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10 October 2024

Interacting Euclidean three-dimensional quantum gravity

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(Received 12 July 1991)

We show that Euclidean three-dimensional gravity coupled to a Gaussian scalar massive matter field in the first-order dreibein formalism gives a quantum theory which has a finite perturbative expansion around a nonvanishing background. We also discuss a possible mechanism to generate a nontrivial background metric starting from Rovelli-Smolin loop observables.

PACS number(s): 04.60.+n, 12.25.+e

I. INTRODUCTION

Some time ago, Witten [1] showed that pure three-dimensional (3D) quantum gravity (QG) in the first-order dreibein formalism is a finite off-shell (topological) theory when expanding around a vanishing background, i.e., $\langle e_\mu^a \rangle = 0$. This result comes from the fact that Einstein 3D theory is off shell and hence it is dependent on the variables that represent the gravitational field. Later, Deser *et al.* [2] extended Witten's result showing that the theory remains finite even when expanding around a flat background gravitational field $\langle e_\mu^a \rangle \propto \delta_\mu^a$ (in Euclidean space). Of course, we now have that $\mathrm{Det}\langle e_\mu^a \rangle \neq 0$.

In this work, we shall demonstrate that, as far as a perturbative theory is concerned, Euclidean 3D gravity coupled to a Gaussian scalar massive matter field still yields to a finite quantum theory in the first-order dreibein formalism.

In the end, we shall discuss a possible "quantum" mechanism for generating as an "order parameter" a nontrivial metric background from some gauge-invariant and diffeomorphism-invariant nonlocal observables of the pure topological theory, i.e. of 3D QG itself. These observables and their algebra were first introduced by Rovelli and Smolin [3] in the frame of Ashtekar's reformulation of canonical 4D general relativity [4] recently specialized to the case of (2+1)-dimensional Einstein gravity [5].

II. PATH-INTEGRAL FORMULATION OF THE THEORY

First-order dreibein gravity with a Euclidean signature is described by the action

$$I_E = \int d^3x \, \varepsilon^{\mu\nu\lambda} e_{\mu a} [\partial_{[\nu}\omega_{\lambda]}^a + \varepsilon_{bc}^a \omega_\nu^b \omega_\lambda^c] , \qquad (1)$$

where we have absorbed a k^{-1} factor into the dreibein e_μ^a and the spin connection $\omega_\mu^a = \varepsilon^{abc}\omega_{\mu bc}$ is an independent variable. In the following we shall consider the coupling of Eq. (1) to a real scalar massive matter field which has the first-order action [6]

$$I_M = \tfrac{1}{2} \int d^3x [\Phi^a \sqrt{e}\, e_a^\mu \partial_\mu\varphi + \tfrac{1}{2}(\Phi^a)^2 + em^2\varphi^2]$$

where Φ is a Lagrange multiplier, e_a^μ is formally the inverse matrix $[e_\mu^a]^{-1}$, $e \equiv \mathrm{Det}^{-1}(e_a^\mu)$ and, of course, we assume that $\mathrm{Det}(e_{\mu a}) \neq 0$. Notice that the Euclidean metric $g_{\mu\nu}$ is given by $g_{\mu\nu} = e_\mu^a e_\nu^b \delta_{ab}$. This equation can be rewritten in the form

$$I_M = \tfrac{1}{2} \int d^2x (\hat{e}_a^\mu \hat{e}^{\nu a} \partial_\mu\varphi\partial_\nu\varphi + em^2\varphi^2) , \qquad (2)$$

after using the equation of motion of Φ and setting $\hat{e}_a^\mu = \sqrt{e}\, e_a^\mu$, where $\hat{e}_a^\mu$ are tensor densities of weight $\tfrac{1}{2}$. Now, we need to fix a gauge. We choose a Landau-type gauge:

$$\partial^\mu e_\mu^a = 0 = \partial^\mu \omega_\mu^a . \qquad (3)$$

The resultant ghost and gauge-fixing action is then

$$I_{\mathrm{FP+GF}} = \int d^3x \, \{ \mathcal{C}_a \partial^\mu \omega_\mu^a + \mathcal{D}_a \partial^\mu e_\mu^a + \overline{c}_a \partial^\mu [(\partial_\mu\delta_b^a + \varepsilon_{cb}^a \omega_\mu^c)c^b] + \overline{d}_a \partial^\mu(\varepsilon^{acb}e_{\mu c}c_b) + \overline{d}_a \partial^\mu [(\partial_\mu\delta_b^a + \varepsilon_{cb}^a \omega_\mu^c)d^b] \} , \qquad (4)$$

where $\mathcal{C}_a$ and $\mathcal{D}_a$ are Lagrange multipliers, $c_a, \overline{c}^b$ and $\overline{d}_a, d^b$ are Faddeev-Popov ghosts. The sum of Eq. (1), Eq. (2), and Eq. (4) gives the total quantum action I. The corresponding Euclidean path integral has the form

$$\int \mathcal{D}e_\mu^a \mathcal{D}\omega_\mu^a \mathcal{D}\varphi \mathcal{D}\overline{c}_a \mathcal{D}c^b \mathcal{D}\overline{d}_a \mathcal{D}d^b \, e^{-I} . \qquad (5)$$

The action I is invariant under the following nilpotent (on-shell) Becchi-Rouet-Stora-Tyutin (BRST) transformation s [7]:

$$s\omega_\mu^a = -(D_\mu c)^a ,$$

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$$sc^a = \tfrac{1}{2}\varepsilon^a_{bc}c^b c^c, \quad s\bar{c}_a = \mathcal{C}_a, \quad s\mathcal{C}_a = 0 ,$$

$$sd^a = \varepsilon^a_{bc}c^b d^c, \quad s\bar{d}_a = \mathcal{D}_a, \quad s\mathcal{D}_a = 0 ,$$

$$se^a_\mu = -(D_\mu d)^a - \varepsilon^a_{bc}e^b_\mu c^c ,$$

$$s\varphi = -e^\mu_a c^a \partial_\mu \varphi ,$$

$$D^a_{\mu b} \equiv (\partial_\mu \delta^a_b + \varepsilon^a_{cb}\omega^c_\mu) ,$$

$$s^2 = 0 .$$

Our purpose is to integrate out, first, the matter field in the functional integral. Exact Gaussian integration of the matter field gives

$$Ne^{-W} \equiv \int \mathcal{D}\varphi \, e^{-I_M} = N[\mathrm{Det}\,(\hat{\Box} + \hat{m}^2)]^{-1/2}$$

$$= N \exp[-\tfrac{1}{2}\ln \mathrm{Det}(\hat{\Box} + \hat{m}^2)] , \quad (6)$$

where $\hat{\Box} = -\hat{e}^\mu_a \partial_\mu \hat{e}^{\nu a}\partial_\nu = -\hat{\partial}_a\hat{\partial}^a$ and $\hat{m}^2 = em^2$. Here, $W = \tfrac{1}{2}\ln \mathrm{Det}(\hat{\Box} + \hat{m}^2)$ is the one-loop effective action of the matter field. Following Birrell and Davies [8], we see that using the DeWitt-Schwinger representation and dimensional regularization, W can be written as

$$W_{m^2} = \int d^n x \, \mathcal{L}_{\mathrm{eff}}(x) \equiv \int d^n x \, \sqrt{g(x)} L_{\mathrm{eff}}(x; m^2) ,$$

where, in n dimensions, the asymptotic (adiabatic) expansion of L_{eff} is

$$L_{\mathrm{eff}} \simeq \tfrac{1}{2}(4\pi)^{-n/2} \sum_{j=0}^{\infty} a_j(x)(m^2)^{(n-2j)/2}\Gamma(j - n/2) . \quad (7)$$

We immediately notice that, in odd dimensions, L_{eff} has only finite terms because $\Gamma(j - n/2) < \infty$ and $a_j(x)$ are geometrical invariants built out of the curvature tensor and its contractions. In three dimensions, for large external momenta (or equivalently for large m^2 in Planck units), only the first two terms of Eq. (7) are sensibly different from zero.

After the functional integration (6), we are left with the

loop expansion constructed from the effective lowest-order action

$$I' \equiv I_E + I_{\mathrm{FP+GF}} + W_{m^2} . \quad (8)$$

In a quantum perturbative treatment of I' (see the next section) one has to consider the modified Feynman rules, coming from W_{m^2}, of $I_E + I_{\mathrm{FP+GF}}$. Indeed, the first two terms of L_{eff} may be regarded as a contribution to the gravitational Lagrangian although they arise from the action of the quantum matter field, since one has that $a_0(x) = e$ and $a_1(x) = (e/6)R(e,w)$, where $R(e,w)$ is the curvature scalar considered in the first-order dreibein formalism [Eq. (1)]. Notice that for large m^2 (in Planck units), $I_E(e,w) + W_{m^2}(e,w)$ is equivalent to a Euclidean non-Abelian Chern-Simons (CS) gauge theory with the gauge group $SO(4) \sim SU(2) \times SU(2)$, assuming that the induced cosmological constant $\Lambda \equiv m^3/12\pi^2$ is positive definite [1]. As a consequence, we may understand the effects of coupling Gaussian matter fields to 3D gravity in the ultraviolet region, i.e., for large momentum of the graviton, as a "dressing" of the pure 3D gravity sector. Thus, one still ends with a Chern-Simons gauge theory which, according to common wisdom [1], gives a finite quantum theory. This is actually the case, as we shall see in the next section by arguments of power counting.

III. PERTURBATIVE EXPANSION

As usual, the exact treatment of Eq. (8) is too hard a thing to cope with and therefore we go on with a perturbative expansion. To establish the finiteness of Eq. (8) around a flat background $e^a_\mu = \delta^a_\mu + h^a_\mu$, $h \ll 1$ (in Euclidean space), we keep the gauge of Eq. (3) that we now write as $\partial^\mu h^a_\mu = 0 = \partial^\mu \omega^a_\mu$. Remembering that we move indices with δ^a_μ, i.e., we identify the metric introduced in the gauge fixing with that introduced by the background, we write the resultant ghost and gauge-fixing action as [2]

$$I_{\mathrm{FP+GF}} = \int d^3x \, \{\mathcal{C}_a \partial^\mu \omega^a_\mu + \mathcal{D}_a \partial^\mu h^a_\mu + \bar{c}_a \partial^\mu [(\partial_\mu \delta^a_b + \varepsilon^a_{cb}\omega^c_\mu)c^b] + \bar{d}_a \partial^\mu [\varepsilon^a_{cb}(\delta^c_\mu + h^c_\mu)c^b + (\partial_\mu \delta^a_b + \varepsilon^a_{cb}\omega^c_\mu)d^b]\} . \quad (9)$$

The vertices of this action and of the pure gravitational one I_E are cubic, while the three basic (off-diagonal) propagators are [2]

$$\langle \omega^b_\nu h^a_\mu \rangle = -i\delta^{ab}\delta_{\mu a}\delta_{\nu \beta}\varepsilon^{\alpha\beta\gamma}\frac{p_\gamma}{p^2} \to \frac{1}{p} \quad \text{as } p \to \infty , \quad (10a)$$

$$\langle \bar{c}_a c^b \rangle = \frac{\delta^b_a}{p^2} = \langle \bar{d}_a d^b \rangle . \quad (10b)$$

In the following we shall represent the graviton propagator $\langle \omega h \rangle$ as in Fig. 1, where the dashed (wavy) line stands for ω (h). In addition, we should have propagators due to the flat background but, for simplicity, we treat them as new vertices beside the cubic ones. We now have to consider the contribution to the above Feynman

rules [Eq. (10)] due to the matter one-loop terms in Eq. (8), i.e., in $\tfrac{1}{2}\ln \mathrm{Det}\{[\hat{\Box} + \hat{m}^2](\delta + h)\}$, which shall be treated in the standard perturbative expansion in the quantum field h^a_μ. This amounts to calculating the effective graviton propagators and vertices given at the lowest order by the insertion of matter one loop terms in the h lines alone. In this connection, we need the φ propagator and the $h\varphi\varphi$ cubic vertex. In this perturbative framework, the two- (three-) point function $\langle \varphi\varphi \rangle$ ($\langle h\varphi\varphi \rangle$) is obtained from Eq. (2) expanding e^a_μ up to order $O(h^2)$; i.e., in the momentum representation picture
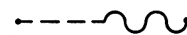

FIG. 1. Graviton propagator $\langle \omega h \rangle$.

FIG. 2. Matter propagator $\langle \varphi\varphi \rangle$.

FIG. 3. Matter-graviton vertex $\langle \varphi\varphi s \rangle$.

one must start from

$$I_M = \int d^3p \left[\tfrac{1}{2}(p^2+m^2)\varphi^2 + s^{\mu\nu}(p_\mu p_\nu - \delta_{\mu\nu}m^2)\varphi^2 \right],$$
$$s^{\mu\nu} = \delta^\mu_a \delta^{\rho\nu} s^a_\rho, \quad s^a_\rho \equiv h^a_\rho - \tfrac{1}{2}\delta^a_\rho h, \quad h \equiv \delta^\rho_a h^a_\rho. \tag{11}$$

As a consequence we get the matter Euclidean Feynman rules

$$\langle \varphi\varphi \rangle = \frac{1}{p^2+m^2}, \quad \langle \varphi\varphi s^{\alpha\beta} \rangle = -2(p_\alpha p_\beta - \delta_{\alpha\beta}m^2). \tag{12}$$

We pictorially associate $\langle \varphi\varphi \rangle$ and $\langle \varphi\varphi s^{\alpha\beta} \rangle$ with Fig. 2

and Fig. 3, respectively. With these Feynman rules, we notice that the first term coming from the h^a_μ expansion of the nonlocal action W defined in Eq. (6) gives a constant contribution to the cosmological constant. The other diagrams, instead, all go like p^3 as $p \to \infty$. In fact, if we take, e.g., the term quadratic in h^a_μ, the self-energy diagram, its lowest order contribution to Eq. (10a), is given by

$$\int_{\omega\to 3/2} d^{2\omega}k \frac{[k_\alpha(k+p)_\beta - \delta_{\alpha\beta}m^2][k_\mu(k+p)_\nu - \delta_{\mu\nu}m^2]}{(k^2+m^2)[(k+p)^2+m^2]}$$
$$= \pi^3 \{ \tfrac{163}{128} p_\alpha p_\beta p_\mu p_\nu (1/\sqrt{p^2}) + (\tfrac{3}{128} A_{\alpha\beta\mu\nu\rho\sigma} p^\rho p^\sigma - \tfrac{5}{128} B_{\alpha\beta\mu\nu\eta\lambda} p^\eta p^\lambda - \tfrac{13}{12} C_{\alpha\beta\mu\nu} p^2)\sqrt{p^2} \} + O(m^2), \tag{13}$$

where

$$A_{\alpha\beta\mu\nu\rho\sigma} = \delta_{\alpha\beta}\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\alpha\nu}\delta_{\rho\mu}\delta_{\sigma\beta} + \delta_{\beta\mu}\delta_{\rho\alpha}\delta_{\sigma\nu} + \delta_{\mu\nu}\delta_{\rho\alpha}\delta_{\sigma\beta},$$
$$B_{\alpha\beta\mu\nu\eta\lambda} = \delta_{\alpha\mu}\delta_{\eta\beta}\delta_{\lambda\nu} + \delta_{\beta\nu}\delta_{\eta\alpha}\delta_{\lambda\mu},$$
$$C_{\alpha\beta\mu\nu} = \delta_{\alpha\beta}\delta_{\mu\nu} + \delta_{\beta\mu}\delta_{\alpha\nu} + \delta_{\alpha\mu}\delta_{\beta\nu}.$$

Its calculation, using dimensional regularization, shows that it is finite and it goes like p^3 for large momentum $p \equiv \sqrt{p^2}$. On the other hand, all the graviton vertex corrections [to I_E, Eq. (1)] induced by $\ln \text{Det}(\hat{\square} + \hat{m}^2)$ in Eq. (8) behave like p^3 when the momentum $p \equiv \sqrt{p^2}$ of one of the gravitons becomes large. Roughly speaking, this is a consequence of the fact that the matter propagator and vertex grow as $(1/p^2)$ and p^2 for $p \to \infty$, respectively. This is the crucial reason why our theory will come out UV finite. Then, the lowest-order effective graviton propagator and cubic vertex are given respectively in Fig. 4 and Fig. 5. They behave as

$$\langle \omega^a_\mu s^b_\nu \rangle_{\text{eff}}(p) \sim \frac{1}{p^3}, \quad p \to \infty, \tag{14}$$

$$\langle \omega^a_\mu \omega^b_\nu s^c_\rho \rangle_{\text{eff}}(p) \sim p^3, \quad p \to \infty. \tag{15}$$

Remember that $s^a_\mu \equiv h^a_\mu - \tfrac{1}{2}\delta^a_\mu(\delta^\rho_b h^b_\rho)$ and p is a Euclidean momentum variable. The off diagonality of the full one-particle-irreducible (1PI) dressed propagator Eq. (14), which comes from a straightforward calculation, may be also explained by the following simple argument. At the

tree level, off diagonality comes from the fact that Euclidean 3D gravity is a (three-dimensional) non-Abelian gauge theory with a *nonsemisimple* Lie group as an internal symmetry, namely ISO(3), and hence it has the Lagrangian structure eR. This is a particular case of the connection (B)–curvature (F) theories (BF theories). In our dimensional-regularization scheme the above gauge invariance is preserved in the full dressed effective action Eq. (8), and therefore it still has a BF structure.

The important consequence of Eq. (14) is that now the graviton fields (ω, h) should be assigned ultraviolet dimension zero in place of the canonical value one of pure gravity, while the dimension of the ghost fields is one-half as usual. Taking into account the above considerations, we find the superficial degree of divergence $\omega(G)$ of an arbitrary diagram G by standard dimensional power counting as

$$\omega_{\text{dim}}(G) = 3 + \sum_V (\omega_V - 3) - 2E_{\bar{d}}, \tag{16}$$

where ω_V is the dimension of the interaction monomial attached to a generic vertex appearing in Eq. (8) and $E_{\bar{d}}$ is the number of external antighosts $\bar{d}$. Notice that in the power counting (16) the graviton external legs in (ω, h) do not contribute since $\dim_{\text{UV}}(h) = \dim_{\text{UV}}(\omega) = 0$.

It is worth mentioning that the power counting (16) makes sense also for Feynman graphs with external lines included, which in our approach involve, in principle, only the graviton and the ghost fields [although only the

FIG. 4. Effective graviton propagator $\langle \omega\omega s \rangle_{\text{eff}}$.

FIG. 5. Effective vertex propagator $\langle \omega\omega s \rangle_{\text{eff}}$.

ghost fields contribute to (16)]. Indeed, our basic idea in this work is to treat the Gaussian quantum matter ϕ field as a correction to the graviton self-energy and therefore to the effective graviton propagator, thus obtaining zero UV dimensions for the graviton field. From a technical point of view this is always possible, since we can exactly integrate out the boson ϕ in the vacuum amplitude functional integral (5).[1] As a consequence, our perturbative Feynman rules follow from the effective action (8), and hence do not contain the ϕ field anymore.

There is also a formal reason that suggests that there are no more renormalizations for the scalar field N-point functions than those required by the power counting (16). If at the beginning (i.e., before the ϕ-functional integration) we pick an N-point function of the ϕ field, after the introduction of a scalar density source in the path integral, by using the fact that the integral is still Gaussian in ϕ, we find that the scalar N-point function is given by the functional integral with respect to the effective measure $De\, D\omega\, D(\text{ghosts})e^{-I'}$, with I' defined by Eq. (8), of a sum $\sum$ of products of scalar field propagators. It turns out that $\sum$ may be factorized out of the functional integral. Hence, we are left only with the renormalization of the vacuum amplitude functional integral as before.

We now turn to the study of possible divergences. First, only one-loop diagrams can be constructed because of the off diagonality of the propagator [Eq. (14)] and the dependence on the field variables of the vertices [2]. In addition, all the interaction monomials have dimension $\omega_V = 3$ for $V = \langle h\omega\omega \rangle$ and $V = [\ln \text{Det}(\hat{\Box} + \hat{m}^2) - O(h^2)]$ or $\omega_V < 3$ for the ghost vertices ($\omega_{\text{ghosts}} = 2$). This tells us that our theory is at least power-counting renormalizable. As a matter of fact, considering, at first, graphs without ghost vertices, we see that we have a superficial cubic degree of divergence for any number n of external (graviton) legs. However, they vanish using the dimensional regularization scheme which implies that $\int [d^{2d}k / (2\pi)^{2d}](k^2)^{\beta-1} = 0$ for $\beta = 0, 1, 2, \ldots$ and any d ('t Hooft–Veltman conjectures) [11] and in our case $\beta = 1$, $d \to (\frac{3}{2})$. In $D = 3$ neither quadratic nor logarithmic divergences are possible for parity reasons; only linear ones remain. In this case, diagrams consist of two ghost vertices of the type $\langle \omega c \bar{c} \rangle$ or $\langle \omega d \bar{d} \rangle$. As we have already noticed, in the dimensional-regularization scheme, linearly divergent graphs are set to zero. Therefore, the quantum theory of 3D gravity coupled to a free scalar massive matter field is *finite* off shell in the first-order dreibein formalism when it is expanded around a nondegenerate flat background.

In the end, we should like to observe the following two things. First, we notice that the perturbative renormalizability can also be reached in the framework of a BRST quantization scheme. Indeed, one could show that the possible divergent part Γ_{div} of the effective action I', Eq. (8), satisfies (in our Landau-like gauge) the Ward identity

$$\mathcal{G}\Gamma_{\text{div}} = \left[s_{\text{eff}} e_\mu^a \frac{\delta}{\delta e_\mu^a} + s_{\text{eff}} \omega_\mu^a \frac{\delta}{\delta \omega_\mu^a} \right] \Gamma_{\text{div}} = 0 \,,$$

where $(s_{\text{eff}} e_\mu^a, s_{\text{eff}} \omega_\mu^a)$ are the BRST transformations that leave the effective action I' invariant. Here we have used the fact that, as has been observed, all 1PI diagrams containing external ghost lines are convergent in the dimensional regularization scheme and/or for parity-symmetry reasons. Therefore, Γ_{div} does not depend on the ghosts. Then, this Ward identity tells us that Γ_{div} is a BRST invariant functional of (e, ω) alone. Since the divergent part is local and of dimension three at most, the only possible form Γ_{div} is thus given by the 3D Einstein action (in first-order dreibein formalism) itself.

Second, we would like to notice that in the above renormalization discussion we have assumed that the effects of Lorentz anomaly terms (if any) such as

$$I_{\text{LCS}} = \frac{ik'}{8\pi^2} \int_M \varepsilon^{\mu\nu\rho} (R_{\mu\nu ab} \Gamma_\rho^{ab} + \tfrac{2}{3} \Gamma_{\mu a}^b \Gamma_{\nu b}^c \Gamma_{\rho c}^a) \,, \quad (17)$$

in Euclidean signature may be taken into account by enlarging [12] the spin-connection gauge group, which at the classical Euclidean level is $SO(3) \sim SU(3)$. In Eq. (17), Γ and R are the Levi-Civita connection and the curvature respectively for the dreibein field e_μ^a, which is the fundamental variable. I_{LCS} can be interpreted as a CS term for an $SO(3) \sim SU(2)$ gauge connection Γ. Thus, in the first-order dreibein formalism, this is equivalent to starting with the $SO(3) \oplus SO(3)$ Lie-algebra-valued connection $\omega \oplus \Gamma$, or equivalent to considering the complex gauge group $SO(3,\mathbb{C}) \sim SL(2,\mathbb{C})$ as "internal Lorentz" symmetry. In any case I_{LCS} is a topological invariant $CS(M^3)$, for a close oriented Riemann manifold M^3, which takes values on the circle $(\mathbb{R}/\frac{1}{2}\mathbb{Z}) \sim S^1$ [13] if M^3 is homeomorphic to a closed hyperbolic three-manifold.[2] Therefore, it does not participate to the local short-distance scale structure of 3D QG and, hence, to the above computation of UV divergences.

IV. CLASSICAL BACKGROUND METRIC FROM GLOBAL OBSERVABLES OF PURE THREE DIMENSIONAL QUANTUM GRAVITY

In this final part of the paper, we suggest a possible way out of the conceptual problem raised by Witten [15] on how to introduce a background space-time metric as some expectation value of gauge and diffeomorphism-invariant observables, the Wilson lines [16], of the topological pure 3D quantum gravity (QG). Clearly, this problem underlies the coupling of matter degrees of freedom to 3D gravity as discussed, for instance, in the previ-

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[1]A similar idea was implemented by Tomboulis [9] in the $(1/N)$ expansion of 4D QG coupled to N massless fermions, and later on extended by Smolin [10] to d dimensions.

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[2]According to a famous conjecture [14] about three-manifolds, almost all interesting (irreducible) three-manifolds have a "geometrical decomposition" into (closed) hyperbolic three-varieties.

ous sections. Indeed, the gravitational coupling of "non-topological" matter fields only makes sense in the "broken phase" of general relativity where there is a Riemannian space-time with distances and light cones, while the topological nature of the theory depends upon having an unbroken phase $\bar{e}_\mu^a = \langle e_\mu^a \rangle = 0$ without metric or Riemannian interpretation.

Pure 3D QG has a set of observables, the so-called Rovelli and Smolin [3] (also Nelson and Regge [15]) observables. One of them will play a fundamental role in explaining a pure "quantum" mechanism that leads to a nontrivial background space-time metric as an "order parameter" for the diffeomorphism group. Let us show this mechanism by considering the pure Euclidean 3D QG where the Lorentz group is SO(3)~SU(2). Following Rovelli and Smolin, we shall use for this purpose a part of their observable called T^1 written here as $T^1 \equiv \mathrm{Tr}\mathcal{W}_\mu[C](s)$, where Tr is the trace in the fundamental representation of SU(2). Naively, one may understand $\mathcal{W}_\mu[C](s)$ as the parallel displacement generator of the dreibein e_μ^a along a loop $C = C(s)$. It is defined as follows. Let us assume for simplicity that the Euclidean space-time M^3 is homeomorphic to $\mathbf{R}^3$. Then, for any loop (knot) $C \in \mathbf{R}^3$ and loop parameter s, $\mathcal{W}_\mu[C](s)$ is given by inserting $E_\mu(x) \equiv e_\mu^a(x)\tau_a$, where τ_a are the generators of SU(2), along the holonomy $\mathcal{H}(c)$ of C at the point $x = C(s)$, i.e.,

$$\mathcal{W}_\mu[C](s) = E_\mu(C(s))\mathcal{H}(c) \equiv \mathcal{W}_\mu^a[C](s)\tau_a \ ,$$

$$\mathcal{H}(C) \equiv \mathcal{P}\exp\left[\oint_C dx^\mu \, \omega_\mu^a(x)\tau_a\right] \ . \tag{18}$$

Here, $\mathcal{P}$ stands for path-ordering and ω_μ^a is the spin connection, i.e., the gauge-connection for the Euclidean Lorentz group. T^1 is not reparametrization invariant
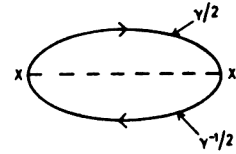

FIG. 6. The loop $\gamma = \gamma/2 \cup \gamma^{-1}/2$.

since it depends on a preferred value of the loop parameter s. However, it is reparametrization covariant in the sense that $\mathcal{W}_\mu[C'](s) = \mathcal{W}_\mu[C][f(s)]$, where $C'(s) = C[f(s)]$, $f'(s) > 0$, is a reparametrizaton of C with the same orientation. Thus, our basic idea for measuring in a "gauge-invariant way" the distance between two points x and x', is to connect them with two half-paths $(\gamma/2)$ and $(\gamma^{-1}/2)$, together forming a loop γ as in Fig. 6, where $x = \gamma(s)$ and $x' = \gamma(s')$.

Then, we define[3] as a background classical space-time metric $g_{\mu\nu}^{cl}(x)$, which must be by definition a c number, the following expectation value of the trace of the product of two composite operators $\mathcal{W}_\mu[\gamma](s)$ evaluated at the points $x = \gamma(s)$ and $x' = \gamma(s')$:

$$g_{\mu\nu}^{cl}(x) = \lim_{\substack{x' \to x \\ x = \gamma(s) \\ x' = \gamma(s')}} \langle \mathrm{Tr}\{\mathcal{W}_\mu^a[\gamma/2](s)\tau_a \mathcal{W}_\nu^c[\gamma^{-1}/2](s')\tau_c\}\rangle \ . \tag{19}$$

Notice that the operator inside the expectation value of (19) is called by Rovelli-Smolin (RS) T^2 observable. The evaluation of Eq. (19) is quite complicated and for our aim it is sufficient to limit ourselves to calculating it in the tree-approximation. Thus, at lowest order in the expansion of the path-ordered $\mathcal{H}$ in $\mathcal{W}_\mu^a[\gamma]$ we get

$$-\tfrac{1}{4}\lim_{\varepsilon \to 0}\oint_\gamma dz^\rho \, \langle e_\mu^a(x_\varepsilon)\omega_\rho^b(z)\rangle \lim_{\varepsilon \to 0}\oint_\gamma dw^\sigma \, \langle e_\nu^c(x_\varepsilon)\omega_\sigma^d(w)\rangle \mathrm{Tr}\,(\tau_a\tau_b\tau_c\tau_d) \ . \tag{20}$$

Here, we have used the fact that the trace of an odd product of Pauli matrices vanishes and that $\langle ee \rangle = 0 = \langle \omega\omega \rangle$. Notice that in this tree approximation and at lowest order in the expansion of the path-ordered approximation, Eq. (20) looks like (up to a numerical factor) the product of two $\langle T^1 \rangle$ evaluated in the same approximation. Furthermore, in order to regularize $\langle e \oint_\gamma \omega \rangle$, we have taken (see Fig. 7) the point x, which in the following will be denoted by x_ε, on the "framed path" γ_f defined by

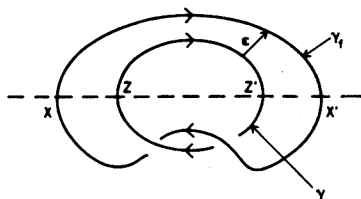$$\gamma_f = \{x^\mu + \varepsilon n^\mu(t): |n(t)| = 1, \ \varepsilon > 0\} \ , \tag{21}$$

FIG. 7. An example of framing. The outer curve represents the framed contour.

where n^μ is a vector orthogonal to γ obtained by shifting the path γ on which we calculate the holonomy $\mathcal{H}(\gamma)$. In Eq. (20) we shall use the formula

$$\lim_{\varepsilon \to 0}\oint_\gamma dz^\rho \, \langle e_\mu^a(x_\varepsilon)\omega_\rho^b(z)\rangle = \lim_{\varepsilon \to 0}\varepsilon_{\mu\rho\nu}\partial_{x_\varepsilon}^\nu \oint_\gamma dz^\rho \, \frac{1}{|x_\varepsilon - z|}\delta^{ab} \ . \tag{22}$$

The result (22) has been obtained by using the expansion of e_μ^a around the "topological vacuum"

$$e_\mu^a = h_\mu^a, \quad \bar{e}_\mu^a = \langle e_\mu^a \rangle = 0$$

and the form of the propagator $\langle h_\mu^a \omega_\rho^b \rangle$, Eq. (10a). Clearly in Eq. (22) there exists and underlying definition

[3]A similar idea of detecting the space-time geometry from the ISO(2,1)-Wilson lines has been recently suggested by S. Carlip [17].

of distance through the modulus. But following Witten [1,14], we may assume to fix *a priori* an external space-time metric $\delta_{\mu\nu}$ which will be identified in a "self-consistent" way, in the end, with a flat-order approximation of $g^{cl}_{\mu\nu,tree}(x)=\delta_{\mu\nu}+x^\rho\partial_\rho g^{cl}_{\mu\nu,tree}(0)+\cdots$. In any case the metric dependence enters only the gauge-fixing procedure and does not affect the physical space. An easy way of understanding this is to recast the first-order dreibein form of the Euclidean 3D Einstein action in an ISO(3) Chern-Simons form [1]. Then one should recognize that the associated symmetric energy-momentum tensor T is given by the commutator with the BRST charge Q, that is, $T=[Q,(\cdots)]$, where the other member of the commutator is not relevant here [18]. Since Q annihilates the physical states, the mean value of T vanishes between physical states. This implies general covariance on the physical space, as it should be. Going

back to Eq. (22), it is identical, up to δ^{ab}, to the potential $\mathcal{A}_\mu(x_\varepsilon)$ due to a closed magnetic vortex line γ. By the Biot-Savart law, $\mathcal{A}_\mu(x_\varepsilon)$ can also be interpreted as the total magnetic field generated by a steady unitary current flowing through γ and observed at a point x_ε belonging to a curve γ_f that is twisted around γ (see Fig. 7). The twists are necessary in order to have a nontrivial results. For instance, if γ is a circle and γ_f is parametrized (in $\mathbf{R}^3$) by:

$$\gamma_f \begin{cases} x^1=(1+\varepsilon\cos\theta)\cos\theta \,, \\ x^2=(1+\varepsilon\sin\theta)\sin\theta \,, \\ x^3=\varepsilon\sin\theta \,, \end{cases} \tag{23}$$

setting $\varepsilon_{\mu\nu\rho}=\varepsilon_{123}=1$, we find that $\mathcal{A}_\mu$ is equal to

$$\lim_{\varepsilon\to0}\mathcal{A}_1(x_\varepsilon)|_{x_\varepsilon=\gamma_f}=-\lim_{\varepsilon\to0}\frac{1}{\varepsilon\cos\theta}\frac{d}{d\theta}[\cos\theta G_\varepsilon(\theta;\gamma)] \,,$$

$$\lim_{\varepsilon\to0}\mathcal{A}_2(x_\varepsilon)|_{x_\varepsilon=\gamma_f}=\lim_{\varepsilon\to0}\frac{1}{\varepsilon\cos\theta}\frac{d}{d\theta}[\sin\theta G_\varepsilon(\theta;\gamma)] \,, \tag{24a}$$

$$\lim_{\varepsilon\to0}\mathcal{A}_3(x_\varepsilon)|_{x_\varepsilon=\gamma_f}=\lim_{\varepsilon\to0}\left[\frac{1}{\sin\theta(1+2\varepsilon\cos\theta)}\frac{d}{d\theta}[\cos\theta G_\varepsilon(\theta;\gamma)]+\frac{1}{\cos\theta(1+2\varepsilon\sin\theta)}\frac{d}{d\theta}[\sin\theta G_\varepsilon(\theta;\gamma)]\right] \,.$$

Here, $G_\varepsilon(\theta;\gamma)$ is defined as

$$G_\varepsilon(\theta;\gamma)=\frac{4}{B(\varepsilon)}\left[\frac{A(\varepsilon)}{\sqrt{A(\varepsilon)+B(\varepsilon)}}F\left[\frac{\pi}{2},\left[\frac{2B(\varepsilon)}{A(\varepsilon)+B(\varepsilon)}\right]^{1/2}\right]-\sqrt{A(\varepsilon)+B(\varepsilon)}E\left[\frac{\pi}{2},\left[\frac{2B(\varepsilon)}{A(\varepsilon)+B(\varepsilon)}\right]^{1/2}\right]\right] \,, \tag{24b}$$

where F and E are elliptic integrals of the first and second kind and $A(\varepsilon)$, $B(\varepsilon)$ are suitable functions of ε such that

$$\lim_{\varepsilon\to0}A(\varepsilon)=\lim_{\varepsilon\to0}B(\varepsilon)=2 \,. \tag{24c}$$

The same result is obtained from the second factor of Eq. (20). Thus, we see that $g^{cl}_{\mu\nu,tree}(x)$ is given by the symmetric 3×3 matrix

$$g^{cl}_{\mu\nu,tree}\simeq\text{const}\times\begin{bmatrix}\mathcal{A}_1^2 & \mathcal{A}_1\mathcal{A}_2 & \mathcal{A}_1\mathcal{A}_3 \\ & \mathcal{A}_2^2 & \mathcal{A}_2\mathcal{A}_3 \\ & & \mathcal{A}_3^2\end{bmatrix} \,, \tag{25}$$

where in the constant we have absorbed the factor $\frac{1}{4}\delta^{ab}\delta^{cd}\text{Tr}(\tau_a\tau_b\tau_c\tau_d)$. $\mathcal{A}_\mu$, obtained in Eqs. (24), is non-vanishing, nontopological, and depends on the length cutoff ε, so that it is singular when this cutoff is removed.[4] It is this regularization, or framing procedure, that breaks the diffeomorphism-invariant unbroken phase at the level of the "true" ground state of the T^2 observable. In other words, roughly speaking, the square root

of the mean field $\langle T^2\rangle$ is a collective state which plays the role of a classical background dreibein field $\bar{e}^a_\mu(x;\varepsilon)=\delta^a_\mu\mathcal{A}_\mu(x_\varepsilon)$. On the other hand, the above framing procedure and hence the ε dependence are strictly necessary in order to get knot- (link-) invariant quantities [19] starting from the expectation value of the set of RS observables [3] which are the Wilson line operators

$$T^0\equiv\text{Tr}\mathcal{H}(C) \tag{26}$$

with C a knot (link) in $\mathbf{R}^3$.

We should like to conclude with three remarks. The first is that the perturbative nonrenormalizability of 3D QG in the second-order metric formalism comes from the fact that the correlation functions of the operator-valued metric field are actually expectation values of products of composite fields (the T^2 observable). Troubles arise essentially because we use the Feynman rules for the whole composite fields instead of those of the fundamental ones (which are e and ω). This situation perhaps also affects 4D QG. This understanding of perturbative nonrenormalizability was underlined several times by Ashtekar, Rovelli, and Smolin.

Second, we have shown that a classical background Euclidean space-time metric [Eq. (25)] is already induced at the tree-approximation level. Then, following the common wisdom that regards the tree (semiclassical) approxi-

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[4]Notice that ε can be reabsorbed, after a constant Weyl transform, by a renormalization of the Newton coupling constant.

mation as a large-scale (low-momenta) limit, we may understand that $\varepsilon \rightarrow 0$ limit as an infrared one and therefore we may agree with Witten's claim [14] that ". . .this infrared divergence is the birth of macroscopic space-time, starting from microscopic quantum theory."

Finally, we think that the quantum states of the gravitational field constructed starting from the loop observables are of two kinds, "macroscopic" and "microscopic" [20], as happens in the quantum Liouville approach to 2D gravity. The macroscopic states correspond to loop observables such as T^0. The microscopic ones refer instead to operators such as T^1, where the dreibein explicitly appears. The former give the global, topological properties (as generalized knot-link invariants [21]) of gravity, while the latter are associated with the local, metric properties of the gravitational field.

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