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A note on primes in short intervals

Danilo Bazzanella

Abstract. This paper is concerned with the number of primes in short intervals. We present a method to use mean value estimates for the number of primes in $(x, x+x^\theta]$ to obtain the asymptotic behavior of $\psi(x+x^\theta)-\psi(x)$. The main idea is to use the properties of the exceptional set for the distribution of primes in short intervals.

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1. Introduction

This paper is concerned with the asymptotic formula

$$\psi(x+x^\theta)-\psi(x)\sim x^\theta \quad x\rightarrow\infty, \quad (1.1)$$

which estimates the number of primes in the interval $(x, x+x^\theta]$. The prime number theorem implies that (1.1) holds with $\theta\geq 1$. An interval $(x, x+x^\theta]$ with $\theta<1$ is called a short interval. The best known unconditional result about the constant θ is due to Huxley [4] and asserts that (1.1) holds for $\theta>7/12$, which was slightly by Heath-Brown [3] to $7/12-o(1)$. Assuming some well-known hypotheses we can handle smaller θ . In particular under the assumption of the Lindelöf hypothesis, which states that the Riemann Zeta-function satisfies

$$\zeta(\sigma+it)\ll t^\eta \quad (\sigma\geq\frac{1}{2}, t\geq 2),$$

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for any $\eta > 0$, Ingham proved that (1.1) holds for $\theta > 1/2$, see [5].

We can relax our request and investigate if (1.1) holds for “almost all” x . By this we mean that the measure of $x \in [X, 2X]$ for which (1.1) does not hold is $o(X)$. Huxley’s zero density estimate [4], in conjunction with the method of Selberg [7], shows that (1.1) holds for almost all x with $\theta > 1/6$, which was slightly by Zaccagnini [9] to $1/6 - o(1)$.

We observe that a suitable mean value estimate is sufficient to get results for almost all x . Moreover we can use mean value estimates to provide a bound for the exceptional set for the distribution of primes in short intervals but it is never sufficient to prove directly that (1.1) holds for all values of x .

The aim of this paper is to present a method to use a mean value estimate for the number of primes in $(x, x + x^\theta]$ to obtain the asymptotic behavior of $\psi(x + x^\theta) - \psi(x)$.

Our results will depend upon the following hypothesis about a four-power mean value for the Chebyshev’s function $\psi(x)$ in short intervals.

Hypothesis. *There exist a constant X_0 and a function $\Delta(y, T)$ such that, for every $\beta < 1/2$ and $\varepsilon > 0$, we have*

$$\int_X^{2X} \left| \psi\left(y + \frac{y}{T}\right) - \psi(y) - \frac{y}{T} + \Delta(y, T) \right|^4 dy \ll X^{4+\varepsilon} T^{-3} \quad (1.2)$$

and

$$\Delta(y, T) \ll \frac{y}{T \ln y} \quad (1.3)$$

uniformly for $X \geq X_0$, $X^{5/12} \leq T \leq X^\beta$ and $X \leq y \leq 2X$.

As noted above it is known that the asymptotic formula (1.1) holds for $\theta \geq 7/12$.

Our hypothesis essentially says that there are not too many exceptions to the asymptotic formula (1.1), with $1/2 < \theta < 7/12$. Our result is the following.

Theorem 1. *Assume the above hypothesis. Then for every $\theta > 1/2$ the intervals $[x, x + x^\theta]$ contain the expected number of primes for $x \rightarrow \infty$.*

We remark that our hypothesis is weaker than the Lindelöf hypothesis, see Lemma 2, and then we get the following result of Ingham as a corollary.

Corollary. *Assume the Lindelöf hypothesis and let $\theta > 1/2$. The intervals $[x, x + x^\theta]$ contain the expected number of primes for $x \rightarrow \infty$.*

2. The basic lemmas

The first lemma is a result about the structure of the exceptional set for the asymptotic formula (1.1). Let X be a large positive number, $\delta > 0$ and let $|\cdot|$ denote the modulus of a complex number or the Lebesgue measure of a set. We define

$$E_\delta(X, \theta) = \{X \leq x \leq 2X : |\psi(x + x^\theta) - \psi(x) - x^\theta| \geq \delta x^\theta\}.$$

It is clear that (1.1) holds if and only if for every $\delta > 0$ there exists $X_0(\delta)$ such that $E_\delta(X, \theta) = \emptyset$ for $X \geq X_0(\delta)$. Hence for small $\delta > 0$, X tending to ∞ , the set $E_\delta(X, \theta)$ contains the exceptions, if any, to the expected asymptotic formula for the number of primes in short intervals. Moreover, we observe that

$$E_\delta(X, \theta) \subset E_{\delta'}(X, \theta) \quad \text{if } 0 < \delta' < \delta.$$

The following lemma provides the basic structure of the exceptional set $E_\delta(X, \theta)$.

Lemma 1. *Let $0 < \theta < 1$, X be sufficiently large, $0 < \delta' < \delta$ with $\delta - \delta' \geq \exp(-\sqrt{\log X})$. If $x_0 \in E_\delta(X, \theta)$ then $E_{\delta'}(X, \theta)$ contains the interval $[x_0 - cX^\theta, x_0 + cX^\theta] \cap [X, 2X]$, where $c = (\delta - \delta')\theta/5$. In particular, if $E_\delta(X, \theta) \neq \emptyset$ then*

$$|E_{\delta'}(X, \theta)| \gg_\theta (\delta - \delta')X^\theta.$$

This first lemma essentially says that if we have a single exception in $E_\delta(X, \theta)$, with a fixed δ , then we necessarily have an interval of exceptions in $E_{\delta'}(X, \theta)$, with δ' little smaller than δ . The interesting consequence of this lemma is that we can use a suitable bound for the exceptional set to prove the non-existence of the exceptions.

The second lemma concerns the conditional estimate for the four-power mean value of the function $\psi(y)$.

Lemma 2. *Assume the Lindelöf hypothesis and let $\varepsilon > 0$. Then there exists a function $\Delta(y, T)$ such that for every $\varepsilon > 0$ we have*

$$\int_X^{2X} \left| \psi\left(y + \frac{y}{T}\right) - \psi(y) - \frac{y}{T} + \Delta(y, T) \right|^4 dy \ll X^{4+\varepsilon} T^{-3}$$

and

$$\Delta(y, T) \ll \frac{y}{T \ln y}$$

uniformly for $X \geq 2, 1 \leq T \leq X$ and $X \leq y \leq 2X$.

The Lemma 2 implies that our hypothesis is weaker than the Lindelöf hypothesis.

Lemma 1 is part (i) of Theorem 1 of Bazzanella and Perelli, see [2], and Lemma 2 is Lemma B of Yu, see [8].

3. Proof of the Theorem

Our theorem asserts that (1.1) holds with $\theta > 1/2$. For $\theta > 7/12$ the result follows unconditionally by Huxley, see [4], and then we consider only $1/2 < \theta \leq 7/12$. In order to prove the theorem we assume that (1.1) does not hold. Then there exists $\delta_0 > 0$ and a sequence $X_n \rightarrow \infty$ such that

$$|\psi(X_n + X_n^\theta) - \psi(X_n) - X_n^\theta| \geq \delta_0 X_n^\theta.$$

Using the above definition of the exceptional set we have then $X_n \in E_{\delta_0}(X_n, \theta)$.

The use of Lemma 1 with $\delta' = \delta_0/2$ leads to

$$|E_{\delta'}(X_n, \theta)| \gg X_n^\theta. \quad (3.1)$$

On the other hand, assuming our hypothesis, we can get a bound for $|E_{\delta'}(X_n, \theta)|$. To perform this, given any $\varepsilon > 0$, we subdivide the interval $[X, 2X]$ into $\ll X^\varepsilon$ intervals of type $I_j = [X_j, X_j + Y]$ with $X \leq X_j < 2X$ and $Y \ll X^{1-\varepsilon}$. For every $y \in E_{\delta'}(X, \theta)$ we have

$$|\psi(y + y^\theta) - \psi(y) - y^\theta| \gg X^\theta,$$

and then

$$\begin{aligned} |E_{\delta'}(X, \theta)| X^{4\theta} &\ll \int_{E_{\delta'}(N, \theta)} |\psi(y + y^\theta) - \psi(y) - y^\theta|^4 dy \\ &= \sum_j \int_{E_{\delta'}^j(N, \theta)} |\psi(y + y^\theta) - \psi(y) - y^\theta|^4 dy, \end{aligned} \quad (3.2)$$

where $E_{\delta'}^j(X, \theta) = E_{\delta'}(X, \theta) \cap [X_j, X_j + Y]$. Our hypothesis asserts that for X sufficiently large and suitable values of T there exists a function $\Delta(y, T)$ which satisfies (1.2) and (1.3).

Let $T_j = X_j^{1-\theta}$ and let $\Delta_j(y, T_j)$ the functions which satisfy the conditions (1.2) and (1.3) for every j . Applying the Brunn-Titchmarsh inequality we can deduce

$$\left(\psi(y + y^\theta) - \psi(y) - y^\theta \right) - \left(\psi\left(y + \frac{y}{T_j}\right) - \psi(y) - \frac{y}{T_j} + \Delta_j(y, T_j) \right) \ll \frac{y}{T_j \log X},$$

for every j and $y \in E_{\delta'}^j(X, \theta)$, and then from (3.2) it follows that

$$\begin{aligned} |E_{\delta'}(X, \theta)| X^{4\theta} &\ll \sum_j \int_{E_{\delta'}^j(X, \theta)} \left| \psi\left(y + \frac{y}{T_j}\right) - \psi(y) - \frac{y}{T_j} + \Delta_j(y, T_j) \right|^4 dy \\ &\leq \sum_j \int_X^{2X} \left| \psi\left(y + \frac{y}{T_j}\right) - \psi(y) - \frac{y}{T_j} + \Delta_j(y, T_j) \right|^4 dy. \end{aligned}$$

Moreover our hypothesis implies that for every $\varepsilon > 0$ we have

$$|E_{\delta'}(X, \theta)| \ll X^{-4\theta} \sum_j \int_X^{2X} \left| \psi\left(y + \frac{y}{T_j}\right) - \psi(y) - \frac{y}{T_j} + \Delta_j(y, T_j) \right|^4 dy \ll X^{1-\theta+\varepsilon},$$

and this leads to

$$|E_{\delta'}(X_n, \theta)| \ll X_n^{1-\theta+\varepsilon}, \quad (3.3)$$

for n sufficiently large and for every $1/2 < \theta \leq 7/12$.

For X_n sufficiently large, we have a contradiction between (3.1) and (3.3), and this completes the proof of the theorem.

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