Fredholm Factorization for Wedge Problems

Original

Availability:
This version is available at: 11583/1413221 since:

Publisher:
IEEE

Published
DOI:10.1109/APS.2006.1711100

Terms of use:
This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

(Article begins on next page)
Fredholm Factorization for Wedge Problems

Vito G. Daniele(1), Guido Lombardi* (1)
(1) Dipartimento di Elettronica, Politecnico di Torino,
C.so Duca degli Abruzzi 24, 10129 Torino (Italy).
http://www.eln.polito.it/staff/daniele, e-mail: guido.lombardi@polito.it

Introduction
Recently the diffraction by arbitrary impenetrable wedges has been reduced to the
factorization of matrices of order four [1]. This paper provides an efficient and
general factorization technique that is based on the solution of a Fredholm integral
equation of second kind.

Wiener- Hopf solution of the problem

Figure 1 illustrates the problem of the diffraction by a plane wave at skew
incidence on an impenetrable wedge immersed in a medium with permittivity \( \varepsilon \)
and permeability \( \mu \).

The incident field is constituted by a plane wave having the following
longitudinal components:
\[
E^i_z = E_o e^{i \tau_o \rho \cos(\varphi_0 - \varphi)} e^{-j \alpha_o z} \quad H^i_z = H_o e^{i \tau_o \rho \cos(\varphi_0 - \varphi)} e^{-j \alpha_o z}
\]  

(1)
where \( \beta \) and \( \varphi_0 \) are the zenithal and azimuthal angle of the direction of the plane
wave \( \hat{n} \), and \( \tau = k \omega \sqrt{\mu_0 \varepsilon_0} \), \( \alpha_o = k \cos \beta \), \( \tau_o = k \sin \beta \).

The tangential fields are related on the boundaries of the wedge \( \varphi = +\Phi \) (a-face)
and \( \varphi = -\Phi \) (b-face) through the Leontovich conditions:
\[
\begin{bmatrix}
E_z(\rho,\Phi) \\
E_{\rho}(\rho,\Phi)
\end{bmatrix}
= Z_a
\begin{bmatrix}
H_z(\rho,\Phi) \\
-H_{\rho}(\rho,\Phi)
\end{bmatrix}, \quad
\begin{bmatrix}
E_z(\rho,-\Phi) \\
E_{\rho}(\rho,-\Phi)
\end{bmatrix}
= -Z_b
\begin{bmatrix}
H_z(\rho,-\Phi) \\
-H_{\rho}(\rho,-\Phi)
\end{bmatrix}
\]

(2)
where the matrices \( Z_{a,b} \) depends on the wedge material and
$Z_o = \sqrt{\mu / \varepsilon}$ is the free space impedance.

The Wiener-Hopf formulation [1,4] of this problem yields the solution:

$$X_o'(\eta) = G_z^{-1}(\eta)\overline{G}_z(\eta_0) \frac{\overline{T}_o}{\eta - \eta_o}$$  \hspace{1cm} (3)

where:

$$V_{z+}(\eta, \varphi) = \int_0^\infty E_z (\rho, \varphi) e^{i \eta \rho} d\rho, \quad I_{z+}(\eta, \varphi) = \int_0^\infty H_z (\rho, \varphi) e^{i \eta \rho} d\rho$$  \hspace{1cm} (4)

$$V_{\rho+}(\eta, \varphi) = \int_0^\infty E_\rho (\rho, \varphi) e^{i \eta \rho} d\rho, \quad I_{\rho+}(\eta, \varphi) = \int_0^\infty H_\rho (\rho, \varphi) e^{i \eta \rho} d\rho$$  \hspace{1cm} (5)

$$\overline{X}_o(\eta) = \begin{vmatrix}
V_{z+}(\eta, 0) & V_{\rho+}(\eta, 0) & Z_o I_{z+}(\eta, 0) & Z_o I_{\rho+}(\eta, 0)
\end{vmatrix}$$  \hspace{1cm} (6)

and $\eta = \eta(\eta) = -\tau_o \cos \frac{\Phi}{\pi} \left[ \arccos \left( -\frac{\eta}{\tau_o} \right) \right]$.  \hspace{1cm} (7)

For the problem at hand the constants $\overline{T}_o, \overline{\eta}_o$ assume the following expressions:

$$\overline{T}_o = \frac{\pi \sin \frac{\Phi}{\phi}}{\phi} \begin{vmatrix}
 \frac{jE_o}{\phi} & \frac{\alpha_0 \cos \phi_o E_o + kZ_o \sin \phi_o H_o}{\phi} & \tau_o & \frac{\alpha_0 Z_o \cos \phi_o H_o - k \sin \phi_o E_o}{\phi} \\
 & & & \\
jZ_o H_o & & & \\
 \frac{j\alpha_0 Z_o \cos \phi_o H_o}{\phi} & \frac{\tau_o}{\phi} & jZ_o H_o & \tau_o \\
\end{vmatrix}$$  \hspace{1cm} and $\overline{\eta}_o = -\tau_o \cos \frac{\pi}{\phi} \phi_o$.

and the matrix $\overline{G}_z(\eta)$ is the plus factorized matrix of the matrix kernel

$$\overline{G}(\eta) = \overline{G}_z(\eta)\overline{G}_z(\eta), \quad \overline{G}(\eta) = \begin{vmatrix}
g_{11} & g_{12} & g_{13} & g_{14} \\
d^a & d^a & d^a & d^a \\
g_{21} & g_{22} & g_{23} & g_{24} \\
d^a & d^a & d^a & d^a \\
g_{31} & g_{32} & g_{33} & g_{34} \\
d^b & d^b & d^b & d^b \\
g_{41} & g_{42} & g_{43} & g_{44} \\
d^b & d^b & d^b & d^b \\
\end{vmatrix}$$  \hspace{1cm} (7)

where:

$$g_{11} = -k n z^3_{10} \alpha_o \eta - m \eta \alpha^2_o - k^2 \eta \xi + k m z^2_{10} \alpha_o \xi - k z^2_{12} \xi \tau_o^2 - g_{12} = -k n z^3_{12} \tau_o^2 - m \alpha_o \tau_o^2, \quad g_{13} = k n \alpha_o \eta - m \eta z^2_{10} \alpha_o - k^2 n z^2_{12} \xi - k m \alpha_o \xi + z^2_{12} \eta \alpha_o \tau_o^2 , \quad g_{14} = k n \tau_o^2 - z^2_{12} m \alpha_o \tau_o^2 + z^2_{12} \tau_o^4 ,$$

$$g_{21} = k (n \eta - m \xi) \alpha_o z_{11}^2 + (\eta \alpha_o + k z_{11}^2 \xi) \tau_o^2, \quad g_{22} = k n z_{11}^3 \tau_o^2 - \tau_o^4, \quad g_{23} = m z_{11}^3 \alpha_o^2 + k n z_{11}^2 \xi - z_{21}^2 \alpha_o \eta \tau_o^2 + k \xi \tau_o^2, \quad g_{24} = m z_{11}^3 \alpha_o \tau_o^2 + z^2_{21} \tau_o^4 ,$$

$$g_{31} = -k n \tau_o^2 z_{11}^2 + m^2 z_{11}^2 \alpha_o^2 + kn(1 + \Delta^2) \tau_o^2 - m(z^2_{12} + z^2_{21}) \alpha_o \tau_o^2 + z^2_{12} \tau_o^4 , \quad \Delta^2 = z^4_{12} - z^4_{12} - z^2_{12} \eta \xi ,$$

$$g_{41} = k^2 n^2 z_{11}^2 + m^2 z_{11}^2 \alpha_o^2 + kn(1 + \Delta^2) \tau_o^2 - m(z^2_{12} + z^2_{21}) \alpha_o \tau_o^2 + z^2_{12} \tau_o^4 , \quad \Delta^2 = z^4_{12} - z^4_{12} - z^2_{12} \eta \xi .$$
\(d^b, \Delta^b_z, g_{31}, g_{32}, g_{33}, g_{34}, g_{41}, g_{42}, g_{43}, g_{44}\) assume respectively the same expression of \(d^a, \Delta^a_z, g_{11}, g_{12}, g_{13}, g_{14}, g_{21}, g_{22}, g_{23}, -g_{24}\) except for the substitution of the superscript \(a\) with the superscript \(b\). The functions \(\xi, m\) and \(n\) depends on \(\eta\) and are defined by:

\[
\begin{align*}
\xi &= \xi(\eta) = -\tau_o \sin\left(\frac{\Phi}{\pi} \arccos\left(-\frac{\eta}{\tau_o}\right)\right) \\
m &= m(\eta) = \tau_o \cos\left(\frac{\Phi}{\pi} \arccos\left(-\frac{\eta}{\tau_o}\right) + \Phi\right) \\
n &= n(\eta) = \tau_o \sin\left(\frac{\Phi}{\pi} \arccos\left(-\frac{\eta}{\tau_o}\right) + \Phi\right)
\end{align*}
\]

In several important cases the matrix \(\widetilde{G}(\eta)\) can be factorized in closed form [2]. For instance, this property is verified for the whole class of problems that have been solved with the Malyuzhinets-Sommerfeld technique.

**Fredholm factorization of the matrix kernel \(\widetilde{G}(\eta)\)**

By using the technique introduced in [3] the factorized matrix \(\widetilde{G}_+(\eta)\) can be expressed by:

\[
\widetilde{G}_+(\eta) = \frac{1}{\eta - \eta_p} \left| X_{i+}(\eta), X_{2+}(\eta), X_{3+}(\eta), X_{4+}(\eta) \right|^{-1}
\]

where \(\eta_p\) is an arbitrary point with negative imaginary part and the functions \(X_{i+}(\eta), \{i=1,2,3,4\}\) satisfy the following Fredholm integral equation:

\[
\widetilde{G}(\eta) X_{i+}(\eta) + \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{\widetilde{G}(x) - \widetilde{G}(\eta)}{x - \eta} X_{i+}(x) dx = -R_i, \text{ for } \Im[\eta_p] < 0
\]

with the vector constant \(R_i\) given by the canonical basis for the 4D space.

Since \(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{\widetilde{G}(x) - \widetilde{G}(\eta)}{x - \eta} \right|^2 dx d\eta\) is bounded [3], (11) is a Fredholm equation of second kind where it is applicable the well-known in literature powerful solution technique.

We experienced [4] that the convergence of approximate solutions considerably increases when we solve the integral equation in the \(t\) plane defined by the mapping \(\eta = \eta(t) = -\tau_o \cos(jt - \pi/2)\).

**Numerical validation**

To ascertain the correctness of our new methodology we have chosen a well known in literature test case to compare our solution with alternative method [5]: the bistatic far field amplitude evaluation for skew incidence on an impedance half plane. Figure 2 reports the GTD Diffraction Coefficient for \(E_z\) component.
\( (D_e(\phi) = s_e(\phi - \pi) - s_e(\phi + \pi) \), where \( s_e(w) \) is the Sommerfeld function) for the test case with the following problem parameters: \( k=1 \), the incident field \( \phi_0=5\pi/6 \), \( \beta = \pi/3 \), \( E_{z0}=1 \), \( H_{z0}=0 \), the aperture angle \( \Phi=\pi \), the integration parameters \( A=5 \), \( h=1 \) for the discretization of equation (11) after the transformation in the \( r \) plane. Peaks of the GTD Diffraction Coefficients are for \( \phi=\phi_0-\pi \) (incident field) and for \( \phi=2\Phi-\phi_0-\pi \) (reflected field).

![Amplitude of the GTD Diffraction Coefficient (dB)](image)

Several other applications of this technique to wedge problems have been reported in [6]. New examples and convergence tests concerning with new canonical wedge problems will be illustrated in the oral presentation of the paper.

**References:**