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## Statistical mechanics in the context of special relativity. II.

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The special relativity laws emerge as one-parameter (light speed) generalizations of the corresponding laws of classical physics. These generalizations, imposed by the Lorentz transformations, affect both the definition of the various physical observables (e.g., momentum, energy, etc.), as well as the mathematical apparatus of the theory. Here, following the general lines of [Phys. Rev. E **66**, 056125 (2002)], we show that the Lorentz transformations impose also a proper one-parameter generalization of the classical Boltzmann-Gibbs-Shannon entropy. The obtained relativistic entropy permits us to construct a coherent and self-consistent relativistic statistical theory, preserving the main features of the ordinary statistical theory, which is recovered in the classical limit. The predicted distribution function is a one-parameter continuous deformation of the classical Maxwell-Boltzmann distribution and has a simple analytic form, showing power law tails in accordance with the experimental evidence. Furthermore, this statistical mechanics can be obtained as the stationary case of a generalized kinetic theory governed by an evolution equation obeying the  $H$  theorem and reproducing the Boltzmann equation of the ordinary kinetics in the classical limit.

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### I. INTRODUCTION

In high energy physics, various approaches of statistical mechanics and kinetic theory have been adopted to study cosmological models [1,2], multiparticle production in hadron-hadron collisions [3], possible Lorentz invariance violation in extensions of the standard model [4], neutron star matter [5], black hole models [6–9], etc. Common to all such approaches is the explicit or implicit assumption of a very specific form of the entropy, i.e., the celebrated Boltzmann-Gibbs-Shannon entropy, which according to the Jaynes [10] maximum entropy principle leads to the Maxwell-Boltzmann exponential distribution.

On the other hand it has been known for a long time that there exist open problems in high energy physics which cannot be solved within the ordinary Maxwell-Boltzmann statistics, e.g., the cosmic ray spectrum [11–13] and the black hole entropy Bekenstein-Hawking area law [14,15].

In statistical physics, in addition to the ubiquitous exponential distribution  $\exp(-x)$ , nonexponential distributions have been also considered. The most famous ones appear within quantum statistical mechanics, namely, the Fermi-Dirac and the Bose-Einstein distributions, which are forced by the exclusion-inclusion principle. In the last decades particular attention has been addressed to statistical distributions presenting power law tails like  $x^{-a}$ . The latter distributions have been observed experimentally in many fields of physics, including cosmic ray physics [11–13], plasma physics [16], and multiparticle production processes [17].

In order to propose theories based on nonexponential statistical distributions, one can proceed in two different ways. The first is the entropic one consisting of a proper generalization of the system entropy. After fixing the entropy, the Jaynes maximum entropy principle univocally determines

the form of the distribution function. The problem then is essentially reduced to the choice of the right entropic form.

The second way is the kinetic one which is very familiar in nonequilibrium statistical mechanics. It is well known that the classical statistical mechanics can be obtained as a particular case of the ordinary kinetic theory for  $t \rightarrow \infty$ , within the Boltzmann or Fokker-Planck picture. Generalized kinetic theories necessarily lead to generalized statistical mechanics producing nonexponential distributions.

Both the entropic and the kinetic paths lead to the conclusion that it is possible to develop, in addition to the classical statistical mechanics, other coherent and self-consistent statistical theories [18–24].

It is remarkable that the Maxwell-Boltzmann distribution does not emerge within the statistical mechanics. It is more proper to say that the classical statistical mechanics is built starting from the Maxwell-Boltzmann distribution. It is commonly accepted that this distribution emerges within Newtonian mechanics. Indeed, numerical simulation of classical molecular dynamics unequivocally leads to the exponential distribution. At this point, the question naturally arises as to whether the Maxwell-Boltzmann distribution obtains also in the case when the microscopic dynamics is governed by the special relativity laws.

In relativistic statistical theory the collision invariants are imposed by the laws of special relativity. Hence the first difference between the classical and the relativistic distribution functions regards the different forms of the collision invariants, in the argument of the distribution function. A second difference could originate from the form of the entropy of the relativistic many body system, which could be different with respect to the Boltzmann-Gibbs-Shannon entropy.

In the ordinary relativistic statistical mechanics it is accepted that the entropy has the same form as in classical statistical mechanics. Then the Jaynes principle yields an exponential distribution which reproduces exactly the Maxwell-Boltzmann distribution in the rest frame. The most

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famous and important relativistic particle system is undoubtedly the cosmic rays [11–13]. The cosmic ray spectrum has a very large extension (13 decades in energy and 33 decades in particle flux) and presents a power law asymptotic behavior. Unfortunately the Maxwell-Boltzmann distribution fails to explain this experimental spectrum. The same power law asymptotic behavior of the distribution function was observed also in other relativistic systems [16,17]. Hence for relativistic systems, the experimental evidence suggests a nonexponential distribution function with power law tails. This distribution can be originated exclusively by an entropy manifestly different from the Boltzmann-Gibbs-Shannon one.

We recall that all the classical physical observables (momentum, energy, etc.) when considered within special relativity are properly generalized. Any relativistic formula can be viewed as a one-parameter (light speed) generalization or deformation of the corresponding classical formula. Consequently it is natural to assume that the entropy of a relativistic system could also be a one-parameter generalization of the classical entropy. In this case the logarithm appearing in the expression of the Boltzmann-Gibbs-Shannon entropy must be replaced by another one-parameter generalized logarithm. Then also the ordinary exponential function appearing in the Maxwell-Boltzmann distribution must be replaced by a one-parameter generalized exponential. These two generalized functions were proposed heuristically in Ref. [18] and are given by

$$\exp_{\{\kappa\}}(t) = (\sqrt{1 + \kappa^2 t^2} + \kappa t)^{1/\kappa}, \quad (1.1)$$

$$\ln_{\{\kappa\}}(t) = \frac{t^\kappa - t^{-\kappa}}{2\kappa}. \quad (1.2)$$

Note that in the classical limit  $\kappa \rightarrow 0$  one easily recovers the ordinary exponential and logarithm, namely,  $\exp_{\{0\}}(t) = \exp(t)$ ,  $\ln_{\{0\}}(t) = \ln(t)$ . Subsequently in Ref. [19] it was shown that a statistical theory developed starting from these generalized functions emerges within the framework of special relativity. The entropy for a relativistic many body system assumes the form

$$S(f) = - \int d^3p f \ln_{\{\kappa\}}(f) = - \langle \ln_{\{\kappa\}}(f) \rangle, \quad (1.3)$$

while the relevant statistical distribution in the rest frame is given by

$$f = \alpha \exp_{\{\kappa\}}\left(-\frac{W - \mu}{\lambda k_B T}\right), \quad (1.4)$$

$W$  being the relativistic kinetic energy, while  $\alpha$  and  $\lambda$  are two constants depending on  $\kappa$ . These two latter relationships are one-parameter generalizations of the Boltzmann-Gibbs-Shannon entropy and of the Maxwell-Boltzmann distribution, respectively, which are recovered in the classical limit  $\kappa \rightarrow 0$ .

The main goal of the present contribution is to show that the functions  $\exp_{\{\kappa\}}(t)$  and  $\ln_{\{\kappa\}}(t)$  leading to the distribution function (1.4) and to the distribution (1.4) emerge naturally

and unequivocally within the special relativity theory. These functions replace the ordinary exponential and logarithm, relegating them to the status of classical functions. It is remarkable that the relativistic statistical theory, based on the entropy (1.3), predicts a power law asymptotic behavior for the spectrum of relativistic particles according to the experimental evidence.

The paper is organized as follows. In Sec. II we recall some concepts of relativistic dynamics, focusing our attention on the deformations in the mathematical formalism enforced by the finite value of the light speed. In Sec. III we show that the form of  $\exp_{\{\kappa\}}(t)$  can be obtained within special relativity by using elementary algebraic calculations, starting from the dispersion relation of free relativistic particles. In Sec. IV we reobtain the exponential  $\exp_{\{\kappa\}}(t)$  starting from the concept of Lorentz invariant integration. In Sec. V starting from the three-dimensional  $\kappa$  sum enforced by the Lorentz transformations we introduce the concept of the generalized differential. In Sec. VI we derive the forms of  $\ln_{\{\kappa\}}(t)$  and  $\exp_{\{\kappa\}}(t)$  within special relativity, starting from the additivity law of relativistic momenta. In Sec. VII we recall some concepts related to the maximum entropy principle, in the context of a class of generalized statistical theories. In Sec. VIII we consider the explicit form of the distribution function within the special relativity. In Sec. IX we consider the main features of the statistical mechanics and of the kinetic theory based on the entropy (1.3). In Sec. X we consider some applications of the present theory to specific physical problems and compare the results with the ones obtained within the classical statistical mechanics. Finally in Sec. XI some concluding remarks are reported, while the mathematical properties of the functions  $\exp_{\{\kappa\}}(t)$  and  $\ln_{\{\kappa\}}(t)$  are collected in the Appendix.

## II. DEFORMED SUMS IN SPECIAL RELATIVITY

In this section we introduce the special relativity in an alternative way, starting from the well known formulas of hyperbolic trigonometry ([25], p. 58)

$$\operatorname{arsinh} x = \operatorname{arccosh} \sqrt{1 + x^2} = \operatorname{arctanh} \frac{x}{\sqrt{1 + x^2}}. \quad (2.1)$$

We set  $x = \kappa q$ ,  $q$  being the dimensionless variable corresponding to the momentum and  $\kappa$  a dimensionless deformation parameter. After introducing the functions

$$u(q) = \frac{q}{\sqrt{1 + \kappa^2 q^2}}, \quad (2.2)$$

$$\mathcal{W}(q) = \frac{1}{\kappa^2} \sqrt{1 + \kappa^2 q^2} - \frac{1}{\kappa^2}, \quad (2.3)$$

corresponding to the dimensionless velocity and kinetic energy, respectively, we can write Eq. (2.1) in the form

$$\operatorname{arsinh}(\kappa q) = \operatorname{arccosh}(1 + \kappa^2 \mathcal{W}) = \operatorname{arctanh}(\kappa u), \quad (2.4)$$

which in the classical limit  $\kappa \rightarrow 0$  reproduces the dispersion relation  $q = \sqrt{2} \mathcal{W} = u$  of Newtonian mechanics. Finally, after introducing the dimensionless total energy

$$\mathcal{E}(\mathcal{W}) = \mathcal{W} + \frac{1}{\kappa^2}, \quad (2.5)$$

one can write Eq. (2.1) also in the form

$$\operatorname{arcsinh}(\kappa q) = \operatorname{arccosh}(\kappa^2 \mathcal{E}) = \operatorname{arctanh}(\kappa u). \quad (2.6)$$

It is easy to verify the validity of the following formulas:

$$q(u) = \frac{u}{\sqrt{1 - \kappa^2 u^2}}, \quad (2.7)$$

$$q(\mathcal{W}) = \sqrt{2\mathcal{W} + \kappa^2 \mathcal{W}^2}, \quad (2.8)$$

$$q(\mathcal{E}) = \sqrt{\kappa^2 \mathcal{E}^2 - \frac{1}{\kappa^2}}, \quad (2.9)$$

$$u(\mathcal{W}) = \frac{\sqrt{2\mathcal{W} + \kappa^2 \mathcal{W}^2}}{1 + \kappa^2 \mathcal{W}}, \quad (2.10)$$

$$u(\mathcal{E}) = \frac{\sqrt{\kappa^2 \mathcal{E}^2 - 1/\kappa^2}}{\kappa^2 \mathcal{E}}, \quad (2.11)$$

$$\mathcal{W}(u) = \frac{1}{\kappa^2 \sqrt{1 - \kappa^2 u^2}} - \frac{1}{\kappa^2}, \quad (2.12)$$

$$\mathcal{W}(\mathcal{E}) = \mathcal{E} - \frac{1}{\kappa^2}, \quad (2.13)$$

$$\mathcal{E}(q) = \frac{1}{\kappa^2} \sqrt{1 + \kappa^2 q^2}, \quad (2.14)$$

$$\mathcal{E}(u) = \frac{1}{\kappa^2 \sqrt{1 - \kappa^2 u^2}}. \quad (2.15)$$

At this point one can define the physical variables velocity  $v$ , momentum  $p$ , and total energy  $E$  through

$$\frac{v}{u} = \frac{p}{mq} = \sqrt{\frac{E}{m\mathcal{E}}} = \kappa c = v^*, \quad (2.16)$$

and the kinetic energy as  $W = E - mc^2$ ,  $c$  being the light speed. We impose that  $\lim_{c \rightarrow \infty, \kappa \rightarrow 0} v^* < \infty$ , in order to preserve the validity of the definitions of the physical variables in the classical limit also. After inserting the physical variables in the above obtained formulas, one recovers all the formulas of the one-particle dynamics within the special relativity. For instance Eqs. (2.7) and (2.14) transform

$$p = \frac{mv}{\sqrt{1 - (v/c)^2}}, \quad (2.17)$$

$$E = \sqrt{m^2 c^4 + p^2 c^2}, \quad (2.18)$$

and so on.

Let us consider in the one-dimensional frame  $\mathcal{S}$  two identical particles of rest mass  $m$ . We suppose that the first par-

ticle moves toward the right with velocity  $u_1$  while the second particle moves toward the left with velocity  $u_2$ . The momenta of the two particles are given by  $q_1 = q(u_1)$  and  $q_2 = q(u_2)$ , respectively, where  $q(u)$  is defined through Eq. (2.7). The total energies of the two particles are given by  $\mathcal{E}_1 = \mathcal{E}(u_1)$  and  $\mathcal{E}_2 = \mathcal{E}(u_2)$ , respectively, with  $\mathcal{E}(u)$  given by Eq. (2.15). Analogously the kinetic energies of the two particles are given by  $\mathcal{W}_1 = \mathcal{W}(u_1)$  and  $\mathcal{W}_2 = \mathcal{W}(u_2)$ , respectively, with  $\mathcal{W}(u)$  defined by Eq. (2.12).

We consider now the same particles in a new frame  $\mathcal{S}'$  that moves at constant speed  $u_2$  toward the left with respect to the frame  $\mathcal{S}$ . In this frame, which is the rest frame of the second particle, the two particles have velocities given by

$u'_1 = u_1 \oplus u_2$  and  $u'_2 = 0$ , respectively, where

$$u_1 \oplus u_2 = \frac{u_1 + u_2}{1 + \kappa^2 u_1 u_2} \quad (2.19)$$

is the well known relativistic additivity law for dimensionless velocities. In the same frame  $\mathcal{S}'$  the particle momenta are given by  $q'_1 = q(u'_1)$  and  $q'_2 = 0$ . Analogously, in  $\mathcal{S}'$ , the particle kinetic energies are given by  $\mathcal{W}'_1 = \mathcal{W}(u'_1)$  and  $\mathcal{W}'_2 = 0$ , while the total particle energies result as  $\mathcal{E}'_1 = \mathcal{E}(u'_1)$  and  $\mathcal{E}'_2 = 1/\kappa^2$ .

A very interesting result follows from the relation  $\mathcal{E}'_2 = 1/\kappa^2$ , regarding the physical meaning of the parameter  $\kappa$ . It is evident that the quantity  $1/\kappa^2$  represents the dimensionless rest energy of the relativistic particle.

Let us ask now the following questions: Is it possible to obtain the value of the momentum  $q'_1$  (or of the total energy  $\mathcal{E}'_1$ , or of the kinetic energy  $\mathcal{W}'_1$ ) starting directly from the values of the momenta  $q_1$  and  $q_2$  (or of the total energies  $\mathcal{E}_1$  and  $\mathcal{E}_2$ , or of the kinetic energies  $\mathcal{W}_1$  and  $\mathcal{W}_2$ ) in the frame  $\mathcal{S}$ ? The answers to the above questions are affirmative as we will see in the following.

First we consider the case of the relativistic momentum  $q'_1$ . Clearly we can write

$$q'_1 = q(u'_1) = q(u_1 \oplus u_2).$$

Then following Ref. [19] we have

$$\begin{aligned} q(u_1 \oplus u_2) &= \frac{u_1 \oplus u_2}{\sqrt{1 - \kappa^2 (u_1 \oplus u_2)^2}} \\ &= \frac{u_1 \oplus u_2}{\sqrt{1 - \kappa^2 \left( \frac{u_1 + u_2}{1 + \kappa^2 u_1 u_2} \right)^2}} \\ &= (u_1 \oplus u_2) \frac{1 + \kappa^2 u_1 u_2}{\sqrt{(1 - \kappa^2 u_1^2)(1 - \kappa^2 u_2^2)}} \\ &= \frac{u_1 + u_2}{\sqrt{(1 - \kappa^2 u_1^2)(1 - \kappa^2 u_2^2)}} \\ &= \frac{u_1}{\sqrt{1 - \kappa^2 u_1^2}} \sqrt{1 + \frac{\kappa^2 u_2^2}{1 - \kappa^2 u_2^2}} \end{aligned}$$

$$\begin{aligned}
& + \frac{u_2}{\sqrt{1-\kappa^2 u_2^2}} \sqrt{1 + \frac{\kappa^2 u_1^2}{1-\kappa^2 u_1^2}} \\
& = q_1 \sqrt{1 + \kappa^2 q_2^2} + q_2 \sqrt{1 + \kappa^2 q_1^2}. \quad (2.20)
\end{aligned}$$

Hence we can write

$$\begin{aligned}
& q(u_1 \oplus u_2) \\
& = q(u_1) \oplus q(u_2), \quad (2.21)
\end{aligned}$$

where

$$q_1 \oplus q_2 = q_1 \sqrt{1 + \kappa^2 q_2^2} + q_2 \sqrt{1 + \kappa^2 q_1^2} \quad (2.22)$$

is the  $\kappa$  sum of relativistic momenta introduced in Ref. [18]. In words, the relativistic momentum  $q'_1$  of the first particle, in the rest frame of the second particle  $\mathcal{S}'$ , is the  $\kappa$ -deformed sum of the momenta  $q_1$  and  $q_2$  of the two particles in the frame  $\mathcal{S}$ . The  $\kappa$  sum of the relativistic momenta and the relativistic sum of the velocities are intimately related. They reduce to the standard sum as the velocity  $c$  approaches infinity (or equivalently the parameter  $\kappa$  approaches zero). The deformations in the above sums, in both cases, are relativistic

effects and originate from the fact that  $c$  has a finite value.

We proceed now to calculate the total energy  $\mathcal{E}'_1$  of the first particle in the frame  $\mathcal{S}'$  starting directly from the values, in the frame  $\mathcal{S}$ , of the total energies  $\mathcal{E}_1$  and  $\mathcal{E}_2$ . Preliminarily we recall the following mathematical properties of the inverse hyperbolic trigonometric functions ([25], p. 58):

$$\operatorname{arcsinh} x_1 \pm \operatorname{arcsinh} x_2 = \operatorname{arcsinh}(x_1 \sqrt{1+x_2^2} \pm x_2 \sqrt{1+x_1^2}), \quad (2.23)$$

$$\operatorname{arccosh} y_1 \pm \operatorname{arccosh} y_2 = \operatorname{arccosh}[y_1 y_2 \pm \sqrt{(y_1^2-1)(y_2^2-1)}], \quad (2.24)$$

$$\operatorname{arctanh} z_1 \pm \operatorname{arctanh} z_2 = \operatorname{arctanh} \frac{z_1 \pm z_2}{1 \pm z_1 z_2}. \quad (2.25)$$

After setting  $x = \kappa q$ ,  $y = \kappa^2 \mathcal{E}$ ,  $z = \kappa u$ , and taking into account Eq. (2.6), we are able to define the  $\kappa$  sum of the relativistic total energies as follows:

$$\mathcal{E}_1 \oplus \mathcal{E}_2 = \kappa^2 \mathcal{E}_1 \mathcal{E}_2 + \frac{1}{\kappa^2} \sqrt{(\kappa^4 \mathcal{E}_1^2 - 1)(\kappa^4 \mathcal{E}_2^2 - 1)}. \quad (2.26)$$

After noticing that

$$\begin{aligned}
\mathcal{E}(u_1) \oplus \mathcal{E}(u_2) &= \frac{1}{\kappa^2} \frac{1}{\sqrt{1-\kappa^2 u_1^2}} \oplus \frac{1}{\kappa^2} \frac{1}{\sqrt{1-\kappa^2 u_2^2}} = \frac{1}{\kappa^2} \frac{1}{\sqrt{(1-\kappa^2 u_1^2)(1-\kappa^2 u_2^2)}} + \frac{1}{\kappa^2} \sqrt{\left(\frac{1}{1-\kappa^2 u_1^2} - 1\right)\left(\frac{1}{1-\kappa^2 u_2^2} - 1\right)} \\
&= \frac{1}{\kappa^2} \frac{1}{\sqrt{(1-\kappa^2 u_1^2)(1-\kappa^2 u_2^2)}} + \frac{u_1 u_2}{\sqrt{(1-\kappa^2 u_1^2)(1-\kappa^2 u_2^2)}} = \frac{1}{\kappa^2} \frac{1 + \kappa^2 u_1 u_2}{\sqrt{(1-\kappa^2 u_1^2)(1-\kappa^2 u_2^2)}} = \frac{1}{\kappa^2} \frac{1 + \kappa^2 u_1 u_2}{\sqrt{(1 + \kappa^2 u_1 u_2)^2 - \kappa^2 (u_1 + u_2)^2}} \\
&= \frac{1}{\kappa^2} \frac{1}{\sqrt{1 - \kappa^2 \left(\frac{u_1 + u_2}{1 + \kappa^2 u_1 u_2}\right)^2}} = \frac{1}{\kappa^2} \frac{1}{\sqrt{1 - \kappa^2 (u_1 \oplus u_2)^2}} = \mathcal{E}(u_1 \oplus u_2), \quad (2.27)
\end{aligned}$$

we can conclude that the total energy  $\mathcal{E}'_1 = \mathcal{E}(u_1 \oplus u_2)$  of the first particle, in the rest frame of the second particle  $\mathcal{S}'$ , is the  $\kappa$  sum, defined through Eq. (2.26), of the total energies  $\mathcal{E}_1$  and  $\mathcal{E}_2$  of the particles in the frame  $\mathcal{S}$ :

$$\mathcal{E}(u_1 \oplus u_2) = \mathcal{E}(u_1) \oplus \mathcal{E}(u_2). \quad (2.28)$$

The  $\kappa$  sum for the relativistic kinetic energies can be defined starting from

$$\mathcal{W}_1 \oplus \mathcal{W}_2 = \mathcal{E}_1 \oplus \mathcal{E}_2 - \frac{1}{\kappa^2} \quad (2.29)$$

and assumes the form

$$\begin{aligned}
\mathcal{W}_1 \oplus \mathcal{W}_2 &= \mathcal{W}_1 + \mathcal{W}_2 + \kappa^2 \mathcal{W}_1 \mathcal{W}_2 \\
&+ \sqrt{\mathcal{W}_1 \mathcal{W}_2 (2 + \kappa^2 \mathcal{W}_1)(2 + \kappa^2 \mathcal{W}_2)}. \quad (2.30)
\end{aligned}$$

After taking into account Eqs. (2.12), (2.19), and (2.30) one immediately obtains

$$\mathcal{W}(u_1 \oplus u_2) = \mathcal{W}(u_1) \oplus \mathcal{W}(u_2). \quad (2.31)$$

In words the  $\kappa$  sum, defined through Eq. (2.30), of the particle kinetic energies  $\mathcal{W}_1$  and  $\mathcal{W}_2$  in the  $\mathcal{S}$  frame, gives the kinetic energy of one of the two particles in the rest frame of the other particle  $\mathcal{S}'$ .

In addition to the relationships (2.21), (2.28), and (2.31), linking the  $\kappa$  sums defined through Eqs. (2.19), (2.22), (2.26), and (2.30), one can easily obtain

$$q(u_1) \oplus q(u_2) = q(u_1 \oplus u_2), \quad (2.32)$$

$$\mathcal{E}(u_1) \oplus \mathcal{E}(u_2) = \mathcal{E}(u_1 \oplus u_2), \quad (2.33)$$

$$\mathcal{W}(u_1) \oplus \mathcal{W}(u_2) = \mathcal{W}(u_1 \oplus u_2), \quad (2.34)$$

$$u(q_1) \oplus u(q_2) = u(q_1 \oplus q_2), \quad (2.35)$$

$$\mathcal{E}(q_1) \oplus \mathcal{E}(q_2) = \mathcal{E}(q_1 \oplus q_2), \quad (2.36)$$

$$\mathcal{W}(q_1) \oplus \mathcal{W}(q_2) = \mathcal{W}(q_1 \oplus q_2), \quad (2.37)$$

$$q(\mathcal{E}_1) \oplus q(\mathcal{E}_2) = q(\mathcal{E}_1 \oplus \mathcal{E}_2), \quad (2.38)$$

$$u(\mathcal{E}_1) \oplus u(\mathcal{E}_2) = u(\mathcal{E}_1 \oplus \mathcal{E}_2), \quad (2.39)$$

$$\mathcal{W}(\mathcal{E}_1) \oplus \mathcal{W}(\mathcal{E}_2) = \mathcal{W}(\mathcal{E}_1 \oplus \mathcal{E}_2), \quad (2.40)$$

$$q(\mathcal{W}_1) \oplus q(\mathcal{W}_2) = q(\mathcal{W}_1 \oplus \mathcal{W}_2), \quad (2.41)$$

$$u(\mathcal{W}_1) \oplus u(\mathcal{W}_2) = u(\mathcal{W}_1 \oplus \mathcal{W}_2), \quad (2.42)$$

$$\mathcal{E}(\mathcal{W}_1) \oplus \mathcal{E}(\mathcal{W}_2) = \mathcal{E}(\mathcal{W}_1 \oplus \mathcal{W}_2). \quad (2.43)$$

Let us make some remarks on the meaning of the  $\kappa$  sums. First we note that in the classical limit the  $\kappa$  sum for the relativistic kinetic energies given by Eq. (2.30) reduces to the following expression of Galilean relativity:

$$\mathcal{W}_1 \oplus \mathcal{W}_2 = \mathcal{W}_1 + \mathcal{W}_2 + 2\sqrt{\mathcal{W}_1 \mathcal{W}_2}. \quad (2.44)$$

Then a deformed sum appears also in classical physics when we change the frame of observation of the particle system. Clearly this happens only for the kinetic energy. For the classical momenta and velocities the ordinary sum holds.

We emphasize that the four  $\kappa$  sums given by Eqs. (2.19), (2.22), (2.26), and (2.30) emerge only when we change the frame of observation of the particles from  $\mathcal{S}$  to  $\mathcal{S}'$ . More precisely these sums give the value of  $q'_1$  (or  $u'_1$ ,  $\mathcal{E}'_1$ ,  $\mathcal{W}'_1$ ) of the first particle in the rest frame  $\mathcal{S}'$  of the second particle, starting directly from the value of  $q_1$  (or  $u_1$ ,  $\mathcal{E}_1$ ,  $\mathcal{W}_1$ ) and  $q_2$  (or  $u_2$ ,  $\mathcal{E}_2$ ,  $\mathcal{W}_2$ ) of the two particles in the old frame  $\mathcal{S}$ .

### III. MINKOWSKI FOUR-VECTORS AND $\kappa$ EXPONENTIAL

From the general discussion in the previous section it appears clear that any relativistic formula can be viewed as a one-parameter deformation of the corresponding classical formula. For instance the kinetic energy given by the relativistic hyperbolic dispersion relation

$$W = \sqrt{m^2 c^4 + p^2 c^2} - mc^2 \quad (3.1)$$

can be viewed as the one-parameter relativistic deformation of the classical parabolic dispersion relation  $W_{cl} = p^2/2m$ . In the present section we will show that the relativistic dispersion relation imposes a one-parameter deformation on the ordinary exponential function, which becomes now as a mathematical tool of classical physics. The deformed exponential, as we will see in the following sections, replaces the ordinary exponential in relativistic physics.

We adopt the metric  $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$  in the Minkowski space and recall that the length of any four-vector is Lorentz invariant. In particular for the four-momentum  $p^\mu = (E/c, \mathbf{p})$ ,  $p_\mu = (E/c, -\mathbf{p})$  one obtains the relativistic dispersion relation

$$p^\mu p_\mu = m^2 c^2, \quad (3.2)$$

which in terms of the total energy assumes the form

$$E = \sqrt{m^2 c^4 + p^2 c^2}. \quad (3.3)$$

We can write the latter equation as follows:

$$\left(\frac{E}{mc^2}\right)^2 - \left(\frac{p}{mc}\right)^2 = 1, \quad (3.4)$$

and then

$$\left(\frac{E}{mc^2} - \frac{p}{mc}\right)\left(\frac{E}{mc^2} + \frac{p}{mc}\right) = 1. \quad (3.5)$$

At this point we can eliminate the variable  $E$  by employing once again Eq. (3.3), obtaining

$$\left[\sqrt{1 + \left(\frac{p}{mc}\right)^2} - \frac{p}{mc}\right]\left[\sqrt{1 + \left(\frac{p}{mc}\right)^2} + \frac{p}{mc}\right] = 1. \quad (3.6)$$

The last relationship, after introducing the dimensionless momentum  $q$ , defined through Eq. (2.16), can be written as

$$(\sqrt{1 + \kappa^2 q^2} - \kappa q)(\sqrt{1 + \kappa^2 q^2} + \kappa q) = 1, \quad (3.7)$$

or equivalently

$$(\sqrt{1 + \kappa^2 q^2} - \kappa q)^{1/\kappa} (\sqrt{1 + \kappa^2 q^2} + \kappa q)^{1/\kappa} = 1. \quad (3.8)$$

We remark that Eq. (3.8) follows directly from the dispersion relation. Interestingly in the classical limit  $\kappa \rightarrow 0$  Eq. (3.8) reduces to  $\exp(-q) \exp(q) = 1$  while the dispersion relation becomes the classical one  $W = p^2/2m$ . In this way we obtain a direct link between the dispersion relation of free classical particles and the ordinary exponential function.

In the light of the above result, we reconsider Eq. (3.8) which, after introducing the function

$$\exp_{\{\kappa\}}(q) = (\sqrt{1 + \kappa^2 q^2} + \kappa q)^{1/\kappa}, \quad (3.9)$$

is written in the form

$$\exp_{\{\kappa\}}(-q) \exp_{\{\kappa\}}(q) = 1. \quad (3.10)$$

The function  $\exp_{\{\kappa\}}(q)$ , reproducing the exponential function  $\exp(q) = \exp_{\{0\}}(q)$  in the classical limit  $\kappa \rightarrow 0$ , represents a

one-parameter relativistic generalization of the ordinary exponential.

#### IV. LORENTZ INVARIANT INTEGRATION AND $\kappa$ EXPONENTIAL

In the present section we present a second path that unambiguously leads to the  $\exp_{\{\kappa\}}(q)$  function. Let us consider the following integral in the three-dimensional momentum space, within the classical physics framework:

$$I_{cl} = \int d^3q F, \quad (4.1)$$

$F$  being an arbitrary quantity. Whenever  $F$  depends only on  $q=|\mathbf{q}|$ , the above integral can be reduced to the following one-dimensional integral:

$$I_{cl} = \int_0^\infty dq 4\pi q^2 F. \quad (4.2)$$

After introducing the physical variable  $\mathbf{p}=mv_*^\infty \mathbf{q}$  the integral (4.1) transforms to

$$I_{cl} = \int \frac{d^3p}{(mv_*^\infty)^3} F. \quad (4.3)$$

In the framework of a relativistic theory it is well known that the integral (4.3) must be replaced by a Lorentz invariant integral  $I_{cl} \rightarrow I_{rel}$ , where

$$I_{rel} = A \int d^4p \theta(p_0) \delta(p^\mu p_\mu - m^2 c^2) F \quad (4.4)$$

and  $A$  is a constant. Now we can introduce the variable  $\mathbf{q}=\mathbf{p}/mv_*$  and take into account that  $p^\mu=(E/c, \mathbf{p})$  and  $E=\sqrt{m^2 c^4 + p^2 c^2}$ . After properly choosing the value of  $A$ , the four-dimensional integral (4.4) can be easily reduced to the following three-dimensional integral:

$$I_{rel} = \int \frac{d^3q}{\sqrt{1 + \kappa^2 q^2}} F. \quad (4.5)$$

We remark that in Eq. (4.4) the integration element  $d^4p$  is a scalar because the Jacobian of the Lorentz transformation is equal to unity. Then  $I_{rel}$  transforms as  $F$ . For this reason in Eq. (4.5) the integration element  $d^3q/\sqrt{1 + \kappa^2 q^2}$  is a scalar. Finally we can reduce Eq. (4.5) to the following one-dimensional integral:

$$I_{rel} = \int_0^\infty \frac{dq}{\sqrt{1 + \kappa^2 q^2}} 4\pi q^2 F. \quad (4.6)$$

It is important to note that in the classical limit  $\kappa \rightarrow 0$ , Eqs. (4.5) and (4.6) reproduce the corresponding classical ones given by Eqs. (4.1) and (4.2), respectively.

We focus now our attention on the classical and relativistic expression of the one-dimensional integrals given by Eqs. (4.2) and (4.6), respectively. One immediately observes that the relativistic integral is obtained directly from the classical one, by making the substitution  $dq \rightarrow d_{\{\kappa\}}q$ , where

$$d_{\{\kappa\}}q = \frac{dq}{\sqrt{1 + \kappa^2 q^2}} \quad (4.7)$$

is the  $\kappa$  differential. The relativistic one-dimensional integral can be written in the form

$$I_{rel} = \int_0^\infty d_{\{\kappa\}}q 4\pi q^2 F. \quad (4.8)$$

Clearly, the  $\kappa$  integral  $\int d_{\{\kappa\}}q$  originates from the fact that the light speed has a finite value.

As a working example involving the  $\kappa$  integration explicitly, we consider the definition of the kinetic energy. In classical mechanics the kinetic energy, using dimensionless variables, is defined as

$$\mathcal{W}_{cl}(q) = \int_0^q dq q. \quad (4.9)$$

It is straightforward to verify that the extension of the above definition in special relativity is given by

$$\mathcal{W}(q) = \int_0^q d_{\{\kappa\}}q q. \quad (4.10)$$

Then the replacement of the ordinary integration with the  $\kappa$  integration in the classical definition (4.9) permits us to recover the relativistic expression (2.3) of the kinetic energy.

Let us consider now the  $\kappa$  derivative related to the above defined  $\kappa$  integration,

$$\frac{d}{d_{\{\kappa\}}q} = \sqrt{1 + \kappa^2 q^2} \frac{d}{dq}. \quad (4.11)$$

Equation (4.10) linking  $\mathcal{W}$  and  $q$  can be written now also in the following differential form:

$$\frac{d}{d_{\{\kappa\}}q} \mathcal{W}(q) = q, \quad (4.12)$$

which after integration with the condition  $\mathcal{W}(0)=0$  yields the relativistic expression of the kinetic energy (2.3). In the limit  $\kappa \rightarrow 0$  the above differential equation reduces to the classical one  $(d/dq)\mathcal{W}(q)=q$ .

Taking into account that the ordinary exponential is an eigenfunction of the ordinary derivative, namely,  $(d/dq)\exp(q)=\exp(q)$ , the question of determining the eigenfunction of the  $\kappa$  derivative naturally arises. After some simple calculation one obtains

$$\frac{d}{d_{\{\kappa\}}q} \exp_{\{\kappa\}}(q) = \exp_{\{\kappa\}}(q), \quad (4.13)$$

so that the  $\kappa$  exponential is an eigenfunction of the  $\kappa$  derivative.

#### V. RELATIVISTIC SUMS AND $\kappa$ DIFFERENTIAL

In this section we explain the physical origin of the mechanism enforcing the deformation in the  $\kappa$  differential. We consider two identical relativistic particles with rest mass

$m$  and velocities  $\mathbf{v}_1$  and  $\mathbf{v}_2$  in the three-dimensional frame  $S$ . The modulus of the relative velocity  $V$  of the particles depends on  $\mathbf{v}_1$  and  $\mathbf{v}_2$  and is given by ([26], p. 20)

$$V(\mathbf{v}_1, \mathbf{v}_2) = \sqrt{(\mathbf{v}_1 \ominus \mathbf{v}_2)^2 - \frac{1}{c^2} \left( \frac{\mathbf{v}_1 \times \mathbf{v}_2}{1 - \mathbf{v}_1 \mathbf{v}_2 / c^2} \right)^2}, \quad (5.1)$$

where

$$\mathbf{v}_1 \ominus \mathbf{v}_2 = \frac{\mathbf{v}_1 - \mathbf{v}_2}{1 - \mathbf{v}_1 \mathbf{v}_2 / c^2}. \quad (5.2)$$

We perform now the calculation of the modulus of the relative momentum related to the modulus of the relative velocity of the two particles according to the relativistic formula

$$P(V) = \frac{mV}{\sqrt{1 - V^2/c^2}}. \quad (5.3)$$

It clearly results that  $P(V) = P(\mathbf{v}_1, \mathbf{v}_2)$  and after taking into account that

$$\mathbf{v} = \frac{\mathbf{p}/m}{\sqrt{1 + \mathbf{p}^2/m^2 c^2}} \quad (5.4)$$

we can conclude that  $P(V) = P(\mathbf{p}_1, \mathbf{p}_1)$ . At this point we introduce the dimensionless variables  $\mathbf{q} = \mathbf{p}/mv_*$  and  $Q = P/mv_*$ . After tedious but straightforward calculation, one arrives at the following expression for the modulus of the dimensionless relative momentum:

$$Q(\mathbf{q}_1, \mathbf{q}_2) = \sqrt{\frac{\kappa (\mathbf{q}_1 \ominus \mathbf{q}_2)^2 - \kappa^2 (\mathbf{q}_1 \times \mathbf{q}_2)^2}{1 - \kappa^4 (\mathbf{q}_1 \times \mathbf{q}_2)^2}}, \quad (5.5)$$

where

$$\mathbf{q}_1 \ominus \mathbf{q}_2 = \mathbf{q}_1 \sqrt{1 + \kappa^2 \mathbf{q}_2^2} - \mathbf{q}_2 \sqrt{1 + \kappa^2 \mathbf{q}_1^2} \quad (5.6)$$

is the  $\kappa$  difference introduced in Ref. [18]. Clearly  $Q(\mathbf{q}_1, \mathbf{q}_2)$  represents the modulus of momentum of the first (second) particle in the rest frame of the second (first) particle.

We suppose now that the two particles in the frame  $S$  have momenta  $\mathbf{q}_1 = \mathbf{q} + d\mathbf{q}$  and  $\mathbf{q}_2 = \mathbf{q}$ , respectively, and calculate the value of  $Q(\mathbf{q} + d\mathbf{q}, \mathbf{q})$ . One easily obtains

$$Q(\mathbf{q} + d\mathbf{q}, \mathbf{q}) = d_{\{\kappa\}}q, \quad (5.7)$$

where  $q = |\mathbf{q}|$  and  $d_{\{\kappa\}}q$  is given by Eq. (4.7).

The physical meaning of the  $\kappa$  differential  $d_{\{\kappa\}}q$  immediately follows. The modulus of the infinitesimal difference of the two particle momenta  $d\mathbf{q}$ , in the frame  $S$ , becomes  $d_{\{\kappa\}}q$ , if this difference is observed in the rest frame of one of the two particles. The meaning of the  $\kappa$  derivative also follows readily. If  $d/dq$  is the derivative in the frame  $S$ , the deformed derivative  $d/d_{\{\kappa\}}q$  represents an ordinary derivative in the rest frame of one of the two particles.

**VI. RELATIVISTIC SUMS AND  $\kappa$ -DEFORMED LOGARITHM AND EXPONENTIAL**

The ordinary logarithm  $h(x) = \ln(x)$  is the only existing function (except for a multiplicative constant) which results

as a solution of the function equation  $h(x_1 x_2) = h(x_1) + h(x_2)$ . Let us consider now the generalization of this equation within the special relativity obtained by substituting the ordinary sum by the  $\kappa$  sum of the dimensionless relativistic momenta,

$$h(x_1 x_2) = h(x_1) \oplus h(x_2). \quad (6.1)$$

We proceed by solving this equation, which assumes the explicit form

$$h(x_1 x_2) = h(x_1) \sqrt{1 + \kappa^2 h(x_2)^2} + h(x_2) \sqrt{1 + \kappa^2 h(x_1)^2}. \quad (6.2)$$

After performing the substitution  $h(x) = \kappa^{-1} \sinh \kappa g(x)$  we obtain that the auxiliary function  $g(x)$  obeys the equation  $g(x_1 x_2) = g(x_1) + g(x_2)$ , and then is given by  $g(x) = A \ln x$ . The unknown function  $h(x)$  becomes

$$h(x) = \frac{\sinh(\kappa \ln x)}{\kappa}, \quad (6.3)$$

where we have set  $A = 1$  in order to recover, in the limit  $\kappa \rightarrow 0$ , the classical solution  $h(x) = \ln(x)$ . The function given by Eq. (6.3) in the following is denoted by  $\ln_{\{\kappa\}}(x)$  and can be written also in the form

$$\ln_{\{\kappa\}}(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa}. \quad (6.4)$$

Setting  $\exp_{\{\kappa\}}[\ln_{\{\kappa\}}(x)] = \ln_{\{\kappa\}}[\exp_{\{\kappa\}}(x)] = x$  it results that  $\ln_{\{\kappa\}}(x)$  is the inverse function of  $\exp_{\{\kappa\}}(x)$  and thus can be viewed as a generalization of the ordinary logarithm  $\ln(x) = \ln_{\{0\}}(x)$  in the framework of special relativity.

It is straightforward to obtain  $\exp_{\{\kappa\}}(x)$ , starting directly from the  $\kappa$  sum of the dimensionless relativistic momenta, as a solution of the functional equation

$$\exp_{\{\kappa\}}(x_1) \exp_{\{\kappa\}}(x_2) = \exp_{\{\kappa\}}(x_1 \oplus x_2), \quad (6.5)$$

which in the classical limit reduces to the functional equation  $\exp(x_1) \exp(x_2) = \exp(x_1 + x_2)$  defining the ordinary exponential.

In the Appendix, the main mathematical properties of the functions  $\kappa$  exponential and  $\kappa$  logarithm are reported.

**VII. MAXIMUM ENTROPY PRINCIPLE**

We start by recalling some elements of classical statistical mechanics. We consider the Boltzmann-Gibbs-Shannon entropy

$$S = - \int d^3p f(p) \ln[f(p)], \quad (7.1)$$

and the constraints functional



$$C = a_0 \left( \int d^3p f(p) - 1 \right) + \mathbf{a} \left( \mathbf{M} - \int d^3p \mathbf{g}(p) f(p) \right), \quad (7.2)$$

where the constants  $a_0$  and  $\mathbf{a} = \{a_1, a_2, \dots, a_l\}$  are the  $l+1$  Lagrange multipliers, while the  $l$  moments  $\mathbf{M} = \{M_1, M_2, \dots, M_l\}$  are the mean values of the  $l$  functions  $\mathbf{g}(p) = \{g_1(p), g_2(p), \dots, g_l(p)\}$ . The variational equation

$$\frac{\delta}{\delta f} (S + C) = 0 \quad (7.3)$$

means that the entropy  $S(p)$  is maximized under the constraints imposing the conservation of the norm of  $f(p)$  and the *a priori* knowledge of the values of the  $l$  moments  $M_j$  of  $f(p)$ , namely,

$$\int d^3p f(p) = 1, \quad M_j = \int d^3p g_j(p) f(p). \quad (7.4)$$

The maximum entropy principle expressed by the variational equation (7.3) yields the following statistical distribution:

$$f(p) = \frac{1}{e} \exp[-\mathbf{a} \cdot \mathbf{g}(p) + a_0]. \quad (7.5)$$

In a classical many body particle system the collisional invariants are essentially the particle number and the kinetic energy  $W(p) = p^2/2m$ . Then we need to involve two Lagrange multipliers  $a_0 = \mu/k_B T$ ,  $a_1 = 1/k_B T$  and one moment  $M_1 = \int d^3p g_1(p) f(p) = \int d^3p W(p) f(p) = \langle W \rangle$ . In this way the distribution function given by Eq. (7.5) reduces to the Maxwell-Boltzmann one of classical statistical mechanics

$$f(p) = \frac{1}{e} \exp\left(-\frac{W(p) - \mu}{k_B T}\right). \quad (7.6)$$

Let us consider now the following generalized trace form entropy:

$$S = - \int d^3p f(p) \Lambda(f(p)), \quad (7.7)$$

$\Lambda(f)$  being a function generalizing the ordinary logarithm. The variational equation (7.3) with  $S$  and  $C$  given by Eqs. (7.7) and (7.2), respectively, produces the distribution function  $f = f(p)$  defined through

$$\frac{\partial}{\partial f} f \Lambda(f) = -\mathbf{a} \cdot \mathbf{g}(p) + a_0. \quad (7.8)$$

The maximum entropy principle, in the case of the ordinary statistical mechanics, relates the Boltzmann-Gibbs-Shannon entropy to the Maxwell-Boltzmann distribution by means of a twofold link. First the distribution is obtained by maximizing the entropy under proper constraints. Second, both the entropy and the distribution are expressed in terms of the same function in direct and inverse form. Indeed the Maxwell-Boltzmann distribution is given in terms of the ordinary exponential while the Boltzmann-Gibbs-Shannon entropy is defined starting from the ordinary logarithm. Clearly this is a very strong interpretation of the maximum entropy

principle. In generalized statistical theories, a weaker interpretation of the maximum entropy principle is customary, relaxing the second link.

Our next task is to use the strong interpretation of the maximum entropy principle holding for the ordinary statistical mechanics in order to minimally generalize it. The requirement of the twofold link between entropy and distribution function in generalized statistical theories also imposes that the function  $\Lambda(f)$  is a solution of the following functional-differential equation:

$$\frac{\partial}{\partial f} f \Lambda(f) = \lambda \Lambda(f/\alpha) + \eta, \quad (7.9)$$

where  $\{\alpha, \lambda, \eta\}$  are three real constants. Indeed the distribution (7.8) assumes now the form

$$f(p) = \alpha \Lambda^{-1} \left( -\frac{\mathbf{a} \cdot \mathbf{g}(p) - a_0 + \eta}{\lambda} \right), \quad (7.10)$$

$\Lambda^{-1}(x)$  being the inverse function of  $\Lambda(x)$ . Note that the constants  $\lambda$  and  $\eta$  introduce a scaling in the Lagrange multipliers. Clearly in the classical limit the result must be  $\Lambda(x) \rightarrow \ln(x)$ ,  $\Lambda^{-1}(x) \rightarrow \exp(x)$ , and  $\{\alpha, \lambda, \eta\} \rightarrow \{1/e, 1, 0\}$ .

The problem related to the determination of the generalized logarithm  $\Lambda(f)$  satisfying Eq. (7.9) and the conditions  $\Lambda(1)=0$ ,  $\Lambda'(1)=1$ , and  $\Lambda(0)=0$ , can be solved easily. In addition to the ordinary logarithm Eq. (7.9) admits other solutions. The general solution of this equation defines a three-parameter family of generalized logarithms indicated by  $\Lambda(f) = \ln_{\{\kappa, r, s\}}(f)$ , where

$$\ln_{\{\kappa, r, s\}}(f) = \frac{s^\kappa f^{\kappa+r} - s^{-\kappa} f^{\kappa-r} - s^\kappa + s^{-\kappa}}{(\kappa+r)s^\kappa + (\kappa-r)s^{-\kappa}}, \quad (7.11)$$

while  $\{\kappa, r, s\}$  are three free parameters. The constants  $\alpha$ ,  $\lambda$ , and  $\eta$  are given by

$$\alpha = \left( \frac{1+r-\kappa}{1+r+\kappa} \right)^{1/2\kappa}, \quad (7.12)$$

$$\lambda = \frac{(1+r-\kappa)^{(r+\kappa)/2\kappa}}{(1+r+\kappa)^{(r-\kappa)/2\kappa}}, \quad (7.13)$$

$$\eta = (\lambda - 1) \frac{s^\kappa - s^{-\kappa}}{(\kappa+r)s^\kappa + (\kappa-r)s^{-\kappa}}. \quad (7.14)$$

Some specific choice for the parameters  $\{\kappa, r, s\}$  generating particular expressions for the function  $\Lambda(f)$  and for its inverse function  $\Lambda^{-1}(x) = \exp_{\{\kappa, r, s\}}(x)$ , have already been considered in mathematical statistics, information theory, and statistical mechanics [20]. As stressed previously, in Refs. [18–23], it is shown that it is possible to develop self-consistent statistical theories starting from any generalized entropy satisfying some standard requirements. Fortunately all these requirements are satisfied by the entropy defined through Eq. (7.7) with  $\Lambda(f) = \ln_{\{\kappa, r, s\}}(f)$ .

Let us focus now our attention on the solution (7.11) corresponding to the particular choice  $\{\kappa, r=0, s=1\}$  [18]. It is easy to verify that the result is  $\ln_{\{\kappa,0,1\}}(x)=\ln_{\{\kappa\}}(x)$  and  $\exp_{\{\kappa,0,1\}}(x)=\exp_{\{\kappa\}}(x)$  with

$$\ln_{\{\kappa\}}(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa}, \quad (7.15)$$

$$\exp_{\{\kappa\}}(x) = (\sqrt{1 + \kappa^2 x^2} + \kappa x)^{1/\kappa}. \quad (7.16)$$

Thus the  $\kappa$  exponential and  $\kappa$  logarithm emerging in special relativity in the place of the ordinary exponential and logarithm reappear also in the statistical theory through the maximum entropy principle. It is easy to verify that the reality and positivity of the parameters  $\alpha$  and  $\lambda$  impose that  $-1 < \kappa < 1$ .

Furthermore, in the case  $\{\kappa, 0, s\}$ , Eq. (7.11) produces a two-parameter generalized logarithm and exponential, which turn out to be the scaled  $\kappa$  logarithm and the scaled  $\kappa$  exponential, respectively, defined as

$$\ln_{\{\kappa,s\}}(x) = \frac{\ln_{\{\kappa\}}(sx) - \ln_{\{\kappa\}}s}{\sqrt{1 + \kappa^2 \ln_{\{\kappa\}}^2 s}}, \quad (7.17)$$

$$\exp_{\{\kappa,s\}}(x) = \frac{1}{s} \exp_{\{\kappa\}}(x \sqrt{1 + \kappa^2 \ln_{\{\kappa\}}^2 s} + \ln_{\{\kappa\}}s). \quad (7.18)$$

We remark that the functions  $\exp_{\{\kappa\}}(x)$  and  $\ln_{\{\kappa\}}(x)$  generalizing the ordinary exponential and logarithm, respectively, emerge both in special relativity and independently in many body physics, through the maximum entropy principle, the cornerstone of statistical mechanics. This interesting fact suggests that we consider with particular attention the statistical theories enforced by the entropies related to the generalized logarithms defined through Eqs. (7.15) and (7.17).

In the next section we consider the statistical theory involving the generalized logarithm (7.15).

### VIII. GENERALIZED DISTRIBUTION FUNCTION

Let us consider the entropy  $S = -\langle \ln_{\{\kappa\}} f \rangle$  related to the generalized logarithm defined through Eq. (7.15). Here the distribution function  $f = f(x, p)$  depends on the four-vectors position  $x$  and momentum  $p$ . The explicit form of this entropy is given by

$$S = - \int d^3p f \ln_{\{\kappa\}}(f), \quad (8.1)$$

and after maximizing under the constraints (7.2) by solving the variational equation

$$\frac{\delta}{\delta f}(S + C) = 0 \quad (8.2)$$

it yields the distribution function

$$f = \alpha \exp_{\{\kappa\}}\left(-\frac{\mathbf{a} \cdot \mathbf{g} - a_0}{\lambda}\right), \quad (8.3)$$

with  $\mathbf{g} = \mathbf{g}(x, p)$  and

$$\alpha = \left(\frac{1 - \kappa}{1 + \kappa}\right)^{1/2\kappa}, \quad (8.4)$$

$$\lambda = \sqrt{1 - \kappa^2}, \quad (8.5)$$

where  $|\kappa| < 1$ . Before proceeding further we explain briefly the meaning of the constants  $\alpha$  and  $\lambda$ , related by  $1/\lambda = \ln_{\{\kappa\}} \times (1/\alpha)$ . We write the entropy in the form  $S = \int_{\mathcal{R}} d^3p \sigma(f)$  where  $\sigma(f) = -f \ln_{\{\kappa\}}(f)$  is the entropy density. It is easy to verify that for  $f = \alpha$  the entropy density assumes its maximum value, which is given by  $\sigma_{max} = \alpha/\lambda$ .

We focus now our attention on the form of  $\mathbf{g} = \mathbf{g}(x, p)$ . In Ref. [27] it is shown that for a relativistic many body system, in the presence of external electromagnetic fields, the more general microscopic invariant has the form  $(p^\nu + qA^\nu/c)U_\nu + \text{const}$ ,  $U_\nu$  being the hydrodynamic four-vector velocity with  $U^\nu U_\nu = c^2$ . Then we can pose  $g_1 = (p^\nu + qA^\nu/c)U_\nu - mc^2$ ,  $a_1 = 1/k_B T$ , and  $a_0 = \mu/k_B T$ . The distribution (8.3) in the case of relativistic statistical systems assumes the form

$$f = \alpha \exp_{\{\kappa\}}\left(-\frac{(p^\nu + eA^\nu/c)U_\nu - mc^2 - \mu}{\lambda k_B T}\right), \quad (8.6)$$

and is quite different from the Juttner distribution of the ordinary relativistic statistical mechanics, where in place of the  $\kappa$  exponential appears the ordinary exponential [27]. The distribution (8.6), in the global rest frame where  $U^\nu = (c, 0, 0, 0)$  and in absence of external forces ( $A^\nu = 0$ ), simplifies as

$$f = \alpha \exp_{\{\kappa\}}\left(-\frac{W - \mu}{\lambda k_B T}\right), \quad (8.7)$$

$W$  being the relativistic kinetic energy

$$W = \sqrt{m^2 c^4 + \mathbf{p}^2 c^2} - mc^2, \quad (8.8)$$

and in the classical limit  $\kappa \rightarrow 0$  it reduces to the classical distribution (7.6). It is remarkable that the above distribution (8.7) presents a power law asymptotic behavior, namely,

$$f \propto W^{-1/\kappa} \quad (W \rightarrow \infty), \quad (8.9)$$

in contrast with the ordinary relativistic distribution (Maxwell-Boltzmann distribution where the energy is given by its relativistic expression) which decays exponentially.

### IX. GENERALIZED STATISTICS AND KINETICS

In this section we report the main features of the statistical theory, based on the entropy (8.1), some of which are already known in the literature.

*Equiprobability.* The non-negative entropy (8.1) achieves its maximum value at equiprobability,  $f(p) = 1/\Omega$  for  $\forall p$ , and this value is  $S = \ln_{\{\kappa\}} \Omega$ .

*Thermodynamic stability.* The entropy (8.1) is concave,

$$S[tf_1 + (1-t)f_2] \geq tS[f_1] + (1-t)S[f_2], \quad (9.1)$$

so that the system, in thermodynamic equilibrium, is stable [19].

*Lesche stability.* The Lesche stability condition is satisfied by any physically meaningful quantity. In particular it is a necessary condition to experimentally detect a physical observable. The entropy given by Eq. (8.1), being a physical quantity depending on a probability distribution, should exhibit a small relative error

$$R = \left| \frac{S[f] - S[f']}{\sup(S[f])} \right|, \quad (9.2)$$

with respect to small changes of the probability distributions

$$D = \|f - f'\|. \quad (9.3)$$

Mathematically this implies that for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $R \leq \varepsilon$  holds for all distribution functions satisfying  $D \leq \delta$ . The Lesche stability condition holds for the Boltzmann-Gibbs-Shannon entropy and is proved also for the entropy (8.1) in Ref. [28]. In the thermodynamic limit  $\delta = \varepsilon^{1/(1-|\kappa|)}$  is obtained.

*Relativistic kinetic equation.* The statistical distribution (8.7) can be viewed as the stationary case of a distribution function  $f = f(x, p)$ , describing a relativistic many body system, governed by the generalized Boltzmann evolution equation [19]

$$p^\nu \partial_\mu f - m F^\nu \frac{\partial f}{\partial p^\nu} = \int \frac{d^3 p'}{p'^0} \frac{d^3 p_1}{p_1^0} \frac{d^3 p'_1}{p'^0_1} G \times [a(f' \otimes_\kappa f'_1) - a(f \otimes_\kappa f_1)]. \quad (9.4)$$

In this equation the standard notations of the ordinary Boltzmann kinetics are used and  $a(f)$  is a positive and increasing arbitrary function, which does not affect the stationary form of  $f$ . The factor  $G$  is the transition rate, depending only on the nature of the two-body particle interaction. The composition law  $\otimes_\kappa$  defined through

$$\ln_{\{\kappa\}}(f \otimes_\kappa h) = \ln_{\{\kappa\}} f + \ln_{\{\kappa\}} h, \quad (9.5)$$

is a one-parameter generalization of the ordinary product having the following properties: (i) associative law,  $(f \otimes_\kappa h) \otimes_\kappa w = f \otimes_\kappa (h \otimes_\kappa w)$ ; (ii) neutral element,  $f \otimes_\kappa 1 = 1 \otimes_\kappa f = f$ ; (iii) inverse element,  $f \otimes_\kappa (1/f) = (1/f) \otimes_\kappa f = 1$ ; (iv) commutative law,  $f \otimes_\kappa h = h \otimes_\kappa f$ . Furthermore we get  $f \otimes_\kappa 0 = 0 \otimes_\kappa f = 0$  and the generalized division  $\oslash_\kappa$  can be defined through  $f \oslash_\kappa h = f \otimes_\kappa (1/h)$ .

*H theorem.* The four-vector entropy  $S^\nu = (S^0, \mathbf{S})$  was defined as

$$S^\nu = - \int \frac{d^3 p}{p^0} p^\nu f \ln_{\{\kappa\}} f. \quad (9.6)$$

The temporal component  $S^0$  of this four-vector coincides with the  $\kappa$  entropy defined previously through Eq. (8.1) while the spatial component  $\mathbf{S}$  is the entropy flow. Starting from the evolution equation (9.4) the  $H$  theorem has been demonstrated, which represents the second law of thermodynamics and states that the entropy production is never negative, and, in equilibrium conditions, there is no entropy pro-

duction. That is, the following relation was obtained:

$$\partial_\nu S^\nu \geq 0, \quad (9.7)$$

which is the local formulation of the relativistic  $H$  theorem and represents the second law of thermodynamics [19].

*Entropy transformation.* After recalling the identity  $d^3 p / p^0 = d^4 p \, 2\theta(p^0) \delta(p^\mu p_\mu - m^2 c^2)$  the definition (9.6) assumes the form

$$S^\nu = - \int d^4 p \, 2\theta(p^0) \delta(p^\mu p_\mu - m^2 c^2) p^\nu f \ln_{\{\kappa\}} f. \quad (9.8)$$

In this expression  $d^4 p$  is a scalar because the Jacobian of the Lorentz transformation is equal to unity. Then since  $p^\nu$  transforms as a four-vector, we can conclude that  $S^\nu$  transforms as a four-vector like the particle four-flow

$$N^\nu = \int d^4 p \, 2\theta(p^0) \delta(p^\mu p_\mu - m^2 c^2) p^\nu f, \quad (9.9)$$

for which the conservation law of the total particle number imposes

$$\partial_\nu N^\nu = 0. \quad (9.10)$$

*Calculation of the parameter  $\kappa$ .* We focus now attention on the general discussion of Sec. II. The results obtained in Sec. II permit us to consider the problem related to the calculation of the parameters from a very different and more physically sound point of view with respect to the one of Ref. [19]. Clearly the velocity  $v_*$  introduced through  $|\kappa| = v_*/c$  does not emerge within the one-particle relativistic theory. In general  $v_*$  could depend on  $c$ , namely,  $v_* = v_*(c)$  with the condition that  $v_*(\infty) < \infty$ . The fact that the condition  $|\kappa| < 1$  and then  $v_* < c$  follows from the maximum entropy principle supports the supposition that  $v_*$  could emerge in the framework of many body theory. Keeping in mind that in statistical mechanics the only emerging velocity is the thermal velocity, in the following we identify  $v_*$  with the thermal velocity of the many body system.

The particle thermal energy, being the mean particle kinetic energy  $\langle W \rangle$ , is related to the thermal velocity  $v_* = \kappa c$  according to the relativistic relationship

$$\langle W \rangle = mc^2 \left( \frac{1}{\sqrt{1 - \kappa^2}} - 1 \right). \quad (9.11)$$

Clearly, in a statistical theory, the mean particle kinetic energy  $\langle W \rangle$  is considered a known quantity. Then the expression of  $\kappa$  in terms of  $\langle W \rangle$  follows immediately,

$$\kappa^2 = 1 - \left( 1 + \frac{\langle W \rangle}{mc^2} \right)^{-2}, \quad (9.12)$$

so that the theory does not contain free parameters and in the classical limit  $c \rightarrow \infty$ , we get  $\kappa = 0$ . From Eq. (9.12) it follows that  $\lim_{c \rightarrow \infty} \kappa c = \sqrt{2\langle W \rangle / m} < \infty$  as required from the theory developed in Sec. II. Note that when we know experimentally the distribution function, the value of  $\kappa$  and consequently, by using Eq. (9.12), the value of the ratio  $\langle W \rangle / mc^2$  can be obtained from the slope of the distribution tails in a log-log plot. The normalization condition for the distribution

function  $f$  and the definition of the particle mean kinetic energy, starting from  $f$ , permit us to determine univocally  $T$  and  $\mu$ . In this way one can link  $\langle W \rangle$  and then  $\kappa$  to the temperature of the system.

Bearing in mind the meaning of the parameter  $\kappa$  we rewrite the distribution function (8.7) in the form

$$f = \alpha \exp_{\{\kappa\}} \left( - \frac{W - \mu}{k_B T'} \right), \quad (9.13)$$

$W$  being the relativistic kinetic energy while the temperature  $T'$  is given by

$$T' = \sqrt{1 - \frac{v_*^2}{c^2}} T. \quad (9.14)$$

We remark that the temperature  $T'$  is contracted with respect to the temperature  $T$  by a factor which, curiously, has the same form as the one entering in the definition of the relativistic temperature suggested by Einstein [29] and Planck [30].

### X. APPLICATIONS

In the last four years the present theory has been used successfully to study some systems which manifestly cannot be treated within the ordinary statistical mechanics. Clearly the most natural applications regard relativistic many body systems.

The first application, considered in Ref. [19], concerns cosmic rays. For a long time it is known that the cosmic ray spectrum, which extends over 13 decades in energy, from a few hundred MeV ( $10^8$  eV) to a few hundred EeV ( $10^{20}$  eV) and spans 33 decades in particle flux, from  $10^4$  to  $10^{-29}$  ( $\text{m}^2 \text{sr s GeV}^{-1}$ ), is not exponential and, thus, it violates the Boltzmann equilibrium statistical distribution, proportional to  $\exp(-E/k_B T)$  [11–13]. On the other hand, it is known that the particles composing the cosmic rays are essentially normal nuclei as in the standard cosmic abundance of matter. Then the cosmic rays can be viewed as an equivalent statistical system of identical relativistic particles with masses near the mass of the proton (938 MeV). These characteristics (relativistic particles with a very large extension for both their flux and energy) make the cosmic ray spectrum an ideal physical system for a preliminary test of the correctness and predictability of any relativistic theory. In Ref. [19], a high quality agreement between the predictions of the present theory and the observed data in the whole cosmic ray spectrum has been found. This agreement over so many decades is quite remarkable.

Other physical applications of the theory regard the formation of a quark-gluon plasma [31] and kinetic models describing a gas of interacting atoms and photons [32].

On the other hand the theory has been applied successfully also to the study of natural or artificial systems exhibiting a limiting velocity in the propagation of the information (like the one imposed by light speed in physics) where a mechanism analogous to the relativistic one can emerge, deforming the distribution function. In Ref. [33] the problem of the fracture propagation has been considered within the

present theory. Other applications have been considered in games theory [34], in economics for study of income distribution [35], and in constructing financial models [36], etc.

We come back to the high energy physics being the more proper field of application of the theory. In the following as a further application we consider the open problem of black hole physics regarding the intriguing question of the Bekenstein-Hawking area law [14,15]. This law asserts that the entropy of a black hole scales as the area  $A$  of the event horizon, namely,  $S \propto A$ . It is well known that the ordinary Boltzmann statistics and thermodynamics fail to justify this law and contrarily predict that the entropy scales as the volume  $V$  of the spatial region delimited by the event horizon, namely,  $S \propto V$ .

Indeed if we indicate with  $N$  the degrees of freedom or microscopic components in the volume  $V$  and with  $M$  the number of states accessible by each component, within the Boltzmann statistics the number of microstates  $\Omega$  of the whole system is given by  $\Omega \propto M^N$  with  $N \gg 1$ . After introducing the notation

$$n(0) = \exp N, \quad m(0) = \ln M, \quad (10.1)$$

the number of microstates can be written also in the form

$$\Omega \propto n(0)^{m(0)}. \quad (10.2)$$

Customarily one assumes a uniform distribution of the components in the volume  $V$  obtaining  $N \propto V$ . Thus the entropy

$$S = \ln \Omega \quad (10.3)$$

of the black hole results as

$$S \propto V. \quad (10.4)$$

This result of the Boltzmann statistics, already known in the literature, is manifestly in contradiction with the Bekenstein-Hawking area law.

Very recently in Ref. [37] it has been suggested to treat this problem within nonstandard statistics. Even if the authors limit their discussion to the nonextensive [38] aspects of the problem the proposed approach has a general validity and can be adopted also in the present case.

Clearly the black hole is a relativistic system and it appears natural to write its entropy in the microcanonical picture as

$$S = \ln_{\{\kappa\}} \Omega. \quad (10.5)$$

Furthermore the present formalism generalizes Eqs. (10.1) and (10.2) as follows:

$$n(\kappa) = \exp_{\{\kappa\}} N, \quad m(\kappa) = \ln_{\{\kappa\}} M, \quad (10.6)$$

$$\Omega \propto n(\kappa)^{m(\kappa)}. \quad (10.7)$$

After recalling the property (A15) the number of microstates becomes

$$\Omega \propto \exp_{\{\kappa/m(\kappa)\}} [m(\kappa)N], \quad (10.8)$$

while the black hole entropy can be written as

$$S \propto \ln_{\{\kappa\}}\{\exp_{\{\kappa/m(\kappa)\}}[m(\kappa)N]\}. \quad (10.9)$$

In the thermodynamic limit  $N \gg 1$ , then Eqs. (10.8) and (10.9) predict that the number of microstates and the entropy scale as power laws

$$\Omega \propto N^{m(\kappa)/\kappa}, \quad (10.10)$$

$$S \propto N^{m(\kappa)}. \quad (10.11)$$

We remark that the power law behavior of the various physical quantities is a feature of the present theory in contrast with the exponential and logarithmic behaviors in classical statistical mechanics. Finally we assume, as is customary, a uniform distribution of the components in the volume  $V$ , namely,  $N \propto V$ , and obtain the following expression of the entropy:

$$S \propto V^{m(\kappa)}. \quad (10.12)$$

After setting

$$m(\kappa) = 2/3, \quad (10.13)$$

the Bekenstein-Hawking law, namely,  $S \propto A$ , follows immediately and thus we can conclude that the present statistical theory is consistent with the area law of black hole entropy. This result is in agreement with Ref. [37] where the authors propose a relationship analogous to the one given by Eq. (10.13).

## XI. CONCLUSION

We have shown that within the standard framework of special relativity, the Lorentz transformations lead to a proper one-parameter generalization of the entropy just as for other physically meaningful quantities (e.g., momentum, energy, etc.).

This entropy generates a coherent and self-consistent relativistic statistical theory (both statistical mechanics as well as kinetic theory) preserving the main features (maximum entropy principle, thermodynamic stability, continuity condition,  $H$  theorem, etc.) of the ordinary statistical theory which reemerges in the classical limit.

The distribution function predicted by the present theory is a one-parameter, continuous deformation of the classical Maxwell-Boltzmann distribution exhibiting a power law tail in accordance with the experimental evidence in several relativistic many body systems.

## APPENDIX

We start by considering the  $\kappa$  sum for the dimensionless relativistic momenta

$$t_1 \oplus t_2 = t_1 \sqrt{1 + \kappa^2 t_2^2} + t_2 \sqrt{1 + \kappa^2 t_1^2}, \quad (A1)$$

and after setting  $t_1 = t + dt$  and  $t_2 = -t$  we can introduce the  $\kappa$  differential through

$$d_{\{\kappa\}}t = (t + dt) \ominus t = (t + dt) \oplus (-t) = \frac{dt}{\sqrt{1 + \kappa^2 t^2}}. \quad (A2)$$

The  $\kappa$  derivative  $d/d_{\{\kappa\}}t$  admits as eigenfunction the  $\kappa$  exponential, namely,

$$\frac{d}{d_{\{\kappa\}}t} \exp_{\{\kappa\}}(t) = \exp_{\{\kappa\}}(t), \quad (A3)$$

where

$$\exp_{\{\kappa\}}(t) = (\sqrt{1 + \kappa^2 t^2} + \kappa t)^{1/\kappa}, \quad (A4)$$

$$\exp_{\{\kappa\}}(t) = \exp\left(\frac{1}{\kappa} \operatorname{arcsinh}(\kappa t)\right). \quad (A5)$$

The  $\kappa$  exponential can be viewed as a generalization of the ordinary exponential

$$\exp_{\{0\}}(t) = \exp(t), \quad (A6)$$

$$\exp_{\{-\kappa\}}(t) = \exp_{\{\kappa\}}(t), \quad (A7)$$

and has the properties

$$\exp_{\{\kappa\}}(t) \in C^\infty(\mathbf{R}), \quad (A8)$$

$$\frac{d}{dt} \exp_{\{\kappa\}}(t) > 0, \quad (A9)$$

$$\frac{d^2}{dt^2} \exp_{\{\kappa\}}(t) > 0, \quad (A10)$$

$$\exp_{\{\kappa\}}(-\infty) = 0^+, \quad (A11)$$

$$\exp_{\{\kappa\}}(0) = 1, \quad (A12)$$

$$\exp_{\{\kappa\}}(+\infty) = +\infty, \quad (A13)$$

$$\exp_{\{\kappa\}}(t) \exp_{\{\kappa\}}(-t) = 1. \quad (A14)$$

Furthermore it results that

$$[\exp_{\{\kappa\}}(t)]' = \exp_{\{\kappa/r\}}(rt), \quad (A15)$$

$$\exp_{\{\kappa\}}(t_1) \exp_{\{\kappa\}}(t_2) = \exp_{\{\kappa\}}(t_1 \oplus t_2), \quad (A16)$$

$$\exp_{\{\kappa\}}(t_1) \otimes_{\kappa} \exp_{\{\kappa\}}(t_2) = \exp_{\{\kappa\}}(t_1 + t_2), \quad (A17)$$

where the product  $\otimes$  is defined through Eq. (9.5). The following asymptotic relations also hold:

$$\exp_{\{\kappa\}}(t) \underset{t \rightarrow 0}{\sim} 1 + t + \frac{t^2}{2} + (1 - \kappa^2) \frac{t^3}{3!}, \quad (A18)$$

$$\exp_{\{\kappa\}}(t) \underset{t \rightarrow \pm\infty}{\sim} |2\kappa t|^{\pm 1/|\kappa|}. \quad (A19)$$

Finally after tedious but straightforward calculations the following formula can be obtained:

$$\int_0^\infty t^{r-1} \exp_{\{\kappa\}}(-t) dt = \frac{|2\kappa|^{-r} \Gamma(1/|2\kappa| - r/2)}{1 + r|\kappa| \Gamma(1/|2\kappa| + r/2)} \Gamma(r), \quad (\text{A20})$$

holding for  $r|\kappa| < 1$ . Note that this integral is given also in [19] by Eq. (5.31), which unfortunately contains a typing error [the first minus in the denominator of the right hand side must be replaced by a plus so that after some simple algebra Eq. (5.31) of [19] reproduces the present result].

The  $\kappa$  logarithm  $\ln_{\{\kappa\}}(t)$ , defined as the inverse function of the  $\kappa$  exponential, namely,  $\ln_{\{\kappa\}}(\exp_{\{\kappa\}} t) = \exp_{\{\kappa\}}(\ln_{\{\kappa\}} t) = t$ , is given by

$$\ln_{\{\kappa\}}(t) = \frac{t^\kappa - t^{-\kappa}}{2\kappa}, \quad (\text{A21})$$

$$\ln_{\{\kappa\}}(t) = \frac{1}{\kappa} \sinh(\kappa \ln t), \quad (\text{A22})$$

and can be viewed as a generalization of the ordinary logarithm

$$\ln_{\{0\}}(t) = \ln(t), \quad (\text{A23})$$

$$\ln_{\{-\kappa\}}(t) = \ln_{\{\kappa\}}(t). \quad (\text{A24})$$

The  $\kappa$  logarithm, just like the ordinary logarithm, has the properties

$$\ln_{\{\kappa\}}(t) \in C^\infty(\mathbf{R}^+), \quad (\text{A25})$$

$$\frac{d}{dt} \ln_{\{\kappa\}}(t) > 0, \quad (\text{A26})$$

$$\frac{d^2}{dt^2} \ln_{\{\kappa\}}(t) < 0, \quad (\text{A27})$$

$$\ln_{\{\kappa\}}(0^+) = -\infty, \quad (\text{A28})$$

$$\ln_{\{\kappa\}}(1) = 0, \quad (\text{A29})$$

$$\ln_{\{\kappa\}}(+\infty) = +\infty, \quad (\text{A30})$$

$$\ln_{\{\kappa\}}(1/t) = -\ln_{\{\kappa\}}(t). \quad (\text{A31})$$

The  $\kappa$  logarithm has also the properties

$$\ln_{\{\kappa\}}(t^r) = r \ln_{\{\kappa\}}(t), \quad (\text{A32})$$

$$\ln_{\{\kappa\}}(t_1 t_2) = \ln_{\{\kappa\}}(t_1) \oplus_{\kappa} \ln_{\{\kappa\}}(t_2), \quad (\text{A33})$$

$$\ln_{\{\kappa\}}(t_1 \otimes_{\kappa} t_2) = \ln_{\{\kappa\}}(t_1) + \ln_{\{\kappa\}}(t_2), \quad (\text{A34})$$

while the following asymptotic formulas hold:

$$\ln_{\{\kappa\}}(1+t) \sim t - \frac{t^2}{2} + \left(1 + \frac{\kappa^2}{2}\right) \frac{t^3}{3}, \quad (\text{A35})$$

$$\ln_{\{\kappa\}}(t) \underset{t \rightarrow 0^+}{\sim} -\frac{1}{2|\kappa|} t^{-|\kappa|}, \quad (\text{A36})$$

$$\ln_{\{\kappa\}}(t) \underset{t \rightarrow +\infty}{\sim} \frac{1}{2|\kappa|} t^{|\kappa|}. \quad (\text{A37})$$

Furthermore, the very useful formula

$$\frac{d^2}{dt^2} t \ln_{\{\kappa\}}(t) > 0 \quad (\text{A38})$$

holds, and finally one obtains [19]

$$\int_0^1 dt \left( \ln_{\{\kappa\}} \frac{1}{t} \right)^{r-1} = \frac{|2\kappa|^{1-r} \Gamma(1/|2\kappa| - (r-1)/2)}{1 + (r-1)|\kappa| \Gamma(1/|2\kappa| + (r-1)/2)} \Gamma(r). \quad (\text{A39})$$

It is evident that the above used procedure to introduce the  $\kappa$  exponential and  $\kappa$  logarithm, starting from the additivity law of the momenta, can be used to obtain further functions if we start from other generalized additivity laws. For instance, starting from the additivity law of the velocities we have  $d_{\{\kappa\}} t = (1 - \kappa^2 t^2)^{-1} dt$ , while instead of the  $\kappa$  exponential we obtain the function

$$\varphi(t) = \left( \frac{1 + \kappa t}{1 - \kappa t} \right)^{1/2\kappa} \quad (\text{A40})$$

$$= \exp\left( \frac{1}{\kappa} \operatorname{arctanh}(\kappa t) \right), \quad (\text{A41})$$

whose inverse function assumes the form

$$\varphi^{-1}(t) = \frac{1}{\kappa} \frac{t^\kappa - t^{-\kappa}}{t^\kappa + t^{-\kappa}} \quad (\text{A42})$$

$$= \frac{1}{\kappa} \tanh(\kappa \ln t). \quad (\text{A43})$$

The function  $\varphi(t) \in C^\infty(I)$  with  $I = ]-\kappa^{-1}, +\kappa^{-1}[$  is connected to  $\exp_{\{\kappa\}}(t)$  through

$$\varphi(t) = \exp_{\{\kappa\}}\left( \frac{t}{\sqrt{1 - \kappa^2 t^2}} \right). \quad (\text{A44})$$

At this point one could be tempted to replace  $\exp_{\{\kappa\}}(t)$  with the function  $\varphi(t)$  as the right one for generalizing the ordinary exponential.

There are several reasons to consider  $\exp_{\{\kappa\}}(t)$  as the more proper generalization of the ordinary exponential. In addition to the motivations emerging within the one-particle relativistic dynamics, there is the one related to the maximum entropy principle. This principle, the cornerstone of statistical physics, selects without any ambiguity (see Sec. VII) just the  $\kappa$  exponential as the more natural generalization of the ordinary exponential. More precisely the function  $\exp_{\{\kappa\}}(t)$  is the only one existing that emerges simultaneously both in the one-particle relativistic dynamics as well as in many body physics through the maximum entropy principle.

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