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# On submanifolds whose shape operator is unipotent

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## Abstract

The object of this article is to characterize submanifolds  $M \subset \mathbb{R}^n$  of the Euclidean space whose shape operator  $A^\xi$  satisfies the equation  $(A^\xi)^2 = k\|\xi\|^2 Id$ , where  $k > 0$  is constant.

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**Key Words:** unipotent shape operator, constant principal curvatures, normal holonomy group.

## 1 Introduction.

Characterization of submanifolds of the Euclidean space by imposing conditions on its extrinsic or intrinsic invariants is a classical subject of the differential geometry ([7], [2]) e.g. minimal submanifolds, isometric immersions of real space forms, isoparametric submanifolds, etc.

The shape operator  $A^\xi$  of a submanifold  $M \subset \tilde{M}$  is called *unipotent* if it satisfies the equation

$$(A^\xi)^2 = k\|\xi\|^2 Id,$$

where  $\xi$  belongs to the normal bundle  $T^\perp M$  of the submanifold  $M$ . The following theorem was proved in [1, Theorem 14].

**Theorem** *Let  $M^m$  be a Kähler submanifold of the complex space form  $S_c^n = \mathbb{C}P^n, \mathbb{C}^n, \mathbb{C}H^n$  ( $n > m > 1$ ) with  $c = 4, 0, -4$  and  $A^\xi$  its shape operator. Then, the following conditions are equivalent:*

- (i)  $(A^\xi)^2 = f\|\xi\|^2 Id$ , where  $f$  is a positive function on  $M$ ,
- (ii)  $c = 4$  and  $M$  is an open subset of the complex quadric  $Q^m := \{[z_0 : \cdots : z_{m+1}] \in \mathbb{C}P^{m+1} \mid z_0^2 + \cdots + z_{m+1}^2 = 0\}$ . In particular  $f = (\frac{4}{c})^2$  is a constant.

The object of this article is to give a similar characterization for submanifolds of the Euclidean space. We get the following result.

**Theorem 1.1** *Let  $M^m \subset \mathbb{R}^n$  be a submanifold of the Euclidean space and let  $A^\xi$  its shape operator. Then, the following facts are equivalent:*

- (i)  $(A^\xi)^2 = k\|\xi\|^2 Id$ , where  $k > 0$  is constant;
- (ii)  $M^m$  is an open subset of a hypersphere or an open subset of the Veronese embedding of the real, complex, quaternionic or Cayley projective planes in the sphere  $S^{3m+1}$  where  $m = 1, 2, 4, 8$ , i.e. a focal manifold of an isoparametric hypersurface of a sphere with three distinct principal curvatures.

It is interesting to remark that the proof uses techniques related with the normal holonomy group of the given submanifold. Normal holonomy groups were classified by C. Olmos [6] and they are very useful for the study of the geometry of homogeneous submanifolds [2].

## 2 Submanifolds with constant principal curvatures.

A submanifold  $M \subset \mathbb{R}^n$  has *constant principal curvatures* if the shape operator  $A^{\xi(t)}$  has constant eigenvalues for any  $\nabla^\perp$ -parallel normal vector field  $\xi(t)$  along any piecewise differentiable curve. If in addition the normal bundle is flat i.e.  $R^\perp \equiv 0$  then  $M$  is called *isoparametric* [7]. We will need the following result.

**Theorem 2.1** [5] *A submanifold  $M$  of Euclidean space has constant principal curvatures if and only if is either isoparametric or a focal manifold to an isoparametric submanifold.*

For a discussion of focal manifolds of an isoparametric submanifold, its related Coxeter group, etc, see [2, Chapter 5].

### 3 Proof of Theorem 1.1.

It is well-known that (ii) implies (i) (see [3] or [4] for details).

Before dwelling into the proof that (i) implies (ii), we briefly explain the main ideas. In first place we will show that the submanifold  $M$  has principal constant curvatures. Then, from Theorem 2.1 there exists an isoparametric submanifold  $N$  such that  $M = N_\xi$  i.e.  $M$  is a parallel submanifold of  $N$ . Then we will show that  $N$  is a hypersurface of a hypersphere. Since  $N$  is a hypersurface it follows that  $M$  is obtained from  $N$  by focalizing an eigendistribution i.e.  $M = N_{\frac{\xi}{\lambda}}$  for a constant principal curvature  $\lambda$  of  $N$ , see [2, pag.119, Example 4.2].

*Proof of Theorem 1.1.* Since  $A^{\xi(t)}$  satisfies a polynomial equation then the shape operator  $A^{\xi(t)}$  with respect to any  $\nabla^\perp$ -parallel normal section  $\xi(t)$  has constant eigenvalues. So the submanifold has constant principal curvatures.

Then, as a consequence of Theorem 2.1, either  $M$  is isoparametric or is a focal manifold of an isoparametric submanifold. If  $M$  is isoparametric then it is not difficult to see that  $M$  is a piece of an hypersphere.

Assume now that  $M$  is a focal submanifold of an isoparametric submanifold  $N$ . Then from the proof of Theorem 2.1 it follows that  $N$  is a holonomy tube  $(M)_{\xi_p}$  through a principal vector  $\xi_p \in T_p^\perp M$ . We claim that  $N$  is a isoparametric hypersurface of a hypersphere. In fact, from [2, pag.126, Remark 4.4.13] it is enough to show that the codimension of a principal orbit  $Hol_p(\nabla^\perp) \cdot \xi_p$  is one. Indeed, if it is greater than one, we will obtain a contradiction. Note first that the Ricci equation implies  $[A^\xi, A^\eta] = 0$  for any two normal vectors  $\xi, \eta$  to a principal orbit of the action of the normal holonomy group. So, we can perform a simultaneous diagonalization of the family  $\{A^\xi : \xi \in \text{the normal space to a principal orbit}\}$ . Let  $\lambda(\xi) = \langle r, \xi \rangle$  be a common eigenvalue. Thus, if  $\eta \perp r$  is in the normal space of the principal orbit then  $(A^\eta)^2 \neq kId$  and we get a contradiction.

Thus,  $N$  is a isoparametric hypersurface of a hypersphere and the submanifold  $M$  is a focal submanifold of  $N$  i.e.  $M = N_{\frac{\xi}{\lambda}}$  where  $\xi$  is the unit normal of  $N$  and  $\lambda$  a constant principal curvature of  $N$ . Since the shape operator  $A^\xi$  of  $M$  has two eigenspaces for any normal vector  $\xi$  then the ‘‘Tube formula’’ (see [2, pag.121, Lemma 4.4.7]) implies that the shape operator of  $N$  must have three principal curvatures.

So from a theorem of Elie Cartan [3] it follows that  $M$  must be one of the cited embeddings (for a beautiful proof of this Cartan’s Theorem see [4]).  $\square$

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