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Fredholm Factorization for Wedge Problems

Vito G. Daniele⁽¹⁾, Guido Lombardi* ⁽¹⁾

(1) Dipartimento di Elettronica, Politecnico di Torino,

C.so Duca degli Abruzzi 24, 10129 Torino (Italy).

<http://www.eln.polito.it/staff/daniele>, e-mail: guido.lombardi@polito.it

Introduction

Recently the diffraction by arbitrary impenetrable wedges has been reduced to the factorization of matrices of order four [1]. This paper provides an efficient and general factorization technique that is based on the solution of a Fredholm integral equation of second kind.

Wiener- Hopf solution of the problem

Figure 1 illustrates the problem of the diffraction by a plane wave at skew incidence on an impenetrable wedge immersed in a medium with permittivity ϵ and permeability μ .

Figure 1: Geometry of the problem

The incident field is constituted by a plane wave having the following longitudinal components:

$$E_z^i = E_o e^{j\tau_o \rho \cos(\varphi - \varphi_o)} e^{-j\alpha_o z} \quad H_z^i = H_o e^{j\tau_o \rho \cos(\varphi - \varphi_o)} e^{-j\alpha_o z} \quad (1)$$

where β and φ_o are the zenithal and azimuthal angle of the direction of the plane wave $\hat{n}_i$ and $k = \omega\sqrt{\mu\epsilon}$, $\alpha_o = k \cos \beta$, $\tau_o = k \sin \beta$.

The tangential fields are related on the boundaries of the wedge $\varphi = +\Phi$ (a-face) and $\varphi = -\Phi$ (b-face) through the Leontovich conditions:

$$\begin{bmatrix} E_z(\rho, \Phi) \\ E_\rho(\rho, \Phi) \end{bmatrix} = Z_a \begin{bmatrix} H_\rho(\rho, \Phi) \\ -H_z(\rho, \Phi) \end{bmatrix}, \quad \begin{bmatrix} E_z(\rho, -\Phi) \\ E_\rho(\rho, -\Phi) \end{bmatrix} = -Z_b \begin{bmatrix} H_\rho(\rho, -\Phi) \\ -H_z(\rho, -\Phi) \end{bmatrix} \quad (2)$$

where the matrices $Z_{a,b} = Z_o \begin{bmatrix} z_{11}^{a,b} & z_{12}^{a,b} \\ z_{21}^{a,b} & z_{22}^{a,b} \end{bmatrix}$ depends on the wedge material and

$Z_o = \sqrt{\mu/\varepsilon}$ is the free space impedance.

The Wiener-Hopf formulation [1,4] of this problem yields the solution:

$$\bar{X}_+(\bar{\eta}) = \bar{G}_+^{-1}(\bar{\eta}) \bar{G}_+(\bar{\eta}_o) \frac{\bar{T}_o}{\bar{\eta} - \bar{\eta}_o} \quad (3)$$

where:

$$V_{z+}(\eta, \varphi) = \int_0^\infty E_z(\rho, \varphi) e^{j\eta\rho} d\rho, \quad I_{z+}(\eta, \varphi) = \int_0^\infty H_z(\rho, \varphi) e^{j\eta\rho} d\rho \quad (4)$$

$$V_{\rho+}(\eta, \varphi) = \int_0^\infty E_\rho(\rho, \varphi) e^{j\eta\rho} d\rho, \quad I_{\rho+}(\eta, \varphi) = \int_0^\infty H_\rho(\rho, \varphi) e^{j\eta\rho} d\rho \quad (5)$$

$$\bar{X}_+(\bar{\eta}) = \begin{vmatrix} V_{z+}(\eta, 0) & V_{\rho+}(\eta, 0) & Z_o I_{z+}(\eta, 0) & Z_o I_{\rho+}(\eta, 0) \end{vmatrix}^t \quad (6)$$

and $\eta = \eta(\bar{\eta}) = -\tau_o \cos\left[\frac{\Phi}{\pi} \left[\arccos\left[-\frac{\bar{\eta}}{\tau_o}\right]\right]\right]$.

For the problem at hand the constants $\bar{T}_o$, $\bar{\eta}_o$ assume the following expressions:

$$\bar{T}_o = \frac{\pi}{\Phi} \frac{\sin \frac{\pi}{\Phi} \varphi_o}{\sin \varphi_o} \begin{vmatrix} jE_o \\ j \frac{\alpha_o \cos \varphi_o E_o + kZ_o \sin \varphi_o H_o}{\tau_o} \\ jZ_o H_o \\ j \frac{\alpha_o Z_o \cos \varphi_o H_o - k \sin \varphi_o E_o}{\tau_o} \end{vmatrix} \quad \text{and } \bar{\eta}_o = -\tau_o \cos \frac{\pi}{\Phi} \varphi_o$$

and the matrix $\bar{G}_+(\bar{\eta})$ is the plus factorized matrix of the matrix kernel

$$\bar{G}(\bar{\eta}) = \bar{G}_-(\bar{\eta}) \bar{G}_+(\bar{\eta}), \quad \bar{G}(\bar{\eta}) = \begin{vmatrix} \frac{g_{11}}{d^a} & \frac{g_{12}}{d^a} & \frac{g_{13}}{d^a} & \frac{g_{14}}{d^a} \\ \frac{g_{21}}{d^a} & \frac{g_{22}}{d^a} & \frac{g_{23}}{d^a} & \frac{g_{24}}{d^a} \\ \frac{g_{31}}{d^b} & \frac{g_{32}}{d^b} & \frac{g_{33}}{d^b} & \frac{g_{34}}{d^b} \\ \frac{g_{41}}{d^b} & \frac{g_{42}}{d^b} & \frac{g_{43}}{d^b} & \frac{g_{44}}{d^b} \end{vmatrix} \quad (7)$$

where:

$$\begin{aligned} g_{11} &= -k n z_{11}^a \alpha_o \eta - m \eta \alpha_o^2 - k^2 \eta \xi + k m z_{12}^a \alpha_o \xi - k z_{22}^a \xi \tau_o^2, \quad g_{12} = -k n z_{12}^a \tau_o^2 - m \alpha_o \tau_o^2, \\ g_{13} &= k n \alpha_o \eta - m \eta z_{12}^a \alpha_o^2 - k^2 n z_{12}^a \xi - k m \alpha_o \xi + z_{22}^a \eta \alpha_o \tau_o^2, \\ g_{14} &= k n \tau_o^2 - z_{12}^a m \alpha_o \tau_o^2 + z_{22}^a \tau_o^4, \\ g_{21} &= k (n \eta - m \xi) \alpha_o z_{11}^a + (\eta \alpha_o + k z_{21}^a \xi) \tau_o^2, \quad g_{22} = k n z_{11}^a \tau_o^2 + \tau_o^4, \\ g_{23} &= m z_{11}^a \alpha_o^2 \eta + k^2 n z_{11}^a \xi - z_{21}^a \alpha_o \eta \tau_o^2 + k \xi \tau_o^2, \quad g_{24} = m z_{11}^a \alpha_o \tau_o^2 - z_{21}^a \tau_o^4, \\ d^a &= k^2 n^2 z_{11}^a + m^2 z_{11}^a \alpha_o^2 + k n (1 + \Delta^a) \tau_o^2 - m (z_{12}^a + z_{21}^a) \alpha_o \tau_o^2 + z_{22}^a \tau_o^4, \quad \Delta_z^a = z_{11}^a z_{22}^a - z_{12}^a z_{21}^a \end{aligned}$$

$d^b, \Delta_z^b, g_{31}, g_{32}, g_{33}, g_{34}, g_{41}, g_{42}, g_{43}, g_{44}$ assume respectively the same expression of $d^a, \Delta_z^a, g_{11}, g_{12}, -g_{13}, -g_{14}, g_{21}, g_{22}, -g_{23}, -g_{24}$ except for the substitution of the superscript a with the superscript b .

The functions ξ, m and n depends on $\bar{\eta}$ and are defined by:

$$\begin{cases} \xi = \xi(\bar{\eta}) = -\tau_o \sin\left[\frac{\Phi}{\pi} \left[\arccos\left[-\frac{\bar{\eta}}{\tau_o}\right]\right]\right] \\ m = m(\bar{\eta}) = \tau_o \cos\left[\frac{\Phi}{\pi} \left[\arccos\left[-\frac{\bar{\eta}}{\tau_o}\right] + \Phi\right]\right] \\ n = n(\bar{\eta}) = \tau_o \sin\left[\frac{\Phi}{\pi} \left[\arccos\left[-\frac{\bar{\eta}}{\tau_o}\right] + \Phi\right]\right] \end{cases} \quad (8)$$

In several important cases the matrix $\bar{G}(\bar{\eta})$ can be factorized in closed form [2]. For instance, this property is verified for the whole class of problems that have been solved with the Malyuzhinets-Sommerfeld technique.

Fredholm factorization of the matrix kernel $\bar{G}(\bar{\eta})$

By using the technique introduced in [3] the factorized matrix $\bar{G}_+(\bar{\eta})$ can be expressed by:

$$\bar{G}_+(\bar{\eta}) = \frac{1}{\bar{\eta} - \bar{\eta}_p} [X_{1+}(\bar{\eta}), X_{2+}(\bar{\eta}), X_{3+}(\bar{\eta}), X_{4+}(\bar{\eta})]^{-1} \quad (10)$$

where $\bar{\eta}_p$ is an arbitrary point with negative imaginary part and the functions $X_{i+}(\bar{\eta}), \{i=1,2,3,4\}$ satisfy the following Fredholm integral equation:

$$\bar{G}(\bar{\eta})X_{i+}(\bar{\eta}) + \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{[\bar{G}(x) - \bar{G}(\bar{\eta})]X_{i+}(x)}{x - \bar{\eta}} dx = \frac{R_i}{\bar{\eta} - \bar{\eta}_p}, \quad \text{Im}[\bar{\eta}_p] < 0 \quad (11)$$

with the vector constant R_i given by the canonical basis for the 4D space.

Since $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{\bar{G}(x) - \bar{G}(\bar{\eta})}{x - \bar{\eta}} \right|^2 dx d\bar{\eta}$ is bounded [3], (11) is a Fredholm equation of second kind where it is applicable the well-known in literature powerful solution technique.

We experienced [4] that the convergence of approximate solutions considerably increases when we solve the integral equation in the t plane defined by the

mapping $\bar{\eta} = \bar{\eta}(t) = -\tau_o \cos(jt - \frac{\pi}{2})$.

Numerical validation

To ascertain the correctness of our new methodology we have chosen a well known in literature test case to compare our solution with alternative method [5]: the bistatic far field amplitude evaluation for skew incidence on an impedance half plane. Figure 2 reports the GTD Diffraction Coefficient for Ez component

($D_E(\varphi) = s_E(\varphi - \pi) - s_E(\varphi + \pi)$, where $s_E(w)$ is the Sommerfeld function) for the test case with the following problem parameters: $k=1$, the incident field $\varphi_0=5\pi/6$, $\beta = \pi/3$, $E_{z0}=1$, $H_{z0}=0$, the aperture angle $\Phi=\pi$, the integration parameters $A=5$, $h=1$ for the discretization of equation (11) after the transformation in the t plane. Peaks of the GTD Diffraction Coefficients are for $\varphi = \varphi_0 - \pi$ (incident field) and for $\varphi = 2\Phi - \varphi_0 - \pi$ (reflected field).

Figure 2: Amplitude of the GTD Diffraction Coefficient (dB)

Several other applications of this technique to wedge problems have been reported in [6]. New examples and convergence tests concerning with new canonical wedge problems will be illustrated in the oral presentation of the paper.

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