A note on primes in short intervals

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Danilo Bazzanella

Abstract. This paper is concerned with the number of primes in short intervals. We present a method to use mean value estimates for the number of primes in $(x, x+x^\theta]$ to obtain the asymptotic behavior of $\psi(x+x^\theta)-\psi(x)$. The main idea is to use the properties of the exceptional set for the distribution of primes in short intervals.

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1. Introduction

This paper is concerned with the asymptotic formula

$$\psi(x + x^\theta) - \psi(x) \sim x^\theta \qquad x \to \infty, \tag{1.1}$$

which estimates the number of primes in the interval $(x, x + x^\theta]$. The prime number theorem implies that (1.1) holds with $\theta \geq 1$. An interval $(x, x + x^\theta]$ with $\theta < 1$ is called a short interval. The best known unconditional result about the constant θ is due to Huxley [4] and asserts that (1.1) holds for $\theta > 7/12$, which was slightly by Heath-Brown [3] to $7/12 - o(1)$. Assuming some well-known hypotheses we can handle smaller θ. In particular under the assumption of the Lindelöf hypothesis, which states that the Riemann Zeta-function satisfies

$$\zeta(\sigma + it) \ll t^\eta \quad (\sigma \geq \frac{1}{2}, t \geq 2),$$

for any $\eta > 0$, Ingham proved that (1.1) holds for $\theta > 1/2$, see [5].
We can relax our request and investigate if (1.1) holds for "almost all" x. By this we mean that the measure of $x \in [X, 2X]$ for which (1.1) does not hold is $o(X)$. Huxley's zero density estimate [4], in conjunction with the method of Selberg [7], shows that (1.1) holds for almost all x with $\theta > 1/6$, which was slightly by Zaccagnini [9] to $1/6 - o(1)$.
We observe that a suitable mean value estimate is sufficient to get results for almost all x. Moreover we can use mean value estimates to provide a bound for the exceptional set for the distribution of primes in short intervals but it is never sufficient to prove directly that (1.1) holds for all values of x.
The aim of this paper is to present a method to use a mean value estimate for the number of primes in $(x, x + x^\theta]$ to obtain the asymptotic behavior of $\psi(x + x^\theta) - \psi(x)$.
Our results will depend upon the following hypothesis about a four-power mean value for the Chebyshev's function $\psi(x)$ in short intervals.

Hypothesis. *There exist a constant X_0 and a function $\Delta(y, T)$ such that, for every $\beta < 1/2$ and $\varepsilon > 0$, we have*

$$\int_X^{2X} \left|\psi\left(y + \frac{y}{T}\right) - \psi(y) - \frac{y}{T} + \Delta(y, T)\right|^4 dy \ll X^{4+\varepsilon} T^{-3} \tag{1.2}$$

and

$$\Delta(y, T) \ll \frac{y}{T \ln y} \tag{1.3}$$

uniformly for $X \geq X_0, X^{5/12} \leq T \leq X^\beta$ and $X \leq y \leq 2X$.

As noted above it is known that the asymptotic formula (1.1) holds for $\theta \geq 7/12$. Our hypothesis essentially says that there are not too many exceptions to the asymptotic formula (1.1), with $1/2 < \theta < 7/12$. Our result is the following.

Theorem 1. *Assume the above hypothesis. Then for every $\theta > 1/2$ the intervals $[x, x + x^\theta]$ contain the expected number of primes for $x \to \infty$.*

We remark that our hypothesis is weaker than the Lindelöf hypothesis, see Lemma 2, and then we get the following result of Ingham as a corollary.

Corollary. *Assume the Lindelöf hypothesis and let $\theta > 1/2$. The intervals $[x, x + x^\theta]$ contain the expected number of primes for $x \to \infty$.*

2. The basic lemmas

The first lemma is a result about the structure of the exceptional set for the asymptotic formula (1.1). Let X be a large positive number, $\delta > 0$ and let $|\ \ |$ denote the modulus of a complex number or the Lebesgue measure of a set. We define

$$E_\delta(X,\theta) = \{X \leq x \leq 2X : |\psi(x+x^\theta) - \psi(x) - x^\theta| \geq \delta x^\theta\}.$$

It is clear that (1.1) holds if and only if for every $\delta > 0$ there exists $X_0(\delta)$ such that $E_\delta(X,\theta) = \emptyset$ for $X \geq X_0(\delta)$. Hence for small $\delta > 0$, X tending to ∞, the set $E_\delta(X,\theta)$ contains the exceptions, if any, to the expected asymptotic formula for the number of primes in short intervals. Moreover, we observe that

$$E_\delta(X,\theta) \subset E_{\delta'}(X,\theta) \quad \text{if} \quad 0 < \delta' < \delta.$$

The following lemma provides the basic structure of the exceptional set $E_\delta(X,\theta)$.

Lemma 1. *Let $0 < \theta < 1$, X be sufficiently large, $0 < \delta' < \delta$ with $\delta - \delta' \geq \exp(-\sqrt{\log X})$. If $x_0 \in E_\delta(X,\theta)$ then $E_{\delta'}(X,\theta)$ contains the interval $[x_0 - cX^\theta, x_0 + cX^\theta] \cap [X, 2X]$, where $c = (\delta - \delta')\theta/5$. In particular, if $E_\delta(X,\theta) \neq \emptyset$ then*

$$|E_{\delta'}(X,\theta)| \gg_\theta (\delta - \delta')X^\theta.$$

This first lemma essentially says that if we have a single exception in $E_\delta(X,\theta)$, with a fixed δ, then we necessarily have an interval of exceptions in $E_{\delta'}(X,\theta)$, with δ' little smaller than δ. The interesting consequence of this lemma is that we can use a suitable bound for the exceptional set to prove the non-existence of the exceptions.

The second lemma concerns the conditional estimate for the four-power mean value of the function $\psi(y)$.

Lemma 2. *Assume the Lindelöf hypothesis and let $\varepsilon > 0$. Then there exists a function $\Delta(y,T)$ such that for every $\varepsilon > 0$ we have*

$$\int_X^{2X} \left|\psi\left(y + \frac{y}{T}\right) - \psi(y) - \frac{y}{T} + \Delta(y,T)\right|^4 dy \ll X^{4+\varepsilon}T^{-3}$$

and

$$\Delta(y,T) \ll \frac{y}{T \ln y}$$

uniformly for $X \geq 2, 1 \leq T \leq X$ and $X \leq y \leq 2X$.

The Lemma 2 implies that our hypothesis is weaker than the Lindelöf hypothesis.

Lemma 1 is part (i) of Theorem 1 of Bazzanella and Perelli, see [2], and Lemma 2 is Lemma B of Yu, see [8].

3. Proof of the Theorem

Our theorem asserts that (1.1) holds with $\theta > 1/2$. For $\theta > 7/12$ the result follows unconditionally by Huxley, see [4], and then we consider only $1/2 < \theta \le 7/12$. In order to prove the theorem we assume that (1.1) does not hold. Then there exists $\delta_0 > 0$ and a sequence $X_n \to \infty$ such that

$$\left|\psi(X_n + X_n^\theta) - \psi(X_n) - X_n^\theta\right| \ge \delta_0 X_n^\theta.$$

Using the above definition of the exceptional set we have then $X_n \in E_{\delta_0}(X_n, \theta)$. The use of Lemma 1 with $\delta' = \delta_0/2$ leads to

$$|E_{\delta'}(X_n, \theta)| \gg X_n^\theta. \tag{3.1}$$

On the other hand, assuming our hypothesis, we can get a bound for $|E_{\delta'}(X_n, \theta)|$. To perform this, given any $\varepsilon > 0$, we subdivide the interval $[X, 2X]$ into $\ll X^\varepsilon$ intervals of type $I_j = [X_j, X_j + Y]$ with $X \le X_j < 2X$ and $Y \ll X^{1-\varepsilon}$. For every $y \in E_{\delta'}(X, \theta)$ we have

$$|\psi(y + y^\theta) - \psi(y) - y^\theta| \gg X^\theta,$$

and then

$$\begin{aligned} |E_{\delta'}(X,\theta)| X^{4\theta} &\ll \int\limits_{E_{\delta'}(N,\theta)} |\psi(y + y^\theta) - \psi(y) - y^\theta)|^4 dy \\ &= \sum_j \int\limits_{E^j_{\delta'}(N,\theta)} |\psi(y + y^\theta) - \psi(y) - y^\theta|^4 dy \ , \end{aligned} \tag{3.2}$$

where $E^j_{\delta'}(X,\theta) = E_{\delta'}(X,\theta) \cap [X_j, X_j + Y]$. Our hypothesis asserts that for X sufficently large and suitable values of T there exists a function $\Delta(y, T)$ which satisfies (1.2) and (1.3).

Let $T_j = X_j^{1-\theta}$ and let $\Delta_j(y, T_j)$ the functions which satisfy the conditions (1.2) and (1.3) for every j. Applying the Brunn-Titchmarsh inequality we can deduce

$$\Big(\psi(y + y^\theta) - \psi(y) - y^\theta\Big) - \Big(\psi(y + \frac{y}{T_j}) - \psi(y) - \frac{y}{T_j} + \Delta_j(y, T_j)\Big) \ll \frac{y}{T_j \log X},$$

for every j and $y \in E^j_{\delta'}(X,\theta)$, and then from (3.2) it follows that

$$\begin{aligned} |E_{\delta'}(X,\theta)| X^{4\theta} &\ll \sum_j \int\limits_{E^j_{\delta'}(X,\theta)} \left|\psi(y + \frac{y}{T_j}) - \psi(y) - \frac{y}{T_j} + \Delta_j(y, T_j)\right|^4 dy \\ &\le \sum_j \int_X^{2X} \left|\psi(y + \frac{y}{T_j}) - \psi(y) - \frac{y}{T_j} + \Delta_j(y, T_j)\right|^4 dy. \end{aligned}$$

Moreover our hypothesis implies that for every $\varepsilon > 0$ we have

$$|E_{\delta'}(X,\theta)| \ll X^{-4\theta} \sum_j \int_X^{2X} \left| \psi(y + \frac{y}{T_j}) - \psi(y) - \frac{y}{T_j} + \Delta_j(y, T_j) \right|^4 dy \ll X^{1-\theta+\varepsilon},$$

and this leads to

$$|E_{\delta'}(X_n,\theta)| \ll X_n^{1-\theta+\varepsilon}, \tag{3.3}$$

for n sufficiently large and for every $1/2 < \theta \leq 7/12$.

For X_n sufficiently large, we have a contradiction between (3.1) and (3.3), and this completes the proof of the theorem.

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