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Abstract

1 Introduction

The study of rotordynamics has been faced up deeply during last century. However, only after World War II, i.e. with the development of gas turbines in aeronautical field, the need for a better understanding of rotating system dynamics has become an urgent problem.

For a long time the studies were based on the hypothesis that the deformed shaft rotates around the axis of the bearings at the constant angular velocity imposed at the end section of the shaft itself by an external torque. This hypothesis, substantially neglecting the effects of Coriolis' acceleration, could not explain how a shaft might rotate without breaking at angular velocities greater than the first critical velocity.

foep foep and jeff jeff, referring to a rotating disk mounted with non-zero eccentricity on a massless elastic shaft, were the first to introduce the idea that the disk instantaneous axis could not be considered always coincident with the bearing axis, but it had to be determined from the system equilibrium equations. Their idea permitted to overtake the contradiction with experimental outcomes.

It is practically impossible to cite all the researchers acting in this field of mechanics; however it is worth to remember at least three researchers for their important contributes to rotordynamics: dime dime, tond tond and yama yama.

In this paper, the authors analyze the dynamic behaviour of asymmetric rotating shafts having distributed mass and distributed stiffness; the analytical solutions are obtained in a fixed reference frame. It is worth noting that in the case of shafts with non circular sections, a rotating reference frame is commonly adopted (see, e.g., tond's work tond, pp.96–105 and kram's book kram, pp.192–199); anyway also fixed reference approaches can be found in the literature (black black and gane gane). Therefore the originality of the paper consists in the mathematical technique used to abtain analytical solutions that, it must be underlined, are well known.

In addition, in the present work the instability region is explicitly related to the asymmetry itself: therefore the results allow to analyze the shaft dynamical behaviour in a simple and direct way.

A case test is analyzed, considering a shaft mounted on two rigid bearings, possessing uniformly distributed mass and rotating at constant angular speed, in absence of external disturbances.

2 Shaft model

Figure 1 shows a generic section of a vertical shaft, projected on a plane normal to the bearing s-axis; the section is obtained with a plane perpendicular to the tangent to the deformed axis line.

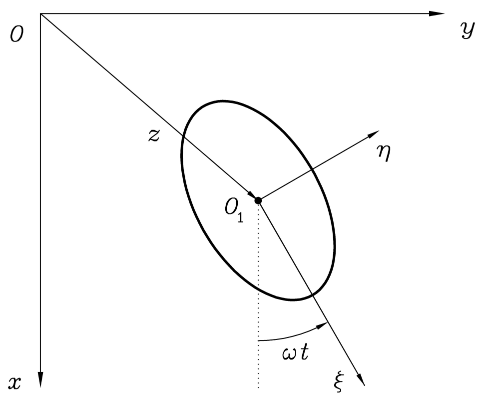

Figure 1: Shaft section

The shaft section is elliptical, with axes ξ and η fixed to the section, coincident with the axes of the geometrical inertial central ellipse.

According to Bernoulli-Euler model, the dynamical equilibrium equations have the following form:

$$ES\frac{\partial^4 z}{\partial s^4} - EDe^{2\imath\omega t}\frac{\partial^4 \tilde{z}}{\partial s^4} + \mu\frac{\partial^2 z}{\partial t^2} = 0 \tag{1}$$

$$ES\frac{\partial^4 \tilde{z}}{\partial s^4} - EDe^{-2\imath\omega t}\frac{\partial^4 z}{\partial s^4} + \mu\frac{\partial^2 \tilde{z}}{\partial t^2} = 0, \tag{2}$$

where E is the shaft elasticity modulus, μ is the mass per unit length, x and y are the coordinates of the section centre, $z = x + \imath y$ is the elastic displacement, $\tilde{z}$ is the complex conjugate of z, s is the distance measured along the bearing axis, ω is the shaft angular speed, I_ξ and I_η are the inertial moments respectively along axis ξ and η, $S = (I_\xi + I_\eta)/2$ and $D = (I_\xi - I_\eta)/2$ (assuming that $I_\xi > I_\eta$).

After dividing Eq. (1) and (2) by $\mu\omega^2$, let

$$\chi = \frac{ES}{\mu\omega^2}; \quad \Delta\chi = \frac{ED}{\mu\omega^2}; \quad \tau = \imath\omega t; \tag{3}$$

it yields:

$$\chi\frac{\partial^4 z}{\partial s^4} - \Delta\chi e^{2\tau}\frac{\partial^4 \tilde{z}}{\partial s^4} - \frac{\partial^2 z}{\partial \tau^2} = 0 \tag{4}$$

$$\chi\frac{\partial^4 \tilde{z}}{\partial s^4} - \Delta\chi e^{-2\tau}\frac{\partial^4 z}{\partial s^4} - \frac{\partial^2 \tilde{z}}{\partial \tau^2} = 0. \tag{5}$$

Let now consider the case in which quantities χ and $\Delta\chi$ are constant and do not depend on coordinate s; therefore, from Eq. (5) it holds:

$$\frac{\partial^8 \tilde{z}}{\partial s^8} = \frac{1}{\chi}\left(\Delta\chi e^{-2\tau}\frac{\partial^8 z}{\partial s^8} + \frac{\partial^2}{\partial \tau^2}\frac{\partial^4 \tilde{z}}{\partial s^4}\right), \tag{6}$$

while from Eq. (4)

$$\frac{\partial^4 \tilde{z}}{\partial s^4} = \frac{e^{-2\tau}}{\Delta\chi}\left(\chi\frac{\partial^4 z}{\partial s^4} - \frac{\partial^2 z}{\partial \tau^2}\right) \tag{7}$$

$$\frac{\partial^8 z}{\partial s^8} = \frac{1}{\chi}\left(\Delta\chi e^{2\tau}\frac{\partial^8 \tilde{z}}{\partial s^8} + \frac{\partial^2}{\partial \tau^2}\frac{\partial^4 z}{\partial s^4}\right). \tag{8}$$

Substituting Eq. (6) and (7) in Eq. (8), it yields:

$$\chi\frac{\partial^8 z}{\partial s^8} = \Delta\chi e^{2\tau}\left\{\frac{1}{\chi}\left[\Delta\chi e^{-2\tau}\frac{\partial^8 z}{\partial s^8} + \frac{\partial^2}{\partial \tau^2}\frac{e^{-2\tau}}{\Delta\chi}\left(\chi\frac{\partial^4 z}{\partial s^4} - \frac{\partial^2 z}{\partial \tau^2}\right)\right]\right\} + \frac{\partial^2}{\partial \tau^2}\frac{\partial^4 z}{\partial s^4}. \tag{9}$$

Finally:

$$\frac{\partial^4 z}{\partial s^4} - 4\frac{\partial^3 z}{\partial \tau^3} + \frac{\partial^2}{\partial \tau^2}\left(4z - 2\chi\frac{\partial^4 z}{\partial s^4}\right) - 4\chi\frac{\partial^4}{\partial s^4}\left(z - \frac{\partial z}{\partial \tau}\right) + \left(\chi^2 - \Delta\chi^2\right)\frac{\partial^8 z}{\partial s^8} = 0. \tag{10}$$

A solution for Eq. (10) can be obtained through separation of variables; let $z(\tau, s) = \varphi(\tau)\psi(s)$, where function $\psi(s)$ satisfy equation:

$$\psi'''' = \nu\psi, \qquad \nu = \text{ constant}. \tag{11}$$

After suitable substitutions, Eq. (10) becomes:

$$\ddot{\ddot{\varphi}} - 4\dddot{\varphi} + (4 - 2\chi\nu)\ddot{\varphi} + 4\chi\nu\dot{\varphi} + \left[\left(\chi^2 - \Delta\chi^2\right)\nu^2 - 4\chi\nu\right]\varphi = 0. \tag{12}$$

It is now necessary to solve the fourth order characteristic equation corresponding to Eq. (12), obtained by letting $\varphi = e^{r\tau}$:

$$r^4 - 4r^3 + (4 - 2\chi\nu)r^2 + 4\chi\nu r + \left[\left(\chi^2 - \Delta\chi^2\right)\nu^2 - 4\chi\nu\right] = 0. \tag{13}$$

It is well known that Eq. (13), having form $ar^4 + br^3 + cr^2 + dr + e = 0$, reduce to $y^4 + 2Ay^2 - By - C = 0$ if $y = r + b/(4a)$. In the case of Eq. (13), it holds:

$$y = r - 1 \tag{14}$$

$$A = \frac{c}{2a} - \frac{3b^2}{16a^2} = -(1+\chi\nu) \tag{15}$$

$$B = \frac{bc}{2a^2} - \frac{d}{a} - \frac{b^3}{8a^3} = 0 \tag{16}$$

$$C = \frac{3b^4}{256a^4} - \frac{b^2c}{16a^3} + \frac{db}{4a^2} - \frac{e}{a} = 2\chi\nu - 1 - \nu^2\left(\chi^2 - \Delta\chi^2\right); \tag{17}$$

hence:

$$y = \pm\sqrt{1 + \chi\nu \pm \sqrt{4\chi\nu + \nu^2\Delta\chi^2}}. \tag{18}$$

Let p_1 and p_2 be defined as follows:

$$p_1 = \sqrt{1 + \chi\nu + \sqrt{4\chi\nu + \nu^2\Delta\chi^2}} \tag{19}$$

$$p_2 = \sqrt{1 + \chi\nu - \sqrt{4\chi\nu + \nu^2\Delta\chi^2}}; \tag{20}$$

then the solutions of Eq. (13) are given by:

$$r = 1 \pm p_1 \quad \text{and} \quad r = 1 \pm p_2. \tag{21}$$

Therefore, function $\varphi(\tau)$ can be written in the form:

$$\varphi(\tau) = A_1 e^{(1+p_1)\tau} + A_2 e^{(1-p_1)\tau} + A_3 e^{(1+p_2)\tau} + A_4 e^{(1-p_2)\tau}, \tag{22}$$

while function $\psi(s)$, solving Eq. (11), becomes:

$$\psi(s) = B_1 \sin\left(\sqrt[4]{\nu}s\right) + B_2 \cos\left(\sqrt[4]{\nu}s\right) + B_3 \sinh\left(\sqrt[4]{\nu}s\right) + B_4 \cosh\left(\sqrt[4]{\nu}s\right). \tag{23}$$

The four integration constants A_i reduce to two, because the solution must satisfy Eq. (4) and (5); hence, from Eq. (4):

$$\chi\psi^{''''}\varphi - \Delta\chi e^{2\tau}\psi^{''''}\tilde{\varphi} - \psi\ddot{\varphi} = 0, \tag{24}$$

where $\tilde{\varphi}$ is the complex conjugate of φ. Substituting Eq. (11) and (22) in Eq. (24) it is possible to determine the following relations between the constants A_i:

$$\frac{A_1}{A_2} = \frac{\nu\Delta\chi}{\chi\nu - (1+p_1)^2} \quad \text{and} \quad \frac{A_3}{A_4} = \frac{\nu\Delta\chi}{\chi\nu - (1+p_2)^2}. \tag{25}$$

With regard to constants B_i, the four boundary conditions, corresponding to null values of the function and of its first or second derivative, lead to four homogeneous equations. Therefore, for any real and distinct value of ν, constants B_i are determined except from a constant that can be either assumed equal to one or included in the A_i constants. Finally, displacement z can be expressed as:

$$z = \sum_{\nu}\sum_{i=1}^{4} A_i e^{r_i\tau} \sum_{j=1}^{4} B_j e^{\sqrt[4]{\nu}s}; \tag{26}$$

hence there are two series of constants $A(\nu)$ necessary and sufficient to define the initial position and velocity of the points of the shaft axis.

3 Case test: pinned–pinned shaft

For a pinned–pinned shaft, the boundary conditions lead to

$$B_2 = B_3 = B_4 = 0 \quad \text{and} \quad \sin\left(\sqrt[4]{\nu}l\right) = 0, \quad \text{i.e. } \nu = n^4\pi^4/l^4. \tag{27}$$

From the analysis of the roots of the temporal function $\varphi(\tau)$ it is possible determine the instability conditions: in fact it results that root p_2 may be complex, hence causing a solution growing with time. This situation occurs when

$$1 + \chi\nu < \sqrt{4\chi\nu + \nu^2 \Delta\chi^2}, \qquad \text{i.e.} \qquad |1 - \chi\nu| < \nu\Delta\chi. \tag{28}$$

By substituting back the inertial moments, it holds:

$$-\nu E \frac{I_\xi - I_\eta}{2\mu\omega^2} < 1 - \nu E \frac{I_\xi + I_\eta}{2\mu\omega^2} < \nu E \frac{I_\xi - I_\eta}{2\mu\omega^2}; \tag{29}$$

hence:

$$\nu E \frac{I_\eta}{\mu} < \omega^2 < \nu E \frac{I_\xi}{\mu}. \tag{30}$$

Therefore, for angular speed inside the range defined by the limit values determined in Eq. (30), the motion of the shaft is unstable. It is immediate to verify that the limit values computed in Eq. (30) represent the natural frequencies corresponding to uniform stiffness respectively equal to EI_ξ and EI_η. Let ω_{n0} be the frequency of a circular shaft; then:

$$\nu = \frac{\mu\omega_{n0}^2}{EI}, \tag{31}$$

where $I = (I_\xi + I_\eta)/2$. Hence Eq. (30) becomes:

$$\begin{cases} \omega^2 < E \dfrac{I_\xi}{\mu} \dfrac{2\mu\omega_{n0}^2}{E\,(I_\xi + I_\eta)} = \omega_{n0}^2 \left(1 + \dfrac{\Delta\chi}{\chi}\right) \\[2ex] \omega^2 > E \dfrac{I_\eta}{\mu} \dfrac{2\mu\omega_{n0}^2}{E\,(I_\xi + I_\eta)} = \omega_{n0}^2 \left(1 - \dfrac{\Delta\chi}{\chi}\right). \end{cases} \tag{32}$$

In the case of a pinned-pinned shaft, the value of ω_{n0}, representing a circular shaft natural frequency, is given by:

$$\omega_{n0}^2 = \frac{EI}{\mu} \frac{n^4\pi^4}{l^4}, \qquad n = 1, 2, \ldots \tag{33}$$

Therefore it is possible to represent the instability regions, for a pinned-pinned shaft, as in Fig. 2.

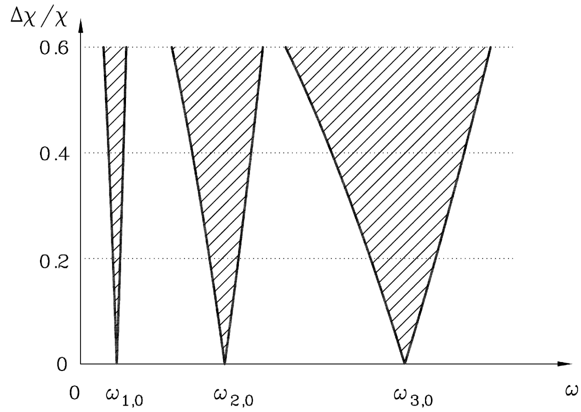

Figure 2: Instability diagram (dashed regions) for a pinned-pinned shaft

In the limit case of angular speed coincident with one of the critical values given by Eq. (30), it results that $p_2 = 0$; hence, since there are two real coincident roots, function $\varphi(\tau)$ grows indefinitely and the motion is unstable.

4 Conclusions

In this work the authors studied the dynamical behaviour of an asymmetric shaft possessing distributed mass and distributed elasticity. The equations of motion are written in a fixed reference frame and solved with an original technique.

Furthermore, it must be observed that the analytical approach developed in the present paper allows to emphasize an important feature of the shaft dynamic behaviour. In fact, the analysis of the results presented in Fig. 2.6,1 of tond's work tond clearly shows the angular velocity intervals corresponding to instability conditions; however the dependence of these regions on the asymmetry ratio $(\Delta\chi/\chi)$ is not highlighted. On the contrary, Fig. 2 of the present work provides a clear and direct answer to this question.

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