^\perp \subset \Phi_p^\perp$.
The proof that $\Phi_p^\perp \subset K^\perp$ is similar to the one given in [Esch, p.7, Theorem 2]. Namely, any transvection of G/K, when extended to the ambient projective space, gives the parallel transport with respect to the normal connection. Then we can approximate any closed curve by a closed geodesic polygon. So we get a composition of isometries which belong to K and, by construction, to $\Phi_p^\perp$.
By taking limits and using the compactness of the involved groups we get the desired inclusion $\Phi_p^\perp \subset K^\perp$. □

As a sum up of the above propositions, we have the following theorem.

Theorem 2.5 *Let $M = G/K \hookrightarrow \mathbb{C}P^N$ be a Hermitian symmetric space embedded into $\mathbb{C}P^N$ with parallel second fundamental form. Assume also that the embedding is full. Then, there exists an irreducible Hermitian symmetric space H/S such that $\Phi_p^\perp = S = K/I$ where $I \subset K$ is a normal subgroup, $\dim_{\mathbb{C}}(N_p(M)) = \dim_{\mathbb{C}}(H/S)$ and $\Phi_p^\perp$ acts on $N_p(M)$ as the isotropy representation of S on $T_{[S]}(H/S)$.*

2.2 Parallel products.

The following result of Nakagawa and Takagi allows us to restrict to parallel embeddings of irreducible Hermitian symmetric spaces. Here we present an alternative proof.

Theorem 2.6 [NaTa, p. 664, Lemma 7.1] *Let M_i be an n_i-dimensional Kähler manifold $(i = 1, 2)$. If the Kähler manifold $M_1 \times M_2$ admits a Kähler immersion into $\mathbb{C}P^{n_1+n_2+p}$ with parallel second fundamental form, then M^i is locally $\mathbb{C}P^{n_i}$ $(i = 1, 2)$.*

Proof. We can assume that the immersion is full otherwise by using Theorem 2.1 we can reduce the codimension. So let us introduce the subbundles $\alpha_{1,1}$, $\alpha_{2,2}$ and $\alpha_{1,2}$ of the normal bundle $N(M)$ as follows:

$$\alpha_{i,j} := \{\alpha(TM_i, TM_j)\},$$

where α is the second fundamental form of M. Since the immersion $M_1 \times M_2 \subset \mathbb{C}P^{n_1+n_2+p}$ has parallel second fundamental form, we get $N(M) = \alpha_{1,1}+\alpha_{1,2}+\alpha_{2,2}$. Indeed, this is a consequence of Theorem 2.1 since the first normal space is parallel. Moreover, a simple application of the equation $(*)$ implies that the above sum is orthogonal, i.e. $N(M) = \alpha_{1,1} \oplus \alpha_{1,2} \oplus \alpha_{2,2}$. Observe that any $\alpha_{i,j}$ is a parallel subbundle with respect to the normal connection. Thus, since $\Phi_p^\perp$ acts irreducibly on $N_p(M)$ by Theorem 2.3, two of the three subbundles $\alpha_{i,j}$ must be trivial. It is not difficult to check that $\alpha_{1,2}$ cannot be trivial (see [AlDi, p. 202, Thm. 17] for details). Indeed, equation $(*)$ with $X \in TM^1$ and $Y \in TM^2$ yields a contradiction. Using again the equation $(*)$, we get that the curvature tensor of each factor agrees with the curvature tensor of the ambient space $\mathbb{C}P^{n_1+n_2+p}$ and we are done. □

2.3 Canonical embeddings.

Let us assume now that $M = G/K$ is an irreducible Hermitian symmetric space and let $f : M \hookrightarrow \mathbb{C}P^N$ be a full holomorphic and isometric embedding. From Calabi rigidity theorem it follows that the embedding $f : M \hookrightarrow \mathbb{C}P^N$ is G-equivariant (see [NaTa, p. 655, Theorem 4.3] or [Take]). Then, from Elie Cartan's work, such embeddings $G/K \hookrightarrow \mathbb{C}P^N$ are well-known and are called *canonical embeddings*.

They can be constructed by means of the representation theory of the simple group G through the so-called Borel-Weil construction (see [BoWe, Take]), which holds, more generally, for homogeneous Kähler manifolds and can be summarized as follows.

Let d be a positive integer and $\rho : G^{\mathbb{C}} \to \mathfrak{gl}(\mathbb{C}^{N_d+1})$ the irreducible representation of the complexification $G^{\mathbb{C}}$ of G with highest weight $d\Lambda_j$, where Λ_j is the fundamental weight corresponding to the simple root α_j. Let p be a highest weight vector corresponding to $d\Lambda_j$. Then the action of $G^{\mathbb{C}}$ on $\mathbb{C}^{N_d+1}$ induces a unitary representation of G whose orbit of the highest weight vector p in $\mathbb{C}P^{N_d}$ yields a full holomorphic embedding f_d of $M = G/K$ into $\mathbb{C}P^{N_d}$, called the *d-th canonical embedding* of M into a complex projective space.

The submanifold M of $\mathbb{C}P^{N_d}$ is the unique complex orbit of the action of G on $\mathbb{C}P^{N_d}$ (or equivalently, the unique compact orbit of the $G^{\mathbb{C}}$-action). The dimension N_d can be calculated explicitly by means of the Weyl's dimension formula. The induced metric on $M \subset \mathbb{C}P^{N_d}$ is Kähler-Einstein.

3 Normal holonomy of parallel submanifolds.

In this section we are going to prove the last sentence of Theorem 1.1 and later Theorem 1.2. The proof of the first part of Theorem 1.1 will be given in Section 4.

Proof of the last part of Theorem 1.1. We compute the third column of Table 1 using Theorem 2.5.

Let us start with the first line, explaining our method. Namely, let us focus on the 1−st canonical embedding $\dfrac{E_6}{T^1 \cdot Spin_{10}} \hookrightarrow \mathbb{C}P^{26}$. By Theorem 2.5, the normal holonomy group $\Phi_p^{\perp}$ is a quotient of $T^1 \cdot Spin_{10}$ whose action on the 10-dimensional normal space agrees with the isotropy representation of a 10-dimensional irreducible Hermitian space. Looking for such an irreducible Hermitian symmetric space (see the first column of Table 1) we see that the only possibility is $\dfrac{SO(12)}{T^1 \cdot SO(10)}$. Thus, $\Phi_p^{\perp}$ acts on $N_p(M)$ as the isotropy representation of $\dfrac{SO(12)}{T^1 \cdot SO(10)}$.

The computation for the second line is similar. Indeed, by Theorem 2.5, the normal holonomy group $\Phi_p^{\perp}$ of the embedding $\dfrac{SO(10)}{U(5)} \hookrightarrow \mathbb{C}P^{15}$ is a quotient of $U(5)$ whose action on the 5-dimensional normal space agrees with a isotropy representation of a 5-dimensional irreducible Hermitian space. Looking for such an irreducible Hermitian symmetric space, the only possibility is $\dfrac{U(6)}{U(5)}$. Thus, $\Phi_p^{\perp}$ acts on $N_p(M)$ as the isotropy representation of $\dfrac{U(6)}{U(5)}$, i.e. as the standard representation of $U(5)$ on $\mathbb{C}^5$.

For the next four classical cases (i.e. the Veronese, the Quadrics, etc) a similar analysis, based on Theorem 2.5, can be done in order to compute the third column. This completes the proof of the last sentence in Theorem 1.1. □

Proof of Theorem 1.2. It is enough to prove that the first normal space does not agree with the full normal space. We will actually show

$$(**)\qquad \dim_{\mathbb{C}} N_p > \frac{m(m+1)}{2}\,, \qquad m = \dim_{\mathbb{C}}(G/K)\,,$$

which implies the above assertion, since the dimension of the first normal space is smaller or equal than $\frac{m(m+1)}{2}$. Indeed, if the first normal space has a complement, any vector in this complement belongs to the nullity of the adapted normal curvature tensor $\tilde{\mathcal{R}}$ (see [AlDi, Ol1]). This is absurd, since the curvature tensor of a Hermitian symmetric space has no nullity, so that the normal holonomy must be transitive on the unit sphere of the normal space.

The inequality (**) is clear for the canonical embeddings of the projective space, namely the rank one case. Assume therefore that rank $G/K > 1$. Observe that the embeddings f_d factor through the Veronese embeddings and the first canonical embedding, i.e., $f_d = \mathrm{Ver} \circ f_1$, where $\mathrm{Ver} : \mathbb{C}P^{N_1} \to \mathbb{C}P^{N_d}$ is the Veronese embedding (see [NaTa, p.659] or [Take, Section 3]). Then, the dimension of the normal space of f_d is greater than $\frac{N_1(N_1+1)}{2}$. Now recall that any canonical embedding is full. Thus, $m < N_1$ and

$$\dim_{\mathbb{C}} N_p > \frac{N_1(N_1+1)}{2} > \frac{m(m+1)}{2}\,.$$

□

Remark 3.1 By using the Weyl's formula [Take, p.189, Remark 2.3] to compute the dimension N of the target projective space $\mathbb{C}P^N$, it is possible to extend the above proof to an arbitrary immersion of a homogeneous Kähler manifold M. Namely, it is enough to check that the codimension of the immersion $M \to \mathbb{C}P^N$ is bigger than $\frac{n(n+1)}{2}$, $n = \dim_{\mathbb{C}}(M)$ to conclude that the normal holonomy group is the full unitary group $U(N-n)$. This observation is another motivation of Conjecture 1.3.

4 Complex parallel submanifolds.

The goal of this section is to simplify the arguments in the classical article [NaTa]. Namely, we are going to avoid the use of the eigenvalues of the curvature operator computed by Calabi-Vesentini [CaVe] and Borel [Borel] which were strongly used in [NaTa].

Here we are going to give a direct proof of the following theorem.

Theorem 4.1 *Let $f_d : G/K \to \mathbb{C}P^{N_d}$ be the d-th canonical embedding of an irreducible Hermitian symmetric space G/K. Assume that the embedding has parallel* *second fundamental form. Then, if f_d is not the Veronese embedding, $f_d = f_1$, that is to say f_d is the first canonical embedding.*

Proof. As we remarked in the proof of Theorem 1.2, the embeddings f_d can be described in terms of the Veronese embeddings and the first canonical embedding. Namely, $f_d = \mathrm{Ver} \circ f_1$, where $\mathrm{Ver} : \mathbb{C}P^{N_1} \to \mathbb{C}P^{N_d}$ is the Veronese embedding. Then the codimension of the embedding f_d is greater than $\frac{N_1(N_1+1)}{2}$ (and one has equality if any only if f_d is the Veronese embedding). Thus, if the codimension of $f_d(G/K)$ is one then f_d is the first canonical embedding of $Gr_2^+(\mathbb{R}^n)$ i.e. a complex quadric. Thus, we can assume that the codimension is greater than one and that f_d is not the Veronese embedding.
Now recall that any canonical embedding is full. Let $n = \dim_{\mathbb{C}}(G/K)$ be the complex dimension of G/K. Thus, $n < N_1$ and we get that the dimension of the first normal space is smaller or equal to $\frac{N_1(N_1+1)}{2}$. Thus, if $d > 1$ then f_d cannot be full since the first normal space is invariant by parallel transport in the normal connection and thus agrees with the normal space, by reduction of the codimension (i.e. by Theorem 2.1). This shows that $f_d = f_1$ and we are done. □

According to the above theorem, in order to get the embeddings of Hermitian spaces with parallel second fundamental form we have to study the first canonical embeddings only. The following theorem gives a sharp description.

Theorem 4.2 *Assume that the first canonical embedding of an irreducible Hermitian symmetric space M of higher rank has parallel second fundamental form. Then* rank $(M) = 2$.

Proof. According to Theorem 2.5, if $M = G/K \hookrightarrow \mathbb{C}P^N$ has parallel second fundamental form then K (or a quotient S) must act on the normal space N_p as the isotropy of an irreducible Hermitian symmetric space. From the classification of irreducible Hermitian symmetric space we will show that if rank $(M) > 2$ then there is no irreducible Hermitian symmetric space H/S of dimension $\dim(N_p)$. This will follow from a case by case analysis on the list in the first column of Table 2.
So, let us start with the rank 3 exceptional Hermitian symmetric space $\dfrac{E_7}{T^1 \cdot E_6}$.
Notice that the codimension of its first canonical embedding is 28 (see the third column of Table 2 constructed using [NaTa, p. 654]). Thus, a simple inspection on the second column of Table 2 implies that there is no irreducible 28-dimensional Hermitian symmetric space whose isotropy is a quotient of $T^1 \cdot E_6$. Then, the first canonical embedding of $\dfrac{E_7}{T^1 \cdot E_6}$ does not have parallel second fundamental form.
Going on on the list, $\dfrac{E_6}{T^1 \cdot Spin_{10}}$ is of rank 2, so we do not need consider it.

Hermitian symmetric space G/K	$\dim_{\mathbb{C}} G/K$	Codimension of its first canonical embedding	Rank of G/K
$\dfrac{E_7}{T^1 \cdot E_6}$	27	28	3
$\dfrac{E_6}{T^1 \cdot Spin_{10}}$	16	10	2
$\dfrac{Sp(n)}{U(n)}$	$\dfrac{n(n+1)}{2}$	$\begin{pmatrix} 2n \\ n \end{pmatrix} - \begin{pmatrix} 2n \\ n-2 \end{pmatrix} - 1 - \dfrac{n(n+1)}{2}$	n
$\dfrac{SO(n+2)}{T^1 \cdot SO(n)}$	n	1	2
$\dfrac{SO(2n)}{U(n)}$	$\dfrac{n(n-1)}{2}$	$2^{n-1} - \dfrac{n(n-1)}{2} - 1$	$[n/2]$
$\dfrac{SU(a+b)}{S(U(a) \times U(b))}$	ab	$\begin{pmatrix} a+b \\ b \end{pmatrix} - ab - 1$	$\min(a,b)$

Table 2: Hermitian symmetric spaces, their dimensions, ranks and the codimension of their first canonical embedding.

Let us consider further $\dfrac{Sp(n)}{U(n)}$: the codimension of its first canonical embedding is $h(n) = \begin{pmatrix} 2n \\ n \end{pmatrix} - \begin{pmatrix} 2n \\ n-2 \end{pmatrix} - 1 - \dfrac{n(n+1)}{2}$. There are two candidates for Hermitian symmetric spaces whose isotropy is $U(n)$: $\dfrac{Sp(n)}{U(n)}$ and $\dfrac{SO(2n)}{U(n)}$. However, some computations show that dimension cannot be equal to h, for any n.
We can skip the case of $\dfrac{SO(n+2)}{T^1 \cdot SO(n)}$, since its rank is two.
The Hermitian symmetric space $\dfrac{SO(2n)}{U(n)}$ has first canonical embedding of codimension $2^{n-1} - \dfrac{n(n-1)}{2} - 1$. Again Hermitian symmetric spaces whose isotropy is $U(n)$ are given by $\dfrac{Sp(n)}{U(n)}$ and $\dfrac{SO(2n)}{U(n)}$, but their dimensions cannot equal $2^{n-1} - \dfrac{n(n-1)}{2} - 1$ for any n.

Finally, for $\dfrac{SU(a+b)}{S(U(a)\times U(b))}$ the codimension of its first canonical embedding is $\begin{pmatrix} a+b \\ b \end{pmatrix} - ab - 1$. Hermitian symmetric spaces whose isotropy is a quotient of $S(U(a)\times U(b))$ are $\dfrac{Sp(n)}{U(n)}$, $\dfrac{SO(2n)}{U(n)}$ with $n=a$ or $n=b$ and $\dfrac{SU(a+b)}{S(U(a)\times U(b))}$, but none of them fits. □

We are now going to show that the converse of Theorem 4.2 also holds. Namely,

Theorem 4.3 *The first canonical embedding of an irreducible Hermitian symmetric space M of rank two has parallel second fundamental form.*

Proof. A first proof can be obtained by an explicit construction of an extrinsic symmetry at each point of M. A second proof was given in [Mok, p.245] by means of a pinching theorem due to A. Ros [Ros1]. Another proof is in Nakagawa-Takagi's paper [NaTa]. □

The above theorem is also a consequence of the following conceptual argument. Indeed, one can see that the list of submanifolds given by the images of the first canonical embedding of an irreducible Hermitian symmetric space of rank two agrees with the list of the unique complex orbits of the isotropy action on the projective space $\mathbb{P}(T_{[K]}G/K)$ i.e. the second column of Table 1. Thus, we just need to show the following proposition.

Proposition 4.4 *The complex orbit of the (projectivized) isotropy representation of an irreducible Hermitian symmetric space G/K has parallel second fundamental form.*

Proof. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. Then, the isotropy action agrees with the restriction to $\mathfrak{p} \cong \mathbb{C}^{N+1}$ of the adjoint action of K.
Let $M = K/K_0$ the unique complex orbit of the isotropy action on the projective space $\mathbb{C}P^N$ and suppose that M is an irreducible Hermitian symmetric space. Let $\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{m}$ be a Cartan decomposition of $\mathfrak{k}$. Then M is isometrically embedded in $\mathbb{C}P^N$ if it is endowed with a multiple of the opposite of the Killing form on $\mathfrak{m}$ and $\mathfrak{m} \cong T_{[p]}M$. A similar computation as in [BCO, Lemma 4.1.5] yields

$$[\mathfrak{m}, N_{[p]}M] \subseteq T_{[p]}M \,.$$

This implies that M has parallel second fundamental form, since it is Hermitian symmetric (cf. [BCO, Lemma 7.2.6]). □

This completes the proof of Theorem 1.1.

Remark 4.5 Observe that Proposition 4.4 can be also proven by using ideas in [Mok, Chapter 6] about characteristic varieties. Furthermore, these orbits can also be described by using the Jordan Algebra approach to Hermitian symmetric spaces, namely, in terms of the so called *tripotents* [Roos] and [Kaup, p. 579].

Remark 4.6 Finally, we recall a problem proposed by A. Ros (Problem 5 in [Ros2, p. 272]), namely to characterize the symmetric submanifolds of $\mathbb{C}P^N$ without using metric notions (that is to say, changing "holomorphic isometry" with "holomorphic transformation"). We remark that Ros' problem could be related to the geometric characterization of the so called *Helwig spaces* see [Ber, p. 58, Problem II.4.5.].

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