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EXISTENCE OF NONTRIVIAL SOLUTIONS FOR SEMILINEAR PROBLEMS WITH STRICTLY DIFFERENTIABLE NONLINEARITY

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The existence of a nontrivial solution for semilinear elliptic problems with strictly differentiable nonlinearity is proved. A result of homological linking under nonstandard geometrical assumption is also shown. Techniques of Morse theory are employed.

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1. Introduction

Since the paper of Amann and Zehnder [1], the existence of nontrivial solutions u for semilinear elliptic problems of the form

$$-\Delta u = g(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.1)$$

with $g(0) = 0$, has been the object of several studies, in which topological and variational methods are successfully applied. We refer the reader to [2, 3, 8, 10]. In particular, since the combination of linking theorems and Morse theory has turned out to be very fruitful, it is customary to impose conditions on g that guarantee that the associated functional $f : H_0^1(\Omega) \rightarrow \mathbb{R}$, given by

$$f(u) = \frac{1}{2} \int_{\Omega} |Du|^2 dx - \int_{\Omega} G(u) dx, \quad G(s) = \int_0^s g(t) dt, \quad (1.2)$$

is of class C^2 .

In a recent paper [12], Perera and Schechter have proved a result of Amann-Zehnder type under assumptions that imply f to be only of class C^1 . More precisely, about the regularity of g , they assume that g is continuous, there exist in \mathbb{R} the limits

$$\lim_{s \rightarrow -\infty} \frac{g(s)}{s}, \quad \lim_{s \rightarrow +\infty} \frac{g(s)}{s}, \quad \lim_{s \rightarrow 0} \frac{g(s)}{s} \quad (1.3)$$

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and that

$$\frac{g(s)}{s} \text{ is Lipschitz continuous in a neighbourhood of } 0. \quad (1.4)$$

One could observe that hypothesis (1.4) allows f not to be of class C^2 , but it does not include every g satisfying the usual assumption that g is of class C^1 and g' is bounded. In particular, condition (1.4) is not stable if we add to g a term of the form

$$\frac{|s|^{3/2}}{1+s^2}. \quad (1.5)$$

The first purpose of this paper is to extend the result of [12] in such a way that also the classical smooth case is included. Our result is the following.

THEOREM 1.1. *Let Ω be a bounded open subset of \mathbb{R}^n and $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying $g(0) = 0$ and*

(a) *there exists $C \geq 0$ such that*

$$|g(s)| \leq C(1 + |s|); \quad (1.6)$$

(b) *there exists $\alpha \in \mathbb{R}$ such that*

$$\lim_{s \rightarrow \pm\infty} \frac{g(s)}{s} = \alpha. \quad (1.7)$$

If we denote by (λ_m) the sequence of the eigenvalues of $-\Delta$ with homogeneous Dirichlet boundary condition, let us assume that $\alpha \neq \lambda_m$ for any $m \in \mathbb{N}$. Moreover, let us suppose that g is strictly differentiable at 0 (see Definition 3.1 below) and that there exists $m \in \mathbb{N}$ with either $g'(0) < \lambda_m < \alpha$ or $g'(0) > \lambda_m > \alpha$.

Then (1.1) admits a nontrivial solution.

Theorem 1.1 is in fact a particular case of a more general result, which will be presented in Section 2.

Remark 1.2. If, as in [12], we have $g(s) = \gamma s$, with γ Lipschitz continuous in a neighbourhood of 0, then it is easy to see that g is strictly differentiable at 0.

A second purpose of the paper is to improve the saddle theorem proved in [11, Theorem 1.4], also mentioned in [12], in which the functional is of class C^2 , but nonstandard geometrical assumptions are considered. We will prove the following.

THEOREM 1.3. *Let H be a Hilbert space such that $H = H_- \oplus H_+$ with $\dim H_- < \infty$ and H_+ closed in H . Let $f : H \rightarrow \mathbb{R}$ be a functional of class C^2 and assume that*

$$c_0 = \inf_{H_+} f > -\infty, \quad c_1 = \sup_{H_-} f < +\infty, \quad (1.8)$$

f satisfies $(PS)_c$ for every $c \in [c_0, c_1]$, $f''(u)$ is a Fredholm operator at every critical point u in $f^{-1}([c_0, c_1])$.

Then there exists a critical point u of f with $c_0 \leq f(u) \leq c_1$ and $m(f, u) \leq \dim H_- \leq m^(f, u)$.*

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In [11] it is only shown that there exist critical points \underline{u} , \bar{u} with $c_0 \leq f(\bar{u}) \leq f(\underline{u}) \leq c_1$ and $m(f, \underline{u}) \leq \dim H_- \leq m^*(f, \bar{u})$, but one cannot say if there exists a critical point $u = \underline{u} = \bar{u}$, as in the case with standard geometrical assumptions (see [8]), or not. Our improvement is related to the fact that, according to Proposition 4.3 below, also under the nonstandard geometrical assumptions of Theorem 1.3, it is possible to recognize a homological linking structure.

The paper is organized as follows: in Section 2 we state the result of existence of nontrivial solutions; Sections 3 and 4 are devoted to prove some auxiliary results, while in Section 5 we prove the main theorems.

2. Existence of a nontrivial solution

Let Ω be a bounded open subset of \mathbb{R}^n and $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying

- (g₀) $g(x, 0) = 0$ for a.e. $x \in \Omega$;
- (g₁) there exists $C \geq 0$ such that $|g(x, s)| \leq C(1 + |s|)$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$;
- (g₂) for a.e. $x \in \Omega$, the function $\{s \mapsto g(x, s)\}$ is strictly differentiable at 0 (see Definition 3.1 below) with $D_s g(\cdot, 0) \in L^\infty(\Omega)$;
- (g₃) there exist $\hat{C} \geq 0$ and $\delta > 0$ such that, for a.e. $x \in \Omega$, we have

$$\forall s, t \in]-\delta, \delta[: |g(x, s) - g(x, t)| \leq \hat{C}|s - t|. \quad (2.1)$$

If we set $G(x, s) = \int_0^s g(x, t) dt$, it is well known that the functional $f : H_0^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$f(u) = \frac{1}{2} \int_{\Omega} |Du|^2 dx - \int_{\Omega} G(x, u) dx \quad (2.2)$$

is of class C^1 .

We denote by $m(f, 0)$ the supremum of the dimensions of the linear subspaces of $H_0^1(\Omega)$ where the quadratic form

$$Q(u) = \int_{\Omega} |Du|^2 dx - \int_{\Omega} D_s g(x, 0) u^2 dx \quad (2.3)$$

is negative definite, and by $m^*(f, 0)$ the supremum of the dimensions of the linear subspaces of $H_0^1(\Omega)$ where Q is negative semidefinite. We call $m(f, 0)$ (resp., $m^*(f, 0)$) the strict (resp., large) Morse index of f at 0.

THEOREM 2.1. *Assume that $H_0^1(\Omega) = X_- \oplus X_+$ with $\dim X_- < \infty$ and X_+ closed in $H_0^1(\Omega)$. Suppose also that*

$$c_0 = \inf_{X_+} f > -\infty, \quad c_1 = \sup_{X_-} f < +\infty, \quad (2.4)$$

and that f satisfies (PS)_c for every $c \in [c_0, c_1]$,

If it is $\dim X_- \notin [m(f, 0), m^*(f, 0)]$, then the problem

$$-\Delta u = g(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (2.5)$$

We changed “(g0)” to “(g₀).” Please check similar cases throughout.

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admits a nontrivial solution u .

Remark 2.2. Under the assumption of **Theorem 1.1**, it is well known that f satisfies $(PS)_c$ for any $c \in \mathbb{R}$ and the geometrical assumptions of **Theorem 2.1**. Since it is clear that also $(g_0)-(g_3)$ are satisfied, **Theorem 1.1** is a consequence of **Theorem 2.1**.

3. Computations of critical groups

Definition 3.1. Let Φ be a map from an open subset U of a normed space X to a normed space Y and let $u \in U$. We say that Φ is *strictly differentiable* at u (*strongly differentiable* in the sense of [6]), if there exists a continuous linear map $L : X \rightarrow Y$ such that

$$\lim_{\substack{(w_1, w_2) \rightarrow (u, u) \\ w_1 \neq w_2}} \frac{\Phi(w_1) - \Phi(w_2) - L(w_1 - w_2)}{\|w_1 - w_2\|} = 0. \quad (3.1)$$

Of course, in such a case Φ is Fréchet differentiable at u and $L = \Phi'(u)$.

Definition 3.2. Let \mathbb{K} be a field, X be a metric space and $f : X \rightarrow \mathbb{R}$ be a continuous function. For $u \in X$ and $c = f(u)$, let us set

$$\forall q \in \mathbb{Z} : C_q(f, u) = H_q(f^c, f^c \setminus \{u\}), \quad (3.2)$$

where $f^c = \{v \in X : f(v) \leq c\}$ and $H_q(A, B)$ denotes the q th singular homology group of the pair (A, B) , with coefficients in \mathbb{K} (see, e.g., [14]). The vector space $C_q(f, u)$ is called *the q th critical group* of f at u . Because of the excision property, we may replace f by $f|_U$ for any neighborhood U of u in X .

Definition 3.3. Let X be a Banach space, U an open subset of X and $f : U \rightarrow \mathbb{R}$ be a function of class C^1 . Let C be a closed subset of X with $C \subseteq U$. We say that f satisfies *the Palais-Smale condition* ((PS) , for short) *on C* , if every sequence (u_h) in C with $f(u_h)$ bounded and $f'(u_h) \rightarrow 0$ admits a convergent subsequence. In the case $C = A = X$, we simply say that f satisfies (PS) .

Let $c \in \mathbb{R}$. We say that f satisfies *the Palais-Smale condition at level c* ($(PS)_c$, for short), if every sequence (u_h) in U with $f(u_h) \rightarrow c$ and $f'(u_h) \rightarrow 0$ admits a convergent subsequence.

Let Ω be a bounded open subset of \mathbb{R}^n ($n \geq 3$), $1 \leq p < (n+2)/(n-2)$ and $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying

(g_1) there exists $C \geq 0$ such that $|g(x, s)| \leq C(1 + |s|^p)$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$.
Let $u_0 \in H_0^1(\Omega)$ be an isolated weak solution of the semilinear problem

$$-\Delta u = g(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (3.3)$$

By regularity theory, we automatically have $u_0 \in L^\infty(\Omega)$. Moreover, let us assume that:

(g_2) for a.e. $x \in \Omega$, the function $\{s \mapsto g(x, s)\}$ is strictly differentiable at $u_0(x)$ and $D_s g(\cdot, u_0) \in L^\infty(\Omega)$;

(g'_3) there exist $\hat{C} \geq 0$ and $\delta > 0$ such that for a.e. $x \in \Omega$

$$\forall s, t \in]-\delta, \delta[: |g(x, u_0(x) + s) - g(x, u_0(x) + t)| \leq \hat{C}|s - t|. \quad (3.4)$$

Let $f : H_0^1(\Omega) \rightarrow \mathbb{R}$ be the functional

$$f(u) = \frac{1}{2} \int_{\Omega} |Du|^2 dx - \int_{\Omega} G(x, u) dx, \quad (3.5)$$

where $G(x, s) = \int_0^s g(x, t) dt$, and let $Q : H_0^1(\Omega) \rightarrow \mathbb{R}$ be the quadratic form

$$Q(u) = \int_{\Omega} |Du|^2 dx - \int_{\Omega} D_s g(x, u_0) u^2 dx. \quad (3.6)$$

Finally, let $m(f, u_0)$ and $m^*(f, u_0)$ be defined as in [Section 2](#).

THEOREM 3.4. *We have that $C_q(f, u_0) = \{0\}$ for every $q \leq m(f, u_0) - 1$ and every $q \geq m^*(f, u_0) + 1$.*

The proof will be given at the end of the section.

As a first step, we approximate the functional f with suitable functionals f_{λ} of class C^1 with f'_{λ} strictly differentiable at u_0 and such that the critical groups of f_{λ} at u_0 are independent of λ .

Let us denote by $\|\cdot\|_q$ the norm of $L^q(\Omega)$ and by $\|\cdot\|_{1,2}$ the norm of $H_0^1(\Omega)$.

Remark 3.5. Up to substitute g with $\tilde{g} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\tilde{g}(x, s) = g(x, u_0(x) + s) - g(x, u_0(x)), \quad (3.7)$$

we may assume that $u_0 = 0$ and that $g(x, 0) = 0$.

LEMMA 3.6. *There exists a constant $\bar{C} > 0$ such that, for a.e. $x \in \Omega$ and for any $s \in \mathbb{R}$, we have*

$$|g(x, s)| \leq \bar{C}(1 + |s|^{p-1})|s|. \quad (3.8)$$

Proof. If $0 < |s| < \delta$, then by (g'_3) it is

$$\left| \frac{g(x, s)}{s} \right| \leq \hat{C}. \quad (3.9)$$

Otherwise, if $|s| \geq \delta$, then it is

$$\left| \frac{g(x, s)}{s} \right| \leq \frac{C(1 + |s|^p)}{|s|} \leq \frac{C}{\delta} + C|s|^{p-1}. \quad (3.10)$$

Hence the assertion follows. \square

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Now let $\delta > 0$ be as in (g'_3) and $\vartheta \in C_c^\infty(\mathbb{R})$ such that $0 \leq \vartheta \leq 1$, $\text{supt}(\vartheta) \subseteq]-\delta, \delta[$ and

$$\begin{aligned} \vartheta(s) &= 1 \quad \text{if } s \in \left[-\frac{\delta}{4}, \frac{\delta}{4}\right], \\ 0 \leq \vartheta \leq \frac{1}{2} \quad &\text{if } s \in [-\delta, \delta] \setminus \left[-\frac{\delta}{2}, \frac{\delta}{2}\right]. \end{aligned} \quad (3.11)$$

For every $\lambda \in [0, 1]$ let us define $g_\lambda(x, s) = g(x, \vartheta(\lambda s)s)$ and let $f_\lambda : H_0^1(\Omega) \rightarrow \mathbb{R}$ be the functional

$$f_\lambda(u) = \frac{1}{2} \int_\Omega |Du|^2 dx - \int_\Omega G_\lambda(x, u) dx, \quad (3.12)$$

where $G_\lambda(x, s) = \int_0^s g_\lambda(x, t) dt$. It is clear that:

- (a) for every $\lambda > 0$ and for a.e. $x \in \Omega$, the function $\{s \mapsto g_\lambda(x, s)\}$ is Lipschitz continuous uniformly with respect to x ;
- (b) for every λ and for a.e. $x \in \Omega$, the function $\{s \mapsto g_\lambda(x, s)\}$ is strictly differentiable at 0 with $D_s g_\lambda(x, 0) = D_s g(x, 0)$;
- (c) for a.e. $x \in \Omega$, the functions $\{(\lambda, s) \mapsto g_\lambda(x, s)\}$ and $\{(\lambda, s) \mapsto G_\lambda(x, s)\}$ are continuous;
- (d) there exists $\bar{C} \geq 0$ such that $|g_\lambda(x, s)| \leq \bar{C}(1 + |s|^p)$, $|G_\lambda(x, s)| \leq \bar{C}(1 + |s|^{p+1})$.

THEOREM 3.7. *The following facts hold:*

- (i) *for every $\lambda \in [0, 1]$, the functional f_λ is of class C^1 ;*
- (ii) *there exists an open bounded neighbourhood U of 0 in $H_0^1(\Omega)$ such that, for every $\lambda \in [0, 1]$, 0 is the only critical point of f_λ in \bar{U} ;*
- (iii) *for every $\lambda \in]0, 1]$, f'_λ is strictly differentiable at 0 with $\langle f''_\lambda(0)v, v \rangle = Q(v)$.*

Proof. It is readily seen that assertion (i) holds.

Let us consider assertion (ii). By contradiction, let us assume that there exist (λ_h) in $[0, 1]$ and (u_h) in $H_0^1(\Omega)$ with $u_h \neq 0$ and $u_h \rightarrow 0$ strongly in $H_0^1(\Omega)$ such that $f'_{\lambda_h}(u_h) = 0$. Up to a subsequence, $\lambda_h \rightarrow \lambda$ in $[0, 1]$. Since u_h is a critical point of f_{λ_h} , we have that u_h is a weak solution of

$$-\Delta u = g_{\lambda_h}(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (3.13)$$

Let

$$a_h = \begin{cases} \frac{g_{\lambda_h}(x, u_h)}{u_h} & \text{where } u_h \neq 0, \\ 0 & \text{where } u_h = 0. \end{cases} \quad (3.14)$$

By **Lemma 3.6** it is

$$|a_h| \leq \left| \frac{g_{\lambda_h}(x, u_h)}{u_h} \right| = \left| \frac{g(x, \vartheta(\lambda_h u_h)u_h)}{u_h} \right| \leq \bar{C}(1 + |\vartheta(\lambda_h u_h)u_h|^{p-1}) \leq \bar{C}(1 + |u_h|^{p-1}). \quad (3.15)$$

Since u_h is bounded in $L^{2n/(n-2)}(\Omega)$, then a_h belongs to $L^q(\Omega)$ with $q > n/2$ and

$$\|a_h\|_q \leq C' \left(1 + \|u_h\|_{2n/(n-2)}^{p-1}\right) \leq M. \quad (3.16)$$

Hence u_h is a weak solution of the linear problem

$$-\Delta u = a_h u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (3.17)$$

By [7, Theorem 3.13.1] $u_h \in L^\infty(\Omega)$ and there exists $C > 0$ such that $\|u_h\|_\infty \leq C\|Du_h\|_2$. Hence $u_h \rightarrow 0$ in $L^\infty(\Omega)$. Since $\vartheta = 1$ on $[-\delta/4, \delta/4]$, for h sufficiently large we have that u_h is a weak solution of (3.3). It follows that 0 is not an isolated solution of (3.3): a contradiction.

Finally, let us consider assertion (iii). Let $L : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ be the continuous linear operator such that

$$\langle Lv, w \rangle = \langle Lw, v \rangle, \quad \langle Lv, v \rangle = Q(v). \quad (3.18)$$

Let $(u_h), (v_h), (w_h)$ in $H_0^1(\Omega)$ be such that $u_h \rightarrow 0$, $w_h \rightarrow 0$ in $H_0^1(\Omega)$ and $\|v_h\|_{1,2} \leq 1$. Up to a subsequence, $w_h \rightarrow 0$ and $u_h \rightarrow 0$ a.e. in Ω . We have that

$$\begin{aligned} & \left| \langle f'_\lambda(w_h), v_h \rangle - \langle f'_\lambda(u_h), v_h \rangle - \langle L(w_h - u_h), v_h \rangle \right| \\ &= \left| \int_{\{x \in \Omega: w_h(x) \neq u_h(x)\}} \left[\frac{g_\lambda(x, w_h) - g_\lambda(x, u_h)}{w_h - u_h} - D_s g(x, 0) \right] (w_h - u_h) v_h dx \right| \\ &\leq C \left(\int_{\{x \in \Omega: w_h(x) \neq u_h(x)\}} \left| \frac{g_\lambda(x, w_h) - g_\lambda(x, u_h)}{w_h - u_h} - D_s g(x, 0) \right|^{n/2} dx \right)^{2/n} \\ &\quad \times \|w_h - u_h\|_{1,2} \|v_h\|_{1,2}. \end{aligned} \quad (3.19)$$

Then it is

$$\begin{aligned} & \frac{\left| \langle f'_\lambda(w_h), v_h \rangle - \langle f'_\lambda(u_h), v_h \rangle - \langle L(w_h - u_h), v_h \rangle \right|}{\|w_h - u_h\|_{1,2}} \\ &\leq C \left(\int_{\{x \in \Omega: w_h(x) \neq u_h(x)\}} \left| \frac{g_\lambda(x, w_h) - g_\lambda(x, u_h)}{w_h - u_h} - D_s g(x, 0) \right|^{n/2} dx \right)^{2/n} \|v_h\|_{1,2} \\ &\leq C \left(\int_\Omega \left| \frac{g_\lambda(x, w_h) - g_\lambda(x, u_h)}{w_h - u_h} - D_s g(x, 0) \right|^{n/2} \chi_{\{x \in \Omega: w_h(x) \neq u_h(x)\}} dx \right)^{2/n}. \end{aligned} \quad (3.20)$$

By (a) and (b) we can apply Lebesgue's theorem, obtaining

$$\left(\int_\Omega \left| \frac{g_\lambda(x, w_h) - g_\lambda(x, u_h)}{w_h - u_h} - D_s g(x, 0) \right|^{n/2} \chi_{\{x \in \Omega: w_h(x) \neq u_h(x)\}} dx \right)^{2/n} \rightarrow 0. \quad (3.21)$$

Therefore

$$\lim_{h \rightarrow +\infty} \frac{\langle f'_\lambda(w_h), v_h \rangle - \langle f'_\lambda(u_h), v_h \rangle - \langle L(w_h - u_h), v_h \rangle}{\|w_h - u_h\|_{1,2}} = 0 \quad (3.22)$$

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and assertion (iii) follows. \square

THEOREM 3.8. *The critical groups $C_q(f_\lambda, 0)$ are independent of λ . In particular*

$$\forall q \in \mathbb{Z} : C_q(f, 0) \approx C_q(f_1, 0). \quad (3.23)$$

Proof. Let U be an open bounded neighbourhood of 0 in $H_0^1(\Omega)$ as in assertion (ii) of **Theorem 3.7**. We claim that if $\lambda_h \rightarrow \lambda$ in $[0, 1]$, then $\|f_{\lambda_h}|_{\overline{U}} - f_\lambda|_{\overline{U}}\|_{1,\infty} \rightarrow 0$. Let (u_h) be a sequence in \overline{U} . Up to a subsequence, $u_h \rightharpoonup u$ in $H_0^1(\Omega)$ and $u_h \rightarrow u$ a.e. in Ω . It is

$$\begin{aligned} f_{\lambda_h}(u_h) - f_\lambda(u_h) &= \int_{\Omega} [G_{\lambda_h}(x, u_h) - G_\lambda(x, u_h)] dx \\ &= \int_{\Omega} [G_{\lambda_h}(x, u_h) - G_\lambda(x, u)] dx + \int_{\Omega} [G_\lambda(x, u) - G_\lambda(x, u_h)] dx. \end{aligned} \quad (3.24)$$

By (c), (d) and Lebesgue's theorem we deduce that

$$\int_{\Omega} [G_{\lambda_h}(x, u_h) - G_\lambda(x, u)] dx \rightarrow 0. \quad (3.25)$$

Therefore $f_{\lambda_h} \rightarrow f_\lambda$ uniformly on \overline{U} .

Now, let $v_h \in H_0^1(\Omega)$ with $\|v_h\|_{1,2} \leq 1$. Up to a subsequence $v_h \rightharpoonup v$ in $H_0^1(\Omega)$, $v_h \rightarrow v$ in $L^{2n/(n-2)}(\Omega)$ and $v_h \rightarrow v$ a.e. in Ω . It is

$$\begin{aligned} &|\langle f'_{\lambda_h}(u_h), v_h \rangle - \langle f'_\lambda(u_h), v_h \rangle| \\ &= \left| \int_{\Omega} [g_{\lambda_h}(x, u_h) - g_\lambda(x, u_h)] v_h dx \right| \\ &= \left| \int_{\Omega} [g(x, \vartheta(\lambda_h u_h) u_h) - g(x, \vartheta(\lambda u_h) u_h)] v_h dx \right| \\ &\leq C \left(\int_{\Omega} |g(x, \vartheta(\lambda_h u_h) u_h) - g(x, \vartheta(\lambda u_h) u_h)|^{2n/(n+2)} dx \right)^{(n+2)/2n} \|v_h\|_{1,2}. \end{aligned} \quad (3.26)$$

As before we have that

$$\int_{\Omega} |g_{\lambda_h}(x, u_h) - g_\lambda(x, u_h)|^{2n/(n+2)} dx \rightarrow 0. \quad (3.27)$$

It follows that $f'_{\lambda_h} \rightarrow f'_\lambda$ uniformly on \overline{U} . Finally, since U is bounded and g has subcritical growth, we have that for every $\lambda \in [0, 1]$ f_λ satisfies (PS) in \overline{U} . By [5, Theorem 5.2] the assertion follows. \square

In the second part of this section we deduce from [6] a generalization of the classical Shifting theorem (see [3, Theorem I.5.4], [10, Theorem 8.4]).

Let H be a Hilbert space, U be an open subset of H , $u_0 \in U$ and $f : U \rightarrow \mathbb{R}$ be a function of class C^1 such that f' is strictly differentiable at u_0 and $f''(u_0)$ is a Fredholm operator. In particular, f' is Lipschitz continuous in a neighbourhood of u_0 . Let $L : H \rightarrow H$

be the linear operator defined by

$$\forall v, w \in H : \langle Lv, w \rangle = \langle f''(u_0)v, w \rangle, \quad (3.28)$$

let $V_0 = \ker L$ and let P_{V_0} be the orthogonal projection on V_0 . We also denote by $m(f, u_0)$ (resp., $m^*(f, u_0)$) the strict (resp., large) Morse index of f at u_0 .

THEOREM 3.9. *Let u_0 be an isolated critical point of f . Then there exist a neighbourhood \hat{U} of $P_{V_0}u_0$ in V_0 and a function $\hat{f} : \hat{U} \rightarrow \mathbb{R}$ of class C^1 with locally Lipschitz gradient such that $P_{V_0}u_0$ is an isolated critical point of \hat{f} and*

$$\forall q \in \mathbb{Z} : C_q(f, u_0) \approx \begin{cases} C_{q-m(f, u_0)}(\hat{f}, P_{V_0}u_0) & \text{if } m(f, u_0) < \infty, \\ \{0\} & \text{if } m(f, u_0) = \infty, \end{cases} \quad (3.29)$$

$$\begin{aligned} \forall q \leq m(f, u_0) - 1 : C_q(f, u_0) &= \{0\}, \\ \forall q \geq m^*(f, u_0) + 1 : C_q(f, u_0) &= \{0\}. \end{aligned} \quad (3.30)$$

Proof. Without loss of generality, we may assume that $u_0 = 0$. From [6, Theorem 1.2] we also see that the generalized Morse lemma holds also in this setting. Arguing as in the proof of [10, Theorem 8.4], we find that (3.29) holds. Actually, in our case f is of class C^{2-0} instead of C^2 , but the proof of [10, Theorem 8.4] remains valid also in this case.

On the other hand, also the proof of [10, Theorem 8.5] can be easily adapted from the C^2 to the C^{2-0} case. Therefore we have that $C_q(\hat{f}, P_{V_0}u_0) = \{0\}$ if $q \geq \dim V_0 + 1$. Since $m^*(f, u_0) = m(f, u_0) + \dim V_0$, the other assertions follow from (3.29). \square

Finally, let us prove **Theorem 3.4**.

Proof. By **Remark 3.5** we may assume that $u_0 = 0$. Let $f_\lambda : H_0^1(\Omega) \rightarrow \mathbb{R}$ be as in (3.12). By **Theorem 3.7** we have that f_1 is of class C^1 with f_1' strictly differentiable at 0 and 0 is an isolated critical point of f_1 . Moreover, $f_1''(0)$ is a Fredholm operator. By **Theorem 3.8** it is

$$\forall q \in \mathbb{Z} : C_q(f, 0) \approx C_q(f_1, 0). \quad (3.31)$$

On the other hand, since $Q(u) = \langle f_1''(0)u, u \rangle$, we have that $m(f, 0) = m(f_1, 0)$ and $m^*(f, 0) = m^*(f_1, 0)$. From **Theorem 3.9** the assertion follows. \square

4. Homological linking

Throughout this section, X will denote a Banach space, $B_r(u)$ the open ball of center $u \in X$ and radius r and $f : X \rightarrow \mathbb{R}$ a function of class C^1 . We set $K = \{u \in X : f'(u) = 0\}$ and, for every $c \in \mathbb{R}$,

$$K_c = \{u \in X : f'(u) = 0, f(u) = c\}. \quad (4.1)$$

We also denote by H_* singular homology.

First of all, let us recall from [4] an extension of the homological linking of [3].

Definition 4.1. Let D, S, A be three subsets of X , $m \in \mathbb{N}$ and \mathbb{K} a field. We say that (D, S) links A homologically in dimension m (over \mathbb{K}), if $S \subseteq D$, $S \cap A = \emptyset$ and there exists $z \in H_m(X, S; \mathbb{K})$ belonging to the image of $i_* : H_m(D, S; \mathbb{K}) \rightarrow H_m(X, S; \mathbb{K})$ but not of $j_* : H_m(X \setminus A, S; \mathbb{K}) \rightarrow H_m(X, S; \mathbb{K})$, where $i : (D, S) \rightarrow (X, S)$ and $j : (X \setminus A, S) \rightarrow (X, S)$ are the inclusion maps.

It is clear that, if (D, S) links A homologically, then $D \cap A \neq \emptyset$.

THEOREM 4.2. *Let D, S, A be three subsets of X such that (D, S) links A homologically in dimension m and let $z \in H_m(X, S; \mathbb{K})$ be as in [Definition 4.1](#). Assume that*

$$\inf_A f > -\infty, \quad \sup_D f < +\infty, \quad \forall u \in S : f(u) < \inf_A f \quad (4.2)$$

and define

$$c = \inf \{ b \in \mathbb{R} : S \subseteq f^b \text{ and } z \text{ belongs to the image of the} \\ \text{homomorphism induced by inclusion } H_m(f^b, S; \mathbb{K}) \longrightarrow H_m(X, S; \mathbb{K}) \}. \quad (4.3)$$

Suppose that f satisfies (PS) and that each element of K_c is isolated in K .

Then $\inf_A f \leq c \leq \sup_D f$ and there exists $u \in K_c$ with $C_m(f, u) \neq \{0\}$.

To prove our main results we need the following.

PROPOSITION 4.3. *Let $X = X_- \oplus X_+$, with $\dim X_- < \infty$ and X_+ closed in X . Assume that*

$$c_0 = \inf_{X_+} f > -\infty, \quad c_1 = \sup_{X_-} f < +\infty \quad (4.4)$$

and that f satisfies $(PS)_c$ for every $c \in [c_0, c_1]$.

Then there exists a compact pair (D, S) in X such that

$$\max_D f \leq c_1, \quad \forall u \in S : f(u) < c_0 \quad (4.5)$$

and such that (D, S) links X_+ homologically in dimension $\dim X_-$ over all \mathbb{K} .

Proof. Since f satisfies $(PS)_c$ for every $c \in [c_0, c_1]$, there exists $r > 0$ such that $K \cap f^{-1}([c_0, c_1]) \subseteq (B_r(0) \cap X_-) \oplus X_+$. Moreover, there exist $\delta, \sigma > 0$ such that

$$\|P_{X_-} u\| \geq r, \\ c_0 - \delta \leq f(u) \leq c_1 + \delta \implies \|f'(u)\| > \sigma, \quad (4.6)$$

where P_{X_-} denotes the projection on X_- induced by the decomposition $X = X_- \oplus X_+$. Let $c > 0$ be such that $\|P_{X_-} u\| \leq c\|u\|$ for any $u \in X$ and let

$$R = c \frac{c_1 - c_0 + \delta}{\sigma} + r + \delta, \quad \rho_1 = 1, \quad \rho_2 = R - r - \delta, \\ C = X \setminus [(B_{r+\rho_1+\rho_2}(0) \cap X_-) \oplus X_+]. \quad (4.7)$$

By [5, Theorem 2.1] applied to the function $f|_{\{u \in X : f(u) \geq c_0 - \delta\}}$, there exist a continuous function

$$\tau : \overline{B_{\rho_1}(C)} \cap \{u \in X : c_0 - \delta \leq f(u) < c_1 + \delta\} \longrightarrow [0, +\infty) \quad (4.8)$$

and a continuous map

$$\eta : \left(\overline{B_{\rho_1}(C)} \cap \{u \in X : c_0 - \delta \leq f(u) < c_1 + \delta\} \right) \times [0, 1] \longrightarrow \{u \in X : f(u) \geq c_0 - \delta\} \quad (4.9)$$

such that

- (a) $\tau(u) = 0 \Leftrightarrow f(u) = c_0 - \delta$;
- (b) $\|\eta(u, t) - u\| \leq \tau(u)t$;
- (c) $f(\eta(u, t)) \leq f(u) - \sigma\tau(u)t$;
- (d) $f(\eta(u, 1)) = c_0 - \delta$.

Let $\vartheta_1 : \mathbb{R} \rightarrow [0, 1]$ be a continuous function such that

$$\vartheta_1(s) = 1 \quad \text{if } s \leq c_1, \quad \vartheta_1(s) = 0 \quad \text{if } s \geq c_1 + \delta/2, \quad (4.10)$$

and let $\vartheta_2 : X \rightarrow [0, 1]$ be a continuous function such that

$$\vartheta_2(u) = 1 \quad \text{if } \|u\| \geq R, \quad \vartheta_2(u) = 0 \quad \text{if } \|u\| \leq R - \delta. \quad (4.11)$$

Let $\mathcal{H} : X \times [0, 1] \rightarrow X$ be the deformation defined by

$$\mathcal{H}(u, t) = \begin{cases} \eta(u, \vartheta_1(f(u))\vartheta_2(P_{X_-}u)t) & \text{if } u \in \overline{B_{\rho_1}(C)}, c_0 - \delta \leq f(u) \leq c_1 + \delta, \\ u & \text{if } f(u) \leq c_0 - \delta, \\ u & \text{if } f(u) \geq c_1 + \delta/2, \\ u & \text{if } \|P_{X_-}u\| \leq R - \delta. \end{cases} \quad (4.12)$$

If $u \in X_-$, we have that

$$\|P_{X_-}\mathcal{H}(u, t) - u\| \leq c\|\mathcal{H}(u, t) - u\| \leq c \frac{f(u) - f(\mathcal{H}(u, t))}{\sigma} \leq c \frac{c_1 - c_0 + \delta}{\sigma} < R - r. \quad (4.13)$$

It follows

$$\begin{aligned} \|P_{X_-}u\| \leq r &\implies \mathcal{H}(u, t) = u, \\ u \in X_-, &\implies f(\mathcal{H}(u, 1)) < c_0, \\ \|u\| \geq R &\implies \|P_{X_-}(\mathcal{H}(u, t))\| \geq r, \quad \forall t \in [0, 1]. \end{aligned} \quad (4.14)$$

It is clear that $(X, (X_- \setminus B_r(0)) \oplus X_+)$ links X_+ homologically in dimension $\dim X_-$ and that the inclusion map

$$i : \left(\overline{B_R(0)} \cap X_-, \partial B_R(0) \cap X_- \right) \longrightarrow (X, (X_- \setminus B_r(0)) \oplus X_+) \quad (4.15)$$

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induces an isomorphism in homology. Let $m = \dim X_-$ and

$$B = \overline{B_R(0)} \cap X_-, \quad E = \partial B_R(0) \cap X_-, \quad F = (X_- \setminus B_r(0)) \oplus X_+. \quad (4.16)$$

Consider now the commutative diagram

$$\begin{array}{ccccc} H_m(B, E) & \longrightarrow & H_m(X, E) & \longleftarrow & H_m(X \setminus X_+, E) \\ \downarrow & & \downarrow & & \downarrow \\ H_m(X, F) & \xrightarrow{Id} & H_m(X, F) & \longleftarrow & H_m(X \setminus X_+, F) \end{array} \quad (4.17)$$

where horizontal rows are induced by the inclusions and the vertical rows are isomorphisms. We have that there exists $z \in H_m(X, E)$ belonging to the image of $H_m(B, E) \rightarrow H_m(X, E)$ such that $i_*(z) \in H_m(X, F)$, but not to the image of $H_m(X \setminus X_+, F) \rightarrow H_m(X, F)$. Let us consider the compact sets $D = \mathcal{H}(B, 1)$ and $S = \mathcal{H}(E, 1)$. We have that

$$\max_D f \leq c_1, \quad \max_S f < c_0, \quad S \subseteq F. \quad (4.18)$$

Consider now the commutative diagram

$$\begin{array}{ccccc} H_m(B, E) & \longrightarrow & H_m(X, E) & & \\ \mathcal{H}_*(\cdot, 1) \downarrow & & \mathcal{H}_*(\cdot, 1) \downarrow & & \\ H_m(D, S) & \longrightarrow & H_m(X, S) & \longleftarrow & H_m(X \setminus X_+, S) \\ \downarrow & & \downarrow & & \downarrow \\ H_m(X, F) & \xrightarrow{Id} & H_m(X, F) & \longleftarrow & H_m(X \setminus X_+, F) \end{array} \quad (4.19)$$

Since $\mathcal{H}(\cdot, 1) : (X, E) \rightarrow (X, F)$ is homotopically equivalent to the identity map, then (D, S) links X_+ homologically in dimension $m = \dim X_-$ and the assertions follows. \square

5. Proof of the main results

proof of Theorem 2.1. By contradiction, let us assume that 0 is the unique solution of (2.5). Since $m = \dim X_- \notin [m(f, 0), m^*(f, 0)]$, by Theorem 3.4 it is $C_m(f, 0) = \{0\}$. By Proposition 4.3 there exists a compact pair (D, S) in $H_0^1(\Omega)$ such that

$$\forall u \in S : f(u) < \inf_{X_+} f \quad (5.1)$$

and (D, S) links X_+ homologically in dimension m over all \mathbb{K} . By Theorem 4.2 there exists a critical point $u \in H_0^1(\Omega)$ of f such that $C_m(f, u) \neq \{0\}$. Hence $u \neq 0$ and u is a weak solution of (2.5): a contradiction. \square

proof of Theorem 1.3. Let (D, S) be as in Proposition 4.3. By [13, Proposition 3.9 and Remark] there exists $\delta > 0$ such that f satisfies $(PS)_c$ for every $c \in [c_0 - \delta, c_1 + \delta]$ and

$f''(u)$ is a Fredholm operator at every critical point u in $f^{-1}([c_0 - \delta, c_1 + \delta])$. Let us argue by contradiction and set

$$\begin{aligned} K_1 &= \{u \in H : c_0 - \delta \leq f(u) \leq c_1 + \delta, f'(u) = 0, m^*(f, u) < \dim H_-\}, \\ K_2 &= \{u \in H : c_0 - \delta \leq f(u) \leq c_1 + \delta, f'(u) = 0, m(f, u) > \dim H_-\}. \end{aligned} \quad (5.2)$$

Then K_1, K_2 are two disjoint compact sets whose union is the critical set of f in $f^{-1}([c_0 - \delta, c_1 + \delta])$. By Marino-Prodi perturbation lemma [9, Teorema 2.2], there exists a functional $\hat{f} : H \rightarrow \mathbb{R}$ of class C^2 such that

$$\inf_{H_+} \hat{f} > c_0 - \delta/2, \quad \sup_{H_-} \hat{f} < c_1 + \delta/2, \quad \max_S \hat{f} < \inf_{H_+} \hat{f}, \quad (5.3)$$

\hat{f} satisfies $(PS)_c$ for every $c \in [c_0 - \delta/2, c_1 + \delta/2]$, \hat{f} has only non-degenerate critical points u in $\hat{f}^{-1}([c_0 - \delta/2, c_1 + \delta/2])$, with either $m(\hat{f}, u) < \dim H_-$ or $m^*(\hat{f}, u) > \dim H_-$. If we apply Theorem 4.2 to \hat{f} , we find a critical point u of \hat{f} with $c_0 - \delta/2 \leq \hat{f}(u) \leq c_1 + \delta/2$ and $C_m(\hat{f}, u) \neq \{0\}$, where $m = \dim H_-$. By the Morse lemma, we have $m(\hat{f}, u) = m$ and a contradiction follows. \square

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