

Polar Actions on Berger Spheres

*Original*

Polar Actions on Berger Spheres / DI SCALA, ANTONIO JOSE'. - In: INTERNATIONAL JOURNAL OF GEOMETRIC METHODS IN MODERN PHYSICS. - ISSN 0219-8878. - STAMPA. - 3:(2006), pp. 1019-1023.  
[10.1142/S0219887806001466]

*Availability:*

This version is available at: 11583/1399749 since: 2015-12-01T11:20:00Z

*Publisher:*

World Scientific Publishing

*Published*

DOI:10.1142/S0219887806001466

*Terms of use:*

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

*Publisher copyright*

(Article begins on next page)

# Polar actions on Berger spheres.

Antonio J. Di Scala

Post print (i.e. final draft post-refereeing) version of an article published on *Int. J. Geom. Methods Mod. Phys.* 3 (2006), no. 5-6, 1019-1023., DOI: 10.1142/S0219887806001466 .

Beyond the journal formatting, please note that there could be minor changes from this document to the final published version. The final published version is accessible from here:

<http://www.worldscientific.com/doi/abs/10.1142/S0219887806001466>

This document has made accessible through PORTO, the Open Access Repository of Politecnico di Torino (<http://porto.polito.it>), in compliance with the Publisher's copyright policy as reported in the SHERPA-ROMEO website: <http://www.sherpa.ac.uk/romeo/issn/0219-8878/>

*Dedicated to Dmitri V. Alekseevsky on the occasion of his sixty-fifth birthday.*

## Abstract

The object of this article is to study a torus action on a so called Berger sphere. We also make some comments on polar actions on naturally reductive homogeneous spaces. Finally, we prove a rigidity-type theorem for Riemannian manifolds carrying a polar action with a fix point.

**Mathematics Subject Classification(2000):** 53C40, 53C35.

**Keywords:** polar actions, Killing vector fields, totally geodesic section, Berger spheres.

## 1 Introduction.

In a generic Riemannian manifold  $(M, g)$  there are neither Killing vector fields (i.e. infinitesimal isometries) nor non trivial totally geodesic submanifolds. The concept of *polar actions* is a good example where both objects come together nicely. A Lie subgroup of isometries  $G \subset I(M, g)$  acts polarly on  $(M, g)$  if there exists a connected closed submanifold  $\Sigma$  meeting all  $G$ -orbits orthogonally. Notice that the sections  $\Sigma$  are totally geodesic submanifolds of  $(M, g)$ . So, both Killing vector fields and totally geodesic submanifolds fit together in the setting of polar actions. Polar actions have been considered by several authors, see for example [6],[10], [1].

Let  $G$  act polarly on  $(M, g)$  and let  $\tilde{g}$  be another metric on  $M$ . Of course,  $G$  does not act isometrically on  $(M, \tilde{g})$  in general. Even in case that the  $G$ -action is still by isometries on  $(M, \tilde{g})$  the change of the metric, usually, changes also the second fundamental form of submanifolds of  $M$ . Thus, a totally geodesic submanifold of

$(M, g)$  is not in general a totally geodesic submanifold of  $(M, \tilde{g})$ . Indeed, if the  $G$ -action on  $(M, \tilde{g})$  is still polar then there are no reasons to think that the *old* section  $\Sigma$  should be still a section after changing the metric.

The first part of this paper is dedicated to give a proof (and some generalizations to naturally reductive spaces) of a theorem stated in [5]. Namely, the  $S^1$ -action on a Berger sphere  $(S^3, \text{Berger}(\varepsilon))$  given by  $\theta.(z, w) := (z, e^{i\theta}w)$  is polar if and only if  $\varepsilon = 1$  i.e. the Berger sphere is just a standard sphere. Our proof is a straightforward corollary of the fact that there are no totally geodesic surfaces in a three dimensional Berger sphere different from the standard one. Also we will show that there is, up to isometry, just one locally polar action on a Berger sphere  $(S^3, \text{Berger}(\varepsilon \neq 1))$ , namely the diagonal action of the torus  $T^2$ .

The proof in [5] is not correct since the authors make the confusion about *old* and *new* section explained above.

It is interesting to note that the above non-polar  $S^1$  action on a fixed Berger sphere ( $\varepsilon > 1$ ) has fixed points  $p \in S^3$ . Thus,  $S^1$  acts also in any geodesic sphere around a fixed point  $p$ . Since any geodesic sphere has dimension 2, this action is locally polar (actually polar) on any geodesic sphere around  $p$ . This contrasts with the well-known fact that in Euclidean spaces (and real space forms) a  $G$ -action with a fixed point  $p \in \mathbb{R}^n$  is polar if and only if is polar in just one (and then in any) sphere centered at  $p \in \mathbb{R}^n$ .

Polar actions with fixed points in general homogeneous spaces were studied in [8]. In the last part of this article we give a generalization of Theorem 3 in [8]. Namely, we get

**Theorem 1.1** *Let  $M$  be a homogeneous Riemannian manifold. Let  $G$  be a Lie group of isometries acting polarly on  $M$ . If the  $G$ -action on  $M$  has a fixed point and a section for the  $G$ -action is a compact locally symmetric space then,  $M$  is locally symmetric.*

**Historical Remark.** According to É. Cartan it was G. Ricci-Curbastro who first observed the interplay between totally geodesic submanifolds and isometries. Let us quote from Cartan's book this *théorème remarquable* [4, pp. 122, 107]:

*S'il existe dans l'espace de Riemann une famille à un paramètre de plans, leurs trajectoires orthogonales établissent entre les différents plans de la famille une correspondance ponctuelle isométrique.*

## 2 Preliminary results.

Let  $M$  be a Riemannian manifold and let  $G \subset I(M)$  be a connected Lie subgroup of isometries. The action of  $G$  on  $M$  is called *polar* if there exists a closed embedded submanifold  $\Sigma \subset M$ , called a section, such that every  $G$ -orbit hits  $\Sigma$  perpendicularly. The  $G$ -action is called *locally polar* if the distribution given by the normal spaces to the principal orbits is integrable. A polar action is locally polar but the converse it is not true. For a detailed discussion of both definitions see [7, p. 6 and Appendix A.].

It is known that if  $G$  acts polarly then the section  $\Sigma$  is a totally geodesic submanifold of  $M$  (see [11] or [1] for a detailed explanation). Indeed, note that if the action is locally polar then the integral leaves of the normal distribution are totally geodesic as a consequence of the Killing equation.

In [12] it is proved that the existence of a totally geodesic hypersurface in an irreducible and simply connected naturally reductive homogeneous space  $M$  implies that  $M$  has constant sectional curvature. As a consequence we get the following proposition.

**Proposition 2.1** *Let  $M$  be a simply connected and irreducible naturally reductive homogeneous space. Let  $X$  be a Killing vector field and let  $\phi^X$  be the corresponding monoparametric Lie group of isometries. If  $\phi^X$  acts locally polar on  $M$  then  $M$  has constant sectional curvature.*

Let  $S^{2n-1} = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_1|^2 + \dots + |z_n|^2 = 1\}$  be the standard unit sphere in  $\mathbb{C}^n = \mathbb{R}^{2n}$ .

Let us now change the standard metric (i.e. the Riemannian structure) on the sphere  $S^{2n-1}$  just by changing (by a positive constant  $\varepsilon$ ) the length of the Hopf vector field  $H(z_1, z_2, \dots, z_n) = (iz_1, iz_2, \dots, iz_n)$ . This new family of metrics  $(S^{2n-1}, \text{Berger}(\varepsilon))$  on the sphere, are called Berger spheres.

**Theorem 2.2** [3] *The Berger spheres  $(S^{2n-1}, \text{Berger}(\varepsilon))$  are naturally reductive Riemannian homogeneous spaces. Indeed, they are geodesic spheres (of a convenient radius) in a complex space form.*

We recall also the following proposition.

**Proposition 2.3** *A Berger sphere  $(S^{2n-1}, \text{Berger}(\varepsilon))$  has constant sectional curvature if and only if  $\varepsilon = 1$  i.e. the Berger sphere is just the standard one.*

We summarize these results as follows.

**Theorem 2.4** *If the flow of a Killing vector field  $X \neq 0$  of a Berger sphere  $(S^{2n-1}, \text{Berger}(\varepsilon))$  acts locally polar then  $\varepsilon = 1$ .*

### 3 Polar actions on Berger spheres.

In this section we give an application of the results of the previous section.

Let  $T^{n-1} = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n-1 \text{ times}}$  act on  $S^{2n-1} \subset \mathbb{C}^n$ , endowed with the standard metric, in the following way

$$(e^{i\theta_2}, \dots, e^{i\theta_n}).(z_1, z_2, \dots, z_n) := (z_1, e^{i\theta_2} z_2, \dots, e^{i\theta_n} z_n).$$

In other words, the action is diagonal, it fixes the first coordinate, and is just a rotation of the other coordinates.

Then is easy to check that this action (by isometries) is polar and a section is given by:

$$\Sigma := S^{2n-1} \cap \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n : \operatorname{Im}(z_2) = \operatorname{Im}(z_3) = \cdots = \operatorname{Im}(z_n) = 0\}.$$

**Theorem 3.1** *Let  $\varepsilon > 0$  be a real number. If the above  $T^{n-1}$ -action is locally polar on  $(S^{2n-1}, \operatorname{Berger}(\varepsilon))$  then  $\varepsilon = 1$ , i.e. the action is (locally) polar just on the standard sphere.*

*Proof.* Let  $\sigma \in U(n)$  be the diagonal matrix  $\sigma := \operatorname{diag}(1, 1, \underbrace{-1, \dots, -1}_{n-2 \text{ times}})$ . Then,

$\sigma$  is an isometry of  $(S^{2n-1}, \operatorname{Berger}(\varepsilon))$ . Thus, the set  $F_\sigma$  of fixed points of  $\sigma$  is a totally geodesic submanifold of  $(S^{2n-1}, \operatorname{Berger}(\varepsilon))$ . Indeed, it is not difficult to check that  $F_\sigma = S^{2n-1} \cap \{(z_1, z_2, 0, 0, \dots, 0) \in \mathbb{C}^n\}$ . Also note that the induced Riemannian metric on  $F_\sigma$  is just a Berger sphere i.e.  $F_\sigma = (S^3, \operatorname{Berger}(\varepsilon))$ . Finally, note that the torus  $T^{n-1}$  leaves  $F_\sigma$  invariant and the induced action is exactly the  $S^1$ -action on  $S^3$  given by  $(z_1, e^{i\theta_2} z_2)$ .

If the torus  $T^{n-1}$  acts locally polar on  $(S^{2n-1}, \operatorname{Berger}(\varepsilon))$  then the  $S^1$ -action is also locally polar on  $F_\sigma = (S^3, \operatorname{Berger}(\varepsilon))$ . Thus, by Theorem 2.4 it follows that  $\varepsilon = 1$  as we claim.  $\square$

Up to conjugation, the torus  $T^2 = \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2})$  is the unique Lie subgroup of isometries of  $(S^3, \operatorname{Berger}(\varepsilon \neq 1))$  which is not transitive on  $S^3$  and acts with codimension 1. So, we get the following theorem.

**Theorem 3.2** *On a Berger sphere  $(S^3, \operatorname{Berger}(\varepsilon \neq 1))$  there is just one locally polar action, up to isometry. Namely, the action of the torus  $T^2$  given by diagonal multiplication.*

## 4 Polar actions with fixed points.

In [8] the authors studied  $G$ -actions with fixed points and they proved the following theorem.

**Theorem 4.1** [8, Theorem 3] *Let  $M$  be a compact Riemannian homogeneous space. Let  $G$  be a compact Lie group acting on  $M$  isometrically and polarly; if the  $G$ -action on  $M$  has a fixed point and the section for the  $G$ -action is flat or a rank one symmetric space then  $M$  is a locally symmetric.*

**Remark 4.2** *Notice that our Theorem 2.4 for  $n = 2$  follows also from the above theorem. Indeed, a section for the  $S^1$ -actions is a totally geodesic homogeneous submanifold of dimension 2 (i.e. a symmetric space of rank one or a flat surface). This forces the Berger sphere to be locally symmetric and this occurs just when  $\varepsilon = 1$ .*

*Proof of Theorem 1.1* Let  $p \in M$  be a fixed point of the  $G$ -action. Then, it is standard to see that any geodesic  $\gamma(t)$  passing through  $p$  is contained in a section. Since sections are compact symmetric spaces any geodesic is contained in a compact flat. Thus,  $M$  is a compact Riemannian homogeneous space all of whose geodesics are contained in a compact flat. So we can apply [9] to get that the universal covering  $\widetilde{M}$  splits as  $\widetilde{M} = \mathbb{R}^n \times C_1 \times \cdots \times C_k \times S$ , where  $S$  is a symmetric space and the manifolds  $C_l$  are manifolds all of whose geodesics are closed. Since  $\widetilde{M}$  is homogeneous any factor  $C_l$  is homogeneous too. Then, from [2, pp. 194, 7.47] we get that the manifolds  $C_l$  are rank one homogeneous symmetric spaces.  $\square$

## Acknowledgements

The author would like to thank S. Console and C. Olmos for the useful comments and suggestions.

## References

- [1] BERNDT, J.; S. CONSOLE AND C. OLMOS.: *Submanifolds and holonomy*, Chapman & Hall/CRC, Research Notes in Mathematics 434 (2003).
- [2] BESSE, A.L.: *Manifolds all of whose geodesics are closed*, Ergebnisse der Mathematik und ihrer Grenzgebiete 93. Springer-Verlag, Berlin-New York, (1978).
- [3] BERNDT, J. AND VANHECKE, L.: *Naturally reductive Riemannian homogeneous spaces and real hypersurfaces in complex and quaternionic space forms*, Differential Geometry and Its Applications, Proc. Conf. Opava (Czechoslovakia), August 24-28, (1992), Silesian University, Opava, 353-364, (1993).

- [4] CARTAN, E.: *Lecons sur la Géométrie des Espaces de Riemann*, Gauthier-Villars (1928).
- [5] CHRISTOPHORIDOU, CH. AND KOBOTIS, A.: *Polar actions on the homogeneous space  $S^{2n-1}$* , Mem. Sec t. Stiin t. Acad. Romnă Ser. IV 23 (2000), 51–56 (2002).
- [6] CONLON, L.: *Variational completeness and  $K$ -transversal domains*, J. Diff. Geometry 5, 135–147 (1971).
- [7] HEINTZE, E.; LIU X. AND OLMOS, C.: *Isoparametric submanifolds and a Chevalley-type restriction theorem*, math.DG/0004028 (2000).
- [8] KOLLROSS, A. AND PODESTÁ, F.: *Homogeneous spaces with polar isotropy*, Manuscripta Math. 110 , no. 4, 487–503 (2003).
- [9] MOLINA, B. AND OLMOS, C.: *Decomposition of spaces with geodesics contained in compact flats*, Proc. Amer. Math. Soc. 129, no. 12, 3701–3709 (2001).
- [10] PALAIS, R.S. AND TERNG, C-L.: *A general theory of canonical forms*, Trans. Amer. Math. Soc. 300, 771–789 (1987).
- [11] PALAIS, R.S. AND TERNG, C-L.: *Critical point theory and submanifold geometry*, Lecture Notes in Mathematics, 1353. Springer-Verlag, Berlin, (1988).
- [12] TSUKADA, K.: *Totally geodesic hypersurfaces of naturally reductive homogeneous spaces*, Osaka J. Math. 33, no. 3, 697–707 (1996).

Dipartimento di Matematica,  
 Politecnico di Torino,  
 Corso Duca degli Abruzzi 24, 10129 Torino, Italy  
 antonio.discal@polito.it