Polar Actions on Berger Spheres

Polar actions on Berger spheres.

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Dedicated to Dmitri V. Alekseevsky on the occasion of his sixty-fifth birthday.

Abstract

The object of this article is to study a torus action on a so called Berger sphere. We also make some comments on polar actions on naturally reductive homogeneous spaces. Finally, we prove a rigidity-type theorem for Riemannian manifolds carrying a polar action with a fix point.

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1 Introduction.

In a generic Riemannian manifold (M, g) there are neither Killing vector fields (i.e. infinitesimal isometries) nor non trivial totally geodesic submanifolds. The concept of *polar actions* is a good example where both objects come together nicely. A Lie subgroup of isometries $G \subset I(M, g)$ acts polarly on (M, g) if there exists a connected closed submanifold Σ meeting all G-orbits orthogonally. Notice that the sections Σ are totally geodesic submanifolds of (M, g). So, both Killing vector fields and totally geodesic submanifolds fit together in the setting of polar actions. Polar actions have been considered by several authors, see for example [6],[10], [1].

Let G act polarly on (M, g) and let $\tilde{g}$ be another metric on M. Of course, G does not act isometrically on $(M, \tilde{g})$ in general. Even in case that the G-action is still by isometries on $(M, \tilde{g})$ the change of the metric, usually, changes also the second fundamental form of submanifolds of M. Thus, a totally geodesic submanifold of

(M, g) is not in general a totally geodesic submanifold of $(M, \tilde{g})$. Indeed, if the G-action on $(M, \tilde{g})$ is still polar then there are no reasons to think that the *old* section Σ should be still a section after changing the metric.

The first part of this paper is dedicated to give a proof (and some generalizations to naturally reductive spaces) of a theorem stated in [5]. Namely, the S^1-action on a Berger sphere $(S^3, Berger(\varepsilon))$ given by $\theta.(z, w) := (z, e^{i\theta} w)$ is polar if and only if $\varepsilon = 1$ i.e. the Berger sphere is just a standard sphere. Our proof is a straightforward corollary of the fact that there are no totally geodesic surfaces in a three dimensional Berger sphere different from the standard one. Also we will show that there is, up to isometry, just one locally polar action on a Berger sphere $(S^3, Berger(\varepsilon \neq 1))$, namely the diagonal action of the torus T^2.

The proof in [5] is not correct since the authors make the confusion about *old* and *new* section explained above.

It is interesting to note that the above non-polar S^1 action on a fixed Berger sphere $(\varepsilon > 1)$ has fixed points $p \in S^3$. Thus, S^1 acts also in any geodesic sphere around a fixed point p. Since any geodesic sphere has dimension 2, this action is locally polar (actually polar) on any geodesic sphere around p. This contrasts with the well-known fact that in Euclidean spaces (and real space forms) a G-action with a fixed point $p \in \mathbb{R}^n$ is polar if and only if is polar in just one (and then in any) sphere centered at $p \in \mathbb{R}^n$.

Polar actions with fixed points in general homogeneous spaces were studied in [8]. In the last part of this article we give a generalization of Theorem 3 in [8]. Namely, we get

Theorem 1.1 *Let M be a homogeneous Riemannian manifold. Let G be a Lie group of isometries acting polarly on M. If the G-action on M has a fixed point and a section for the G-action is a compact locally symmetric space then, M is locally symmetric.*

Historical Remark. According to É. Cartan it was G. Ricci-Curbastro who first observed the interplay between totally geodesic submanifolds and isometries. Let us quote from Cartan's book this *théoréme remarquable* [4, pp. 122, 107]:

S'il existe dans l'espace de Riemann une famille á un paramétre de plans, leurs trajectories orthogonales etablissent entre les differents plans de la famille une correspondence ponctuelle isométrique.

2 Preliminary results.

Let M be a Riemannian manifold and let $G \subset I(M)$ be a connected Lie subgroup of isometries. The action of G on M is called *polar* if there exists a closed embedded submanifold $\Sigma \subset M$, called a section, such that every G-orbit hits Σ perpendicularly. The G-action is called *locally polar* if the distribution given by the normal spaces to the principal orbits is integrable. A polar action is locally polar but the converse it is not true. For a detailed discussion of both definitions see [7, p. 6 and Appendix A.].

It is known that if G acts polarly then the section Σ is a totally geodesic submanifold of M (see [11] or [1] for a detailed explanation). Indeed, note that if the action is locally polar then the integral leaves of the normal distribution are totally geodesic as a consequence of the Killing equation.

In [12] it is proved that the existence of a totally geodesic hypersurface in an irreducible and simply connected naturally reductive homogeneous space M implies that M has constant sectional curvature. As a consequence we get the following proposition.

Proposition 2.1 *Let M be a simply connected and irreducible naturally reductive homogeneous space. Let X be a Killing vector field and let ϕ^X be the corresponding monoparametric Lie group of isometries. If ϕ^X acts locally polar on M then M has constant sectional curvature.*

Let $S^{2n-1} = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 = 1\}$ be the standard unit sphere in $\mathbb{C}^n = \mathbb{R}^{2n}$.

Let us now change the standard metric (i.e. the Riemannian structure) on the sphere S^{2n-1} just by changing (by a positive constant ε) the length of the Hopf vector field $H(z_1, z_2, \cdots, z_n) = (iz_1, iz_2, \cdots, iz_n)$. This new family of metrics $(S^{2n-1}, Berger(\varepsilon))$ on the sphere, are called Berger spheres.

Theorem 2.2 *[3] The Berger spheres $(S^{2n-1}, Berger(\varepsilon))$ are naturally reductive Riemannian homogeneous spaces. Indeed, they are geodesic spheres (of a convenient radius) in a complex space form.*

We recall also the following proposition.

Proposition 2.3 *A Berger sphere $(S^{2n-1}, Berger(\varepsilon))$ has constant sectional curvature if and only if $\varepsilon = 1$ i.e. the Berger sphere is just the standard one.*

We summarize these results as follows.

Theorem 2.4 *If the flow of a Killing vector field $X \neq 0$ of a Berger sphere $(S^{2n-1}, Berger(\varepsilon))$ acts locally polar then $\varepsilon = 1$.*

3 Polar actions on Berger spheres.

In this section we give an application of the results of the previous section.

Let $T^{n-1} = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n-1 \ times}$ act on $S^{2n-1} \subset \mathbb{C}^n$, endowed with the standard metric, in the following way

$$(e^{i\theta_2}, \ldots, e^{i\theta_n}).(z_1, z_2, \cdots, z_n) := (z_1, e^{i\theta_2} z_2, \cdots, e^{i\theta_n} z_n)\,.$$

In other words, the action is diagonal, it fixes the first coordinate, and is just a rotation of the other coordinates.

Then is easy to check that this action (by isometries) is polar and a section is given by:

$$\Sigma := S^{2n-1} \bigcap \{(z_1, z_2, \cdots, z_n) \in \mathbb{C}^n \ : \mathrm{Im}(z_2) = \mathrm{Im}(z_3) = \cdots = \mathrm{Im}(z_n) = 0\}.$$

Theorem 3.1 *Let $\varepsilon > 0$ be a real number. If the above T^{n-1}-action is locally polar on $(S^{2n-1}, Berger(\varepsilon))$ then $\varepsilon = 1$, i.e. the action is (locally) polar just on the standard sphere.*

Proof. Let $\sigma \in U(n)$ be the diagonal matrix $\sigma := diag(1, 1, \underbrace{-1, \cdots, -1}_{n-2 \ times})$. Then, σ is an isometry of $(S^{2n-1}, Berger(\varepsilon))$. Thus, the set F_σ of fixed points of σ is a totally geodesic submanifold of $(S^{2n-1}, Berger(\varepsilon))$. Indeed, it is not difficult to check that $F_\sigma = S^{2n-1} \bigcap \{(z_1, z_2, 0, 0, \cdots, 0) \in \mathbb{C}^n\}$. Also note that the induced Riemannian metric on F_σ is just a Berger sphere i.e. $F_\sigma = (S^3, Berger(\varepsilon))$. Finally, note that the torus T^{n-1} leaves F_σ invariant and the induced action is exactly the S^1-action on S^3 given by $(z_1, e^{i\theta_2} z_2)$.
If the torus T^{n-1} acts locally polar on $(S^{2n-1}, Berger(\varepsilon))$ then the S^1-action is also locally polar on $F_\sigma = (S^3, Berger(\varepsilon))$. Thus, by Theorem 2.4 it follows that $\varepsilon = 1$ as we claim. $\square$

Up to conjugation, the torus $T^2 = diag(e^{i\theta_1}, e^{i\theta_2})$ is the unique Lie subgroup of isometries of $(S^3, Berger(\varepsilon \neq 1))$ which is not transitive on S^3 and acts with codimension 1. So, we get the following theorem.

Theorem 3.2 *On a Berger sphere $(S^3, Berger(\varepsilon \neq 1))$ there is just one locally polar action, up to isometry. Namely, the action of the torus T^2 given by diagonal multiplication.*

4 Polar actions with fixed points.

In [8] the authors studied G-actions with fixed points and they proved the following theorem.

Theorem 4.1 *[8, Theorem 3] Let M be a compact Riemannian homogeneous space. Let G be a compact Lie group acting on M isometrically and polarly; if the G-action on M has a fixed point and the section for the G-action is flat or a rank one symmetric space then M is a locally symmetric.*

Remark 4.2 *Notice that our Theorem 2.4 for $n = 2$ follows also from the above theorem. Indeed, a section for the S^1-actions is a totally geodesic homogeneous submanifold of dimension 2 (i.e. a symmetric space of rank one or a flat surface). This forces the Berger sphere to be locally symmetric and this occurs just when $\varepsilon = 1$.*

Proof of Theorem 1.1 Let $p \in M$ be a fixed point of the G-action. Then, it is standard to see that any geodesic $\gamma(t)$ passing through p is contained in a section. Since sections are compact symmetric spaces any geodesic is contained in a compact flat. Thus, M is a compact Riemannian homogeneous space all of whose geodesics are contained in a compact flat. So we can apply [9] to get that the universal covering $\widetilde{M}$ splits as $\widetilde{M} = \mathbb{R}^n \times C_1 \times \cdots \times C_k \times S$, where S is a symmetric space and the manifolds C_l are manifolds all of whose geodesics are closed. Since $\widetilde{M}$ is homogeneous any factor C_l is homogeneous too. Then, from [2, pp. 194, 7.47] we get that the manifolds C_l are rank one homogeneous symmetric spaces. □

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