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MINIMAL IMMERSIONS OF KÄHLER MANIFOLDS INTO EUCLIDEAN SPACES

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Abstract

We prove that a minimal isometric immersion of a Kähler-Einstein or homogeneous Kähler manifold into an Euclidean space must be totally geodesic. As an application we show that an open subset of the real hyperbolic plane $\mathbb{R}H^2$ cannot be minimally immersed into the Euclidean space. As another application we prove that if an irreducible Kähler manifold is minimally immersed in an Euclidean space then its restricted holonomy group must be $U(n)$, where $n = dim_{\mathbb{C}}M$.

1. *Introduction*

The aim of this article is to prove the following results.

THEOREM 1. *Let M be a Kähler manifold (connected, non necessarily complete) that is either homogeneous or Einstein. Then, any minimal isometric immersion $f : M \to \mathbb{R}^n$ is totally geodesic, i.e. $f(M)$ is an open subset of an affine subspace of $\mathbb{R}^n$.*

Recall that a Riemannian manifold with parallel Ricci curvature tensor is locally a product of Einstein manifolds. Since minimal immersions of Riemannian products into a Euclidean space are product immersions [**9**] we obtain the following corollary.

THEOREM 2. *Let M be a Kähler manifold that is a product $M = H \times E$ of an homogeneous Kähler manifold H and a Kähler manifold E with parallel Ricci tensor. Then, any minimal isometric immersion $f : M \to \mathbb{R}^n$ is totally geodesic.*

Since any orientable surface is naturally a Kähler manifold we obtain the following well-known corollary [**2**, Proposition 4.1], [**10**, Chapter IV].

COROLLARY 1. *Let $U \subset \mathbb{R}H^2$ be an open submanifold of the real hyperbolic plane $\mathbb{R}H^2$. Then, there is no minimal isometric immersion $f : U \to \mathbb{R}^n$.*

Finally, we obtain the following corollary which is a generalization of the result in [**12**] concerning the restricted holonomy group of hypersurfaces in complex Euclidean spaces.

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Corollary 2. *Let M^n be a locally irreducible Kähler manifold of complex dimension n and assume that there exists a minimal isometric immersion $f : M \to \mathbb{R}^{2n+m}$. Then, the restricted holonomy group of M is $U(n)$. In particular, the restricted holonomy group of an irreducible complex submanifold of $\mathbb{C}^{n+m}$ is $U(n)$.*

It interesting to note that a strong version of the last statement of the above corollary also holds for Kähler manifolds M^n isometrically and holomorphically immersed in the complex hyperbolic space $\mathbb{C}H^{n+p}$. Namely, a Kähler manifold M^n isometrically and holomorphically immersed in $\mathbb{C}H^{n+p}$ is locally irreducible and its holonomy group $Hol(\nabla)$ is $U(n)$ (see [**1**]).

2. *Proof of the main theorem and further remarks*

To prove our main theorem we note that Theorem 1.2 from [**6**] and Theorem 1.11 from [**5**] implies the following theorem.

Theorem 3. *[**6**] [**5**] Let M be a simple connected Kähler manifold (not necessary complete) and let $f : M \to \mathbb{R}^n$ be a minimal isometric immersion. Then there exist a minimal isometric immersion $g : M \to \mathbb{R}^n$ such that $\overline{f} : M \to \mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$ is given by $\overline{f}(p) = (\frac{f(p)}{\sqrt{2}}, \frac{g(p)}{\sqrt{2}})$ is isometric and holomorphic with respect to the complex structure $J(u, v) = (-v, u)$ on $\mathbb{R}^n \times \mathbb{R}^n$.*

Recall the following remarkable result by M. Umehara.

Theorem 4. *[**13**] Every Kähler-Einstein submanifold of a complex linear or hyperbolic space is always totally geodesic.*

Notice that the above theorem is of local nature. The proof is by using the so called "diastatic function" introduced by E. Calabi in [**4**].

We need the following result.

Theorem 5. *[**7**] Let M be an homogeneous Kähler manifold and $f : M \to \mathbb{C}^n$ an holomorphic and isometric immersion. Then f is totally geodesic.*

Now we can prove our main theorem.

Proof of Theorem 1. Let $\tilde{M}$ be the universal covering of M and $\tilde{f} = f \circ \pi$, where $\pi : \tilde{M} \to M$ is the canonical projection. Let $\overline{\tilde{f}} : \tilde{M} \to \mathbb{C}^n$ be the isometric and holomorphic immersion given by Theorem 3. Notice that f is totally geodesic if and only if $\overline{\tilde{f}}$ is totally geodesic.

Assume now that M is Kähler-Einstein. Then $\tilde{M}$ is Kähler-Einstein. If $p \in \tilde{M}$ is any point of $\tilde{M}$ then we can restrict $\overline{\tilde{f}}$ to an small neighborhood U of $p \in \tilde{M}$ such that $\overline{\tilde{f}}|_U : U \to \mathbb{C}^n$ is an imbedding. Then Theorem 4 implies that $\overline{\tilde{f}}|_U$ is totally geodesic. So $\overline{\tilde{f}}$ is totally geodesic.

Assume now that M is homogeneous. Then $\tilde{M}$ is homogeneous. Then by Theorem

5 it follows that $\bar{\tilde{f}}$ is totally geodesic. This completes the proof of our main theorem.

□

Proof of Corollary 2. Observe that f can not be totally geodesic because we assume that M is locally irreducible. Theorem 2 shows that the Ricci tensor of M can not be parallel. Let Φ^* be the restricted holonomy group of M. The classification of irreducible restricted holonomy groups by M. Berger shows that either Φ^* is $U(n), SU(n), Sp(n)$ or M is a locally symmetric space [**3**]. If M is a locally symmetric space we get that M has parallel Ricci tensor. If Φ^* is one of the groups $SU(n), Sp(n)$ we get that M is Ricci-flat. Thus $\Phi^* = U(n)$, as we have to prove.

□

Let M be a Riemannian manifold and let $f : M \to \mathbb{R}^n$ be a minimal isometric immersion. It is interesting to note that f must be totally geodesic if one of the following conditions holds.

(1) M is complete, the Ricci tensor of M has constant eigenvalues and f has flat normal bundle (see [**8**]),
(2) M is Ricci-flat (as follows from Gauss equation),
(3) f has codimension 2 and M is Einstein (see [**11**]).

So, it seems natural to pose the following conjecture.

Conjecture. Let M be a Riemannian manifold that is either locally homogeneous or Einstein. Then, any minimal isometric immersion $f : M \to \mathbb{R}^n$ must be totally geodesic.

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