On the accuracy of Haar-based multiresolution time-domain schemes

On the Accuracy of Haar-Based Multiresolution Time-Domain Schemes

S. Grivet-Talocia

***Abstract*—We discuss in this paper the numerical accuracy of multiresolution time-domain (MRTD) schemes based on Haar scaling functions and wavelets. It has been noted that when the first resolution of wavelets is included in the schemes, the discrete difference equations arising from the Maxwell's system do not couple the scaling and wavelet coefficients except at boundary and excitation points. This fact is proved to be a serious drawback, since both a dispersion analysis and numerical tests for terminated and nonterminated schemes show that the addition of wavelets does not improve significantly the numerical accuracy of the underlying coarse-grid FDTD scheme.**

I. INTRODUCTION

WAVELET-based discretizations for Maxwell's equations have received much attention in the very recent literature, since they seem very promising for the reduction of the computational cost of more standard time-domain methods like FDTD [1]. Examples are provided by the so-called multiresolution time-domain (MRTD) schemes based on Battle-Lemarié [3] and Haar wavelets [4]. The main point lies in the intrinsic capability of wavelets to add higher spatial frequency contributions in the representation of the fields. This can be achieved locally by adding details only where the solution has fast variations [6]. This procedure leads naturally to multigrid schemes [7], [8].

This paper focuses on Haar-based MRTD schemes. It is well known that these reduce to the standard FDTD scheme when no resolutions of wavelets are used in the representation of the fields. Conversely, when only one resolution of wavelets is added, the resulting difference equations do not couple the scaling and wavelet coefficients except at boundary and excitation points [4]. We show in the following that this decoupling does not represent an advantage but a serious drawback of the Haar-MRTD scheme. Both theoretical arguments and numerical experiments will show that addition of one resolution of wavelets improves the sampling of the fields but not the accuracy at which the new samples are computed. In order to achieve higher accuracy, coupling between different scales is needed. This can be achieved through addition of higher resolution wavelets (thus also improving treatment of the boundary conditions [9]). However, it will be shown that the last resolution of added wavelets is always wasted since it leads to no accuracy improvement with respect to the coarser resolutions.

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II. HAAR-BASED MRTD SCHEMES

Let us consider the Maxwell's equations in a homogeneous medium

$$\nabla \times \mathbf{H} = \frac{1}{Z_0 c}\frac{\partial \mathbf{E}}{\partial t} \qquad \nabla \times \mathbf{E} = -\frac{Z_0}{c}\frac{\partial \mathbf{H}}{\partial t}$$

where Z_0 is the wave impedance and c the propagation velocity. Each field component is expanded into scaling functions $\phi_\mu(s) = \phi(s/\Delta s - \mu)$ and wavelets $\psi_\mu(s) = \psi(s/\Delta s - \mu)$, where ϕ is equal to one within the unit interval $(0, 1)$ and vanishing elsewhere, and $\psi(s) = \phi(2s) - \phi(2s-1)$. Expansion and testing is performed for each spatial coordinate $s = \{x, y, z\}$ with corresponding discretization indices $\mu = \{k, l, m\}$, as well as for time with rectangular pulses $h_n(t)$. In compact notations, the x-directed electric field component in the staggered Yee grid of size Δx, Δy, Δz is represented as

$$E_x(\underline{r}, t) = \sum_n \sum_{klm} \sum_{\zeta\eta\xi} {}^{x}_{n}E^{\zeta\eta\xi}_{k+1/2,\, l,\, m} \cdot h_n(t)\zeta_{k+1/2}(x)\eta_l(y)\xi_m(z) \tag{1}$$

where the summation over $\zeta\eta\xi$ includes eight terms stemming from all the permutations of scaling functions and wavelets

$$\zeta\eta\xi \in \{\phi\phi\phi,\, \phi\phi\psi,\, \phi\psi\phi,\, \phi\psi\psi,\, \psi\phi\phi,\, \psi\phi\psi,\, \psi\psi\phi,\, \psi\psi\psi\}. \tag{2}$$

The representation of the other field components is easily derived through permutation of the indices and follows the same rule as for standard FDTD scheme [2]. Inserting the above expressions into the first Maxwell's equation and performing a Galerkin test procedure leads [4] to the following expressions for the field coefficients within each cell $\{k, l, m\}$:

$$\begin{aligned} {}^{x}_{1}E^{\zeta\eta\xi}_{h00} - {}^{x}_{0}E^{\zeta\eta\xi}_{h00} &= Z_0 c \Delta t \\ &\cdot \left\{ \frac{{}^{z}_{h}H^{\zeta\eta\xi}_{hh0} - {}^{z}_{h}H^{\zeta\eta\xi}_{h\overline{h}0}}{\Delta y} - \frac{{}^{y}_{h}H^{\zeta\eta\xi}_{h0h} - {}^{y}_{h}H^{\zeta\eta\xi}_{h0\overline{h}}}{\Delta z} \right\} \end{aligned} \tag{3}$$

where $\{0, h, \overline{h}, 1\}$ denote, respectively, $\{\mu, \mu+1/2, \mu-1/2, \mu+1\}$ for each $\mu = \{k, l, m, n\}$. Similar expressions are obtained for the other field coefficients. This set of discretized equations matches the FDTD scheme when only the scaling coefficients are considered, i.e., $\zeta = \eta = \xi = \phi$. We will refer to this scheme as coarse-grid FDTD scheme. However, the same relations are found for any other combination of ζ, η, ξ involving at least one wavelet in any direction. There is no coupling between the eight different sets of equations corresponding to the various permutations of (2). This immediately leads to the conclusion that the stability criterion constrains the time step Δt to a maximum value that is the same as for the underlying coarse-grid FDTD scheme, even if the number of unknowns is eight times larger due to addition of wavelets.

III. Dispersion Analysis

We perform now a dispersion analysis of the Haar-MRTD scheme with one resolution of wavelets. Each field component F is taken as a plane wave:

$$F(\underline{r},\, t) = F_0\, e^{j\omega t} e^{-j\tilde{k}_x x} e^{-j\tilde{k}_y y} e^{-j\tilde{k}_z z}$$

where $\tilde{k}$ is the numerical wave vector. This expression is projected onto the discretization scheme through a standard procedure, not reported here. Obviously, since the numerical scheme (3) is coincident with the coarse-grid FDTD scheme any fixed choice of ζ, η, ξ, the analysis leads to the well-known FDTD dispersion relation

$$\frac{\sin^2 \Omega}{c^2 \Delta t^2} = \frac{\sin^2 K_x}{\Delta x^2} + \frac{\sin^2 K_y}{\Delta y^2} + \frac{\sin^2 K_z}{\Delta z^2} \tag{4}$$

where $\Omega = \omega \Delta t/2$ and $K_\mu = \tilde{k}_\mu \Delta\mu/2$. Note that the dispersion relation published in [5] is not correct. This explains why the numerical experiments in [5] did not agree with analytical results. The above expression controls the accuracy of the numerical scheme. Since all field coefficients, including those related to wavelet functions, are affected by the same numerical dispersion that applies to the scaling functions coefficients, and since the scheme with scaling coefficients only is fully equivalent to the coarse-grid FDTD scheme, no improvement is obtained by adding wavelets. Addition of wavelets corresponds to a refined sampling of the field quantities by a factor of two in each direction, but the added samples are affected by the same numerical errors of the coarse-grid ones.

The above considerations are illustrated through a simple numerical test. We consider a one-dimensional propagation along the z direction of a normalized TEM mode ($Z_0 = 1$, $c = 1$, and $z \in [0,\, 1]$). The Haar-MRTD equations for scaling and wavelet coefficients are, respectively

$${}^{x}_{1}E^{\phi}_{0} = {}^{x}_{0}E^{\phi}_{0} - \frac{Z_0 c \Delta t}{\Delta z}\left({}^{y}_{h}H^{\phi}_{h} - {}^{y}_{h}H^{\phi}_{\bar{h}}\right) \tag{5}$$

$${}^{y}_{h}H^{\phi}_{h} = {}^{y}_{\bar{h}}H^{\phi}_{h} - \frac{c \Delta t}{Z_0 \Delta z}\left({}^{x}_{0}E^{\phi}_{1} - {}^{x}_{0}E^{\phi}_{0}\right) \tag{6}$$

and

$${}^{x}_{1}E^{\psi}_{0} = {}^{x}_{0}E^{\psi}_{0} - \frac{Z_0 c \Delta t}{\Delta z}\left({}^{y}_{h}H^{\psi}_{h} - {}^{y}_{h}H^{\psi}_{\bar{h}}\right) \tag{7}$$

$${}^{y}_{h}H^{\psi}_{h} = {}^{y}_{\bar{h}}H^{\psi}_{h} - \frac{c \Delta t}{Z_0 \Delta z}\left({}^{x}_{0}E^{\psi}_{1} - {}^{x}_{0}E^{\psi}_{0}\right). \tag{8}$$

To avoid any perturbation due to truncations of the computational domain we consider periodic boundary conditions. The initial conditions are set to

$$E_x(z,\, t=0) = \exp\left\{-\frac{(z-0.5)^2}{2\sigma^2}\right\} \qquad H_y(z,\, t=0) = 0$$

with $\sigma = 0.05$. These conditions correspond to a periodized Gaussian pulse that splits into two equal waves propagating in opposite direction. Simulation is performed up to the final time $T = 8$ using $\Delta z = 0.025$, with a stability factor $c\Delta t/\Delta z = 0.8$. The results are shown in Fig. 1, together with the outcome of the fine-grid FDTD obtained by setting to $\Delta z/2$ and $\Delta t/2$ the space and time steps, respectively. The results confirm that addition of wavelets improves sampling of the fields but not the accuracy. It is quite evident that this improved sampling could have been obtained through simple interpolation at the end of the simulation. In summary, the Haar-MRTD with only one resolution of wavelets is equivalent to the coarse-grid FDTD scheme as far as the overall accuracy is concerned, and is not equivalent to the fine-grid FDTD. The latter has the same number of unknowns of Haar-MRTD, requires twice its computing time, but is far more accurate.

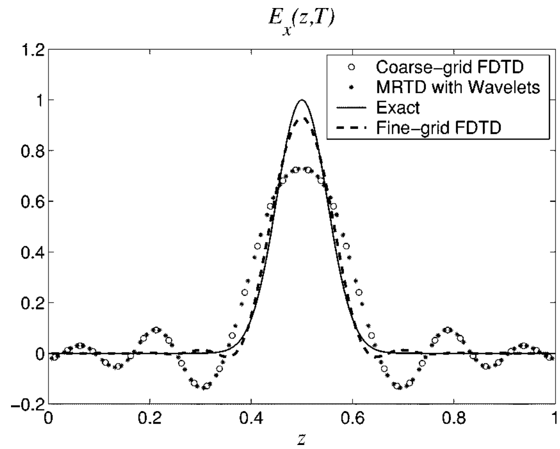

Fig. 1. Dispersion test for Haar-MRTD with one resolution wavelets.

We look now at the behavior of the Haar-MRTD scheme under another perspective. Without loss of generality we restrict our attention to the TEM case illustrated above. We take the sum and difference of (5)–(7), as well as of (6)–(8). It is well known that taking the sum and the difference of scaling and wavelet coefficients leads to the expansion coefficients into the two scaling functions at the next refinement level, corresponding to a double sampling rate of the fields. These two scaling functions have support, respectively, in the left and right half of the considered cell. We label the corresponding coefficients with the superscripts L and R, respectively. We get the following decoupled equations:

$${}^{x}_{1}E^{L}_{0} = {}^{x}_{0}E^{L}_{0} - \frac{Z_0 c \Delta t}{\Delta z}\left({}^{y}_{h}H^{L}_{h} - {}^{y}_{h}H^{L}_{\bar{h}}\right) \tag{9}$$

$${}^{y}_{h}H^{L}_{h} = {}^{y}_{\bar{h}}H^{L}_{h} - \frac{c \Delta t}{Z_0 \Delta z}\left({}^{x}_{0}E^{L}_{1} - {}^{x}_{0}E^{L}_{0}\right) \tag{10}$$

and

$${}^{x}_{1}E^{R}_{0} = {}^{x}_{0}E^{R}_{0} - \frac{Z_0 c \Delta t}{\Delta z}\left({}^{y}_{h}H^{R}_{h} - {}^{y}_{h}H^{R}_{\bar{h}}\right) \tag{11}$$

$${}^{y}_{h}H^{R}_{h} = {}^{y}_{\bar{h}}H^{R}_{h} - \frac{c \Delta t}{Z_0 \Delta z}\left({}^{x}_{0}E^{R}_{1} - {}^{x}_{0}E^{R}_{0}\right). \tag{12}$$

This means that addition of wavelets leads to two equivalent schemes operating, respectively, on the left half and on the right half of each cell. As there is no coupling between the left and right parts, the overall scheme is exactly coincident to two superimposed coarse-grid schemes shifted half cell one from each other. Obviously, the same considerations hold also in three dimensions, in which case there are $2^3 = 8$ interleaved schemes, each equivalent to the coarse-grid FDTD scheme, that operate independently one from each other.

The same considerations hold when more resolutions of wavelets are added to the fields representation. We give here a concise theoretical proof based on abstract formulations. Let us consider a coarse grid FDTD with cell size Δz applied to 1–D problem. The field approximation can be thought to belong to the space $\mathcal{S}_0$ of piecewise constant functions on each cell. If we add J resolutions of Haar wavelets we will get a hierarchical representation of the space $\mathcal{S}_J$ of piecewise constant functions on intervals of size $2^{-J}\Delta z$. The following two representations hold for this space

$$\mathcal{S}_J = \mathcal{S}_0 \oplus \bigoplus_{j=0}^{J-1} \mathcal{W}_j = \mathcal{S}_{J-1} \oplus \mathcal{W}_{J-1}$$

where $\mathcal{W}_j$ is the space spanned by the Haar wavelets at resolution j. Note that $\mathcal{S}_J$ is invariant under translations by $2^{-J}\Delta z$, while the spatial staggering of the coarse FDTD scheme is based on a $2^{-1}\Delta z$ displacement between E and H samples. This means that one resolution of wavelets is sufficient to loose the advantages of a staggered coarse-grid scheme. The last representation on the right shows that a Haar-MRTD scheme with J resolutions of wavelets is equivalent to a Haar MRTD scheme using scaling functions at level $J-1$ (i.e., to a FDTD scheme with cell size $2^{-J+1}\Delta z$) plus one resolution of wavelets. Therefore, based on the considerations in the foregoing paragraphs, the accuracy will be the same of the FDTD scheme without the last resolution of wavelets.

IV. Influence of Boundary Conditions

We discuss now the influence of boundary conditions in case of terminated schemes. A detailed treatment of boundaries based on Lagrange interpolation has been given in [4], and will not be repeated here. Instead, we show with a simple numerical test that the inherent coupling between scaling and wavelet coefficients occurring at the boundaries may allow some accuracy improvement with respect to the underlying coarse-grid FDTD, but only when the number of discretization points is very small. In such cases, the numerical solution of the field equations can be obtained very efficiently with standard algorithms like FDTD, without need of more advanced MRTD schemes. When the number of discretization points increases, the influence of the boundary conditions becomes less and less effective, since the main responsible for accuracy degradation is the dispersion error at the internal nodes.

The above considerations are best illustrated with a numerical test. Fig. 2 shows the relative error on the first resonant frequency of a parallel-plate waveguide. The domain is subdivided into N coarse-grid cells, and a time-domain simulation is performed according to the standard procedure followed in [4]. A Courant number of 0.8 has been used for three different simulations employing, respectively, coarse-grid FDTD, fine-grid FDTD, and Haar-based MRTD with one resolution of wavelets. The number of time steps has been set in all cases large enough to make negligible the frequency error due to limited time sampling. As expected, the accuracy of Haar-MRTD is almost equivalent to that of the coarse-grid FDTD and not to that of the fine-grid FDTD. It should be noted that these results confirm those obtained in the original paper [4] for the resonant frequencies of rectangular cavities.

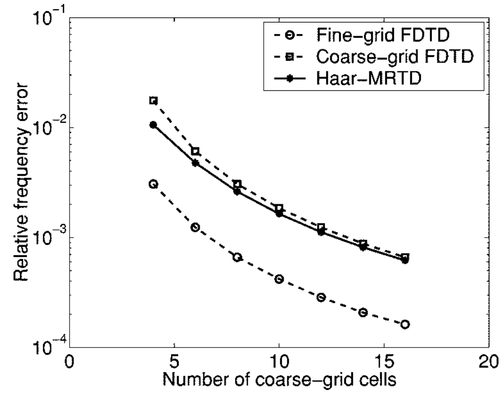

Fig. 2. Relative error on the first resonant frequency of a parallel-plate waveguide computed with FDTD and Haar-based MRTD.

V. Conclusion

We have given both theoretical and numerical evidence that addition of one resolution of wavelets within a Haar-based MRTD framework does not improve significantly the numerical accuracy of the underlying coarse-grid scheme. This holds for both terminated and nonterminated schemes. Similar conclusions hold when an arbitrary number of wavelet resolutions are included. A more effective strategy could be to resort to different (compactly supported) scaling and wavelet systems, such as biorthogonal B-splines. For instance, the results published in [6], [10] showed how higher resolution wavelets can be added to the discretization in order to achieve dynamic adaptivity. In any case, the efficiency of the resulting schemes should be carefully assessed in terms of accuracy and computational cost.

References

[1] K. S. Yee, “Numerical solution of initial boundary value problems involving Maxwell’s equations in isotropic media,” *IEEE Trans. Antennas Propagat.*, vol. AP-14, pp. 302–307, 1966.
[2] M. Krumpholtz, C. Huber, and P. Russer, “A field theoretical comparison of FDTD and TLM,” *IEEE Trans. Microwave Theory Tech.*, vol. 43, pp. 1935–1950, 1995.
[3] M. Krumpholtz and L. P. B. Katehi, “MRTD: New time-domain schemes based on multiresolution analysis,” *IEEE Trans. Microwave Theory Tech.*, vol. 44, pp. 555–571, 1996.
[4] M. Fujii and W. J. R. Hoefer, “A three-dimensional Haar-Wavelet-Based multiresolution analysis similar to the FDTD method—Derivation and application,” *IEEE Trans. Microwave Theory Tech.*, vol. 46, pp. 2463–2475, 1998.
[5] “Numerical dispersion in Haar-Wavelet Based MRTD scheme—Comparison between analytical and numerical results,” in *Proc. 15th Annu. Rev. Progress in Applied Computational Electromagnetics*, Monterey, CA, Mar. 15–20, 1999, pp. 602–607.
[6] S. Grivet-Talocia and F. Canavero, “Wavelet-based adaptive solution for the nonuniform multiconductor transmission lines,” *IEEE Microwave Guided Wave Lett.*, vol. 8, pp. 287–289, 1998.
[7] K. Goverdhanam, L. P. B. Katehi, and A. Cangellaris, “Applications of multiresolution based FDTD multigrid,” in *1997 IEEE MTT-S Int. Microwave Symp. Dig.*, pp. 333–336.
[8] K. Goverdhanam, E. Tentzeris, M. Krumpholtz, and L. P. B. Katehi, “An FDTD multigrid based on multiresolution analysis,” in *Proc. IEEE AP-S*, 1996, pp. 352–355.
[9] K. Goverdhanam and L. P. B. Katehi, “Applications of Haar wavelet based MRTD schemes in the characterization of 3D microwave circuits,” in *1999 IEEE MTT-S Int. Microwave Symp. Dig*.
[10] S. Grivet-Talocia and F. Canavero, “Wavelet-based adaptive simulation of nonuniform interconnects with arbitrary loads,” in *Proc. IEEE 1999 Int. Symp. EMC*, Seattle, WA, pp. 450–455.