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ON THE ASYMPTOTIC FORMULA FOR GOLDBACH NUMBERS IN SHORT INTERVALS

by

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1. INTRODUCTION

Define a Goldbach number (G-number) to be an even number which can be written as a sum of two primes. In the following we denote by \( N \) a sufficiently large integer and let \( L = \log N \). Let further

\[
R(k) = \sum_{N < m \leq 2N} \sum_{N < l \leq 2N} \Lambda(l)\Lambda(m)
\]

be the weighted counting function of G-numbers,

\[
\mathcal{S}(k) = \begin{cases} 
2 \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p \mid k} \left(\frac{p-1}{p-2}\right) & \text{if } k \text{ is even} \\
0 & \text{if } k \text{ is odd}
\end{cases}
\]

be the singular series of Goldbach’s problem and

\[
m(k) = \sum_{N < m \leq 2N} \sum_{N < l \leq 2N} 1.
\]

We recall that a well-known conjecture states that as \( k \to \infty \)

\[
R(k) \sim m(k)\mathcal{S}(k).
\]

In this paper we study the asymptotic formula for the average of \( R(k) \) over short intervals of type \([n, n + H]\). In the extreme case \( H = 1 \), Chudakov [1], van der Corput [2] and Estermann [4] proved that, as \( N \to \infty \), (1) holds for all \( k \in [1, N] \) but

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\[ O(NL^{-A}) \text{ exceptions, for every } A > 0. \] Moreover, the same techniques prove, for \( H \leq L^D \) and \( N \to \infty \), that
\[
\sum_{k \in [n,n+H]} R(k) \sim \sum_{k \in [n,n+H]} m(k) \mathcal{S}(k)
\]
holds for all \( n \in \left( \frac{5}{2}N, \frac{7}{2}N \right] \) but \( O(NL^{-A}) \) exceptions, for every \( A, D > 0 \).

We recall that Montgomery-Vaughan [12] improved Chudakov-van der Corput-Estermann’s result proving that there exists a (small) constant \( \delta > 0 \) such that \( |E(N)| \ll N^{1-\delta} \), where \( E(N) = E \cap [1, N] \) and \( E \) is the exceptional set for Goldbach’s problem. Montgomery-Vaughan’s technique intrinsically does not give any information about the asymptotic formula for \( R(k) \).

On the other hand, using the circle method and Ingham-Huxley’s zero density estimate, Perelli [14] proved that (2) holds as \( n \to \infty \) uniformly for \( H \geq n^{1/6+\varepsilon} \).

Our aim here is to show, using the circle method, that the asymptotic formula (2) holds for almost all \( n \in (\frac{5}{2}N, \frac{7}{2}N] \), uniformly for \( L^D \leq H \leq N^{1/6+\varepsilon} \), for all \( D > 0 \).

Our result is

**Theorem.** Let \( D, \varepsilon > 0 \) be arbitrary constants and \( L^D \leq H \leq N^{1/6+\varepsilon} \). Then, as \( N \to \infty \), (2) holds for all \( n \in (\frac{5}{2}N, \frac{7}{2}N] \) but \( O(NL^{42+\varepsilon}H^{-2}) \) exceptions.

In fact, following the proof of the Theorem, it is easy to see that we have \( O(NL^{f(\theta)}H^{-2}) \) exceptions, where
\[
H = N^\theta \quad \text{and} \quad f(\theta) = \frac{24 - 18\theta}{1 - 3\theta} + \varepsilon.
\]

A direct computation shows that \( f(\theta) \) is an increasing function and hence the exponent 42 in the log-factor of the Theorem follows taking \( \theta = 1/6 + \varepsilon \).

We observe that our result, for \( \theta = 1/6 + \varepsilon \), proves only that the number of exceptions for (2) is \( O(N^{2/3-\varepsilon}) \) while, from Perelli’s [14] result, we know that there are no exceptions.

We recall that Mikawa, see Lemma 4 of [10], proved a slightly weaker, in the log-factor, result without using the circle method. We finally recall that, under the assumption of the Riemann Hypothesis (RH), (2) holds uniformly for \( H \geq \infty (\log^2 n) \), where \( f = \infty (g) \) means \( g = o(f) \), and that, assuming further the Montgomery pair correlation conjecture, (2) holds uniformly for \( H \geq \infty (\log n) \).

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2. **Outline of the method**

Let
\[
Q = \frac{H}{L^\varepsilon}, \quad T = \frac{N}{Q}L^{2+\varepsilon} \quad \text{and} \quad K_H(n) = \sum_{k \in [n,n+H]} e(-k\alpha),
\]
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where \( e(x) = \exp(2\pi i x) \). Let further \( \beta + i\gamma \) denote the generic non-trivial zero of \( \zeta(s) \),

\[
S(\alpha) = \sum_{N < m \leq 2N} \Lambda(m) e(m\alpha), \quad T(\alpha) = \sum_{N < m \leq 2N} e(m\alpha),
\]

\[
T_\rho(\alpha) = \sum_{N < m \leq 2N} a_\rho(m) e(m\alpha), \quad a_\rho(m) = \int_m^{m+1} t^{\rho-1} dt.
\]

Given an interval \( I = [a, b] \subset [1/2, 1] \) we define

\[
\Sigma_b(\alpha) = \sum_{|\gamma| \leq T \notin I} T_\rho(\alpha), \quad \Sigma_g(\alpha) = \sum_{|\gamma| \leq T} T_\rho(\alpha) + \sum_{|\gamma| > T} T_\rho(\alpha) + R(\alpha)
\]

where \( R(\alpha) \) is defined by difference in the approximation

\[
S(\alpha) = T(\alpha) - \Sigma_g(\alpha) - \Sigma_b(\alpha).
\]

Subdivide now \((-1/2, 1/2)\) into \( O(\log Q) \) subintervals of the following form

\[
A_0 = \left(-\frac{1}{Q}, \frac{1}{Q}\right), \quad A_j = \left(-\frac{1}{2j}, -\frac{1}{2j+1}\right] \cup \left[\frac{1}{2j+1}, \frac{1}{2j}\right)
\]

for \( j \in [1, K] \), where \( K = \lceil \log Q / \log 2 \rceil \). Hence we have

\[
\sum_{k \in [n, n+H]} R(k) = \int_{-1/2}^{1/2} S(\alpha)^2 K_H(\alpha) d\alpha = \int_{-1/Q}^{1/Q} S(\alpha)^2 K_H(\alpha) d\alpha
\]

\[
+ \sum_{j=1}^{K} \int_{A_j} S(\alpha)^2 K_H(\alpha) d\alpha = \Sigma_1 + \Sigma_2,
\]

say. We will prove that

\[
\Sigma_1 = \sum_{k \in [n, n+H]} m(k) \mathcal{G}(k) + \int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha) d\alpha + o(HN),
\]

\[
\sum_{\frac{1}{2} N < n \leq \frac{3}{4} N} \left| \int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha) d\alpha \right|^2 \ll N^3 L'(\theta),
\]

and

\[
\Sigma_2 = o(HN).
\]

We will need also that

\[
\sum_{k \in [n, n+H]} m(k) \mathcal{G}(k) \gg HN
\]

which can be obtained immediately using \( \mathcal{G}(2k) \gg 1 \). Since \( \varepsilon > 0 \) is arbitrarily small, our Theorem follows at once from (4)-(8).
3. Preliminary Lemmas

In the following we will need two auxiliary lemmas.

**Lemma 1.** Let \( N(\sigma, T) \) be the number of zeros \( \rho = \beta + i\gamma \) of the Riemann zeta-function such that \( |\gamma| \leq T \) and \( \beta \geq \sigma \), and let \( I \subset [1/2, 1] \) be an interval. Then

\[
\int_N^{2N} \left| \sum_{|\rho| \leq T, \beta \in I} \frac{x^\rho (1 + Q/x)^\rho - 1}{\rho} \right|^2 dx \ll Q^2 L^4 \max_{\sigma \in I} N^{2\sigma - 1} N(\sigma, \frac{N}{Q}).
\]

The proof of Lemma 1 is standard. It can be obtained using, e.g., Saffari-Vaughan’s [15] technique and hence we omit it.

**Lemma 2.** We have, for \( |\gamma| \ll N \) and \( N \) sufficiently large, that

\[ T_\rho(\alpha) \ll N^\beta |\gamma|^{-1/2}. \]

**Proof.** We follow the line of Perelli [13] and hence we give only a brief sketch of the proof. Since

\[
a_\rho(m) = \int_m^{m+1} t^{\rho-1} dt = \frac{m^\rho}{\rho} ((1 + \frac{1}{m})^\rho - 1),
\]

and, for \( P \) sufficiently large but fixed,

\[
(1 + \frac{1}{m})^\rho - 1 = \sum_{j=1}^P \frac{\rho(\rho - 1) \cdots (\rho - j + 1)}{j!} \left( \frac{1}{m} \right)^j + O(N^{-11}),
\]

we can write

\[ T_\rho(\alpha) = T_{\rho,1}(\alpha) + \sum_{j=2}^P \frac{(\rho - 1)(\rho - 2) \cdots (\rho - j + 1)}{j!} T_{\rho,j}(\alpha) + O(N^{\beta - 10}), \] (9)

where

\[ T_{\rho,j}(\alpha) = \sum_{N < m \leq 2N} m^{\rho-j} e(m\alpha). \]

From Abel’s inequality we have

\[
|T_{\rho,j}(\alpha)| \ll N^{\beta - j} \max_{N \leq y \leq 2N} \left| \sum_{N \leq m \leq y} e^{2\pi i f_\rho(m) \alpha} \right|,
\]

where \( f_\rho(\alpha) = \frac{1}{2\pi} \log n + \alpha n \). We can assume that the maximum is attained at \( Y = 2N \), and so, using van der Corput’s second derivative method, see Theorem 2.2 of Graham-Kolesnik [5], we get

\[ T_{\rho,j}(\alpha) \ll N^{\beta - j + 1} |\gamma|^{-1/2}. \] (10)

Lemma 2 now follows inserting (10) in (9).
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4. Estimation of $\Sigma_2$

Letting $S(\alpha) = T(\alpha) + R_1(\alpha)$, where $R_1(\alpha)$ is defined by difference, and using

$$K_H(\alpha) \ll \min(H, \frac{1}{|\alpha|}) \quad \text{for every} \quad \alpha \in [-\frac{1}{2}, \frac{1}{2}],$$

we have

$$\Sigma_2 \ll \sum_{j=1}^{K} \left( \int_{A_j} |T(\alpha)|^2 |K_H(\alpha)| d\alpha + \int_{A_j} |R_1(\alpha)|^2 |K_H(\alpha)| d\alpha \right) \quad (11)$$

$$\ll \sum_{j=1}^{K} 2^j \left( \int_{A_j} |T(\alpha)|^2 d\alpha + \int_{A_j} |R_1(\alpha)|^2 d\alpha \right) = \Sigma_{2,1} + \Sigma_{2,2},$$

say. Using

$$T(\alpha) \ll \min(N, \frac{1}{|\alpha|}) \quad \text{for every} \quad \alpha \in [-\frac{1}{2}, \frac{1}{2}],$$

we obtain

$$\Sigma_{2,1} \ll \sum_{j=1}^{K} 4^j \ll 4^K \ll Q^2 = o(HN). \quad (12)$$

By Gallagher’s lemma, see, e.g., Lemma 1.9 of Montgomery [11], and the Brun-Titchmarsh theorem we get

$$\Sigma_{2,2} \ll \sum_{j=1}^{K} 2^j \int_{-2^{-j}}^{2^{-j}} \left( \sum_{N < m \leq 2N} (\Lambda(m) - 1)e(m\alpha) \right)^2 d\alpha \ll \sum_{j=1}^{K} 2^{-j} (J(N, 2^j) + L^2 2^{3j}), \quad (13)$$

where $J(N, h)$ is the Selberg integral. Inserting the estimate $J(N, h) \ll h^2 N + hNL$ for all $h \geq 1$, see the Lemma in Languasco [7], in (15) we have

$$\Sigma_{2,2} \ll \sum_{j=1}^{K} 2^{-j} (2^{3j} L^2 + 2^{2j} N + 2^{j} NL) \ll L^2 Q^2 + NQ + NL \log Q = o(HN). \quad (14)$$

Hence, inserting (14) and (16) in (12), we finally have that (7) holds.

5. Estimation of $\Sigma_1$

Inserting the identity

$$S(\alpha)^2 = (2S(\alpha)T(\alpha) - T(\alpha)^2) - \Sigma_g(\alpha)^2 - 2T(\alpha)\Sigma_g(\alpha) + 2S(\alpha)\Sigma_g(\alpha) + \Sigma_b(\alpha)^2$$

into the definition of $\Sigma_1$, we obtain

$$\Sigma_1 = \Sigma_{1,1} - \Sigma_{1,2} - \Sigma_{1,3} + \Sigma_{1,4} + \int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha) d\alpha, \quad (17)$$

where

$$\Sigma_{1,1} = \int_{-1/Q}^{1/Q} (2S(\alpha)T(\alpha) - T(\alpha)^2) K_H(\alpha) d\alpha,$$
\[ \Sigma_{1,2} = \int_{-1/Q}^{1/Q} \sigma_0(\alpha)^2 K_H(\alpha) d\alpha, \]

\[ \Sigma_{1,3} = \int_{-1/Q}^{1/Q} 2T(\alpha) \sigma_0(\alpha) K_H(\alpha) d\alpha \quad \text{and} \]

\[ \Sigma_{1,4} = \int_{-1/Q}^{1/Q} 2S(\alpha) \sigma_0(\alpha) K_H(\alpha) d\alpha. \]

In this section we will prove

\[ \Sigma_{1,1} = \sum_{k \in [n, n+H]} m(k) \mathcal{G}(k) + o(HN) \quad (18) \]

and

\[ \Sigma_{1,2} = o(HN), \quad (19) \]

while the estimation of the mean-square of \( \int_{-1/Q}^{1/Q} \sigma_0(\alpha)^2 K_H(\alpha) d\alpha \) will be performed in the next section.

Assuming that (19) holds, the contribution of \( \Sigma_{1,3} \) and \( \Sigma_{1,4} \) can be estimated using the Cauchy-Schwarz inequality and

\[ \int_{-1/Q}^{1/Q} |S(\alpha)|^2 d\alpha \ll N, \quad (20) \]

which can be proved using the same argument in the proof of Corollary 3 of Languasco-Perelli [9]. We obtain

\[ \Sigma_{1,3} = o(HN) \quad \text{and} \quad \Sigma_{1,4} = o(HN). \quad (21) \]

Hence, by (17)-(19) and (21), we have that (5) holds.

Now we proceed to evaluate \( \Sigma_{1,1} \) and \( \Sigma_{1,2} \).

**Contribution of \( \Sigma_{1,1} \)**

Squaring out we obtain

\[ \int_{-1/2}^{1/2} T(\alpha)^2 K_H(\alpha) d\alpha = \sum_{k \in [n, n+H]} m(k) \]

and hence, using (11) and (13), we get

\[ \int_{-1/Q}^{1/Q} T(\alpha)^2 K_H(\alpha) d\alpha = \int_{-1/2}^{1/2} T(\alpha)^2 K_H(\alpha) d\alpha + O(Q^2) = \sum_{k \in [n, n+H]} m(k) + o(HN). \quad (22) \]

Using the Prime Number Theorem, the Cauchy-Schwarz inequality and arguing analogously, we can write

\[ \int_{-1/Q}^{1/Q} S(\alpha) T(\alpha) K_H(\alpha) d\alpha = \sum_{k \in [n, n+H]} m'(k) + o(HN), \quad (23) \]
where
\[
m'(k) = \sum_{N < m \leq 2N} \Lambda(m) \sum_{N < h \leq 2N \atop m + h = k} 1.
\]

Again by the Prime Number Theorem, we get
\[
\sum_{k \in [n, n + H)} m(k) = \sum_{k \in [n, n + H)} m'(k) + o(HN) \tag{24}
\]
and hence, by (22)-(24), we have
\[
\Sigma_{1,1} = \sum_{k \in [n, n + H)} m(k) + o(HN). \tag{25}
\]

Using the Theorem of Languasco [8] and by partial summation, it is easy to prove
\[
\sum_{k \in [n, n + H)} m(k) = \sum_{k \in [n, n + H)} m(k) \mathcal{G}(k) + o(HN) \quad \text{for} \quad H \geq L^{2/3+\varepsilon}. \tag{26}
\]

Now (18) follows from (25) and (26).

**Contribution of \(\Sigma_{1,2}\)**

Since
\[
\Sigma_{2}(\alpha)^2 \ll \left( \sum_{|\gamma| \leq T \atop \beta \notin \mathcal{I}} T_{\rho}(\alpha)^2 \right)^2 |K_H(\alpha)|^2 + |R(\alpha)|^2,
\]
we have
\[
\Sigma_{1,2} \ll A_1 + A_2 + A_3, \tag{27}
\]
where
\[
A_1 = \int_{-1/Q}^{1/Q} | \sum_{|\gamma| \leq T \atop \beta \notin \mathcal{I}} T_{\rho}(\alpha)|^2 |K_H(\alpha)| d\alpha,
\]
\[
A_2 = \int_{-1/Q}^{1/Q} | \sum_{|\gamma| > T \atop \beta \notin \mathcal{I}} T_{\rho}(\alpha)|^2 |K_H(\alpha)| d\alpha
\]
and
\[
A_3 = \int_{-1/Q}^{1/Q} |R(\alpha)|^2 |K_H(\alpha)| d\alpha.
\]

Using (11) and Gallagher’s lemma, we obtain
\[
A_1 \ll \frac{H}{Q^2} \left( \int_{N}^{2N} | \sum_{x < m < x+Q \atop |\gamma| \leq T \atop \beta \notin \mathcal{I}} a_p(m)|^2 |d\alpha + \int_{N-2N}^{N} | \sum_{N < m < x+Q \atop |\gamma| \leq T \atop \beta \notin \mathcal{I}} a_p(m)|^2 |d\alpha
\]
\[
+ \int_{2N-Q}^{2N} | \sum_{x < m \leq 2N \atop |\gamma| \leq T \atop \beta \notin \mathcal{I}} a_p(m)|^2 |d\alpha \right) = A_{1,1} + A_{1,2} + A_{1,3}, \tag{28}
\]
say. Interchanging summation and integration in $A_{1,1}$, we get
\[
A_{1,1} \ll \frac{H}{Q^2} \int_{N}^{2N} \left| \int_{[x]+1}^{[x]+Q} \sum_{|\gamma| \leq T, \beta \notin I} t^{\rho-1} dt \right|^2 dx
\]

\[\ll \frac{H}{Q^2} \int_{N}^{2N} \left( \int_{x}^{x+Q} - \int_{x}^{[x]+1} - \int_{[x]+Q}^{x+Q} \right) \sum_{|\gamma| \leq T, \beta \notin I} t^{\rho-1} dt \right|^2 dx. \tag{29}\]

To bound the contribution of the integral on $[x, [x]+1]$ in (29), we argue as follows. Interchanging summation and integration, we get
\[
\int_{N}^{2N} \left| \int_{x}^{[x]+1} \sum_{|\gamma| \leq T, \beta \notin I} t^{\rho-1} dt \right|^2 dx \ll \sum_{N < n \leq 2N} \int_{x}^{n+1} \sum_{|\gamma| \leq T, \beta \notin I} x^\rho \left( \frac{(n+1)/x}{\rho} - 1 \right)^2 dx
\]

and then, using $\left( \frac{(n+1)/x}{\rho} - 1 \right) \ll \min\left( \frac{1}{N}, \frac{1}{|\gamma|} \right)$, we have
\[
\int_{N}^{2N} \left| \int_{x}^{[x]+1} \sum_{|\gamma| \leq T, \beta \notin I} t^{\rho-1} dt \right|^2 dx \ll L^4 \max_{\sigma \notin I} N^{2\sigma-1} N \left( \sigma, \frac{N}{Q} \right). \tag{30}\]

To estimate the integral on $[x+Q, x+Q]$ in (29) we proceed analogously and hence we get
\[
\int_{N}^{2N} \left| \int_{x+Q}^{[x]+1} \sum_{|\gamma| \leq T, \beta \notin I} t^{\rho-1} dt \right|^2 dx \ll L^4 \max_{\sigma \notin I} N^{2\sigma-1} N \left( \sigma, \frac{N}{Q} \right). \tag{31}\]

Now we treat the integral on $[x, x+Q]$ in (29). Proceeding as above we obtain
\[
\int_{N}^{2N} \left| \int_{x}^{x+Q} \sum_{|\gamma| \leq T, \beta \notin I} t^{\rho-1} dt \right|^2 dx \ll \int_{N}^{2N} \left| \sum_{|\gamma| \leq T, \beta \notin I} x^\rho \left( \frac{1+Q/x}{\rho} - 1 \right)^2 dx \tag{32}\]

\[\ll Q^2 L^4 \max_{\sigma \notin I} N^{2\sigma-1} N \left( \sigma, \frac{N}{Q} \right), \]

where the last inequality follows by Lemma 1. Choosing, in the definition of the interval $I$,
\[
a = \frac{1 + 3\theta}{2} - l \frac{\log L}{L} \quad \text{and} \quad b = \frac{5 - 3\theta}{6} + k \frac{\log L}{L}, \tag{33}\]

where $l > \frac{27 (1-\theta)}{21 - 3\theta}$ and $k$ is a sufficiently large constant, we have, using Ingham-Huxley’s density estimate, see, e.g., Ivić [6], and (29)-(33), that
\[
A_{1,1} \ll HL^4 \max_{\sigma \notin I} N^{2\sigma-1} N \left( \sigma, \frac{N}{Q} \right) = o(HN). \tag{34}\]
Interchanging summation and integration in $A_{1,2}$, we get

$$A_{1,2} \ll \frac{H}{Q^2} \int_{N-Q}^{N} \left| \sum_{|\gamma| \leq T} x^\rho c_{\rho, Q} \right|^2 dx,$$

where $c_{\rho, Q} = \left( \left( \frac{|x+Q|}{x} \right)^\rho - \left( \frac{N}{x} \right)^\rho \right) / \rho$. Splitting the summation according to $|\gamma| \leq N/Q$ and $N/Q \leq |\gamma| \leq T$ and using $c_{\rho, Q} \ll \min\left( \frac{Q}{N}, \frac{1}{|\gamma|} \right)$, we obtain

$$A_{1,2} \ll \frac{H}{Q^2} \left( \frac{Q^2}{N^2} \int_{N-Q}^{N} |\gamma|^2 dx + \int_{N-Q}^{N} \sum_{N/Q \leq |\gamma| \leq T} |\gamma|^2 dx \right) \ll HQ L^4 \max_{\sigma \in I} N^{2\sigma - 2} N(\sigma, \frac{N}{Q})^2.$$

Using Ingham-Huxley’s density estimate, we see that the maximum is attained at $\sigma = 1/2$ and hence we can write

$$A_{1,2} \ll HQL^4 N^{-1} \left( \frac{N}{Q} \right)^2 L^2 = \frac{HN L^6}{Q} = o(HN).$$

(35)

$A_{1,3}$ can be bounded following the lines of the estimation of $A_{1,2}$. We have

$$A_{1,3} = o(HN).$$

(36)

Inserting (34) and (35)-(36) in (28) we obtain

$$A_1 = o(HN).$$

(37)

Now we proceed to estimate $A_2$. By (11) we get

$$A_2 \ll H \int_{-1/Q}^{1/Q} \left| \sum_{N < m \leq 2N} \sum_{|\gamma| > T} a_{\rho}(m) e(m\alpha) \right|^2 d\alpha.$$  

(38)

Using (38), Gallagher’s lemma and the explicit formula for $\psi(x)$, see equations (9)-(10) in ch. 17 of Davenport [3], we have

$$A_2 \ll \frac{H}{Q^2} \int_{N-Q}^{2N} \frac{N^2 L^4}{T^2} dx \ll \frac{HN^3}{Q^2 T^2} L^4 = o(HN).$$

(39)

To bound $A_3$ we use (11), Gallagher’s lemma and the explicit formula for $\psi(x)$, see equation (1) in ch. 17 of Davenport [3]. Hence

$$A_3 \ll \frac{H}{Q^2} \int_{N-Q}^{2N} \left| \sum_{x < m \leq x+Q \atop N < m \leq 2N} \left( \Lambda(m) - 1 + \sum_{\rho} a_{\rho}(m) \right) \right|^2 dx$$

(40)

$$\ll \frac{H}{Q^2} \int_{N-Q}^{2N} L^4 dx \ll \frac{HN L^4}{Q^2} = o(HN).$$

Now (19) follows inserting (37) and (39)-(40) in (27).
6. Mean-square estimate of \( \Sigma_b(\alpha)^2 \)

Squaring out and using the definition of \( \Sigma_b(\alpha) \), we get

\[
\sum_{\frac{1}{2} N < \alpha \leq \frac{3}{2} N} \left| \int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha) \, d\alpha \right|^2
\]

\[
= \sum_{\frac{1}{2} N < \alpha \leq \frac{3}{2} N} \left| \int_{-Q}^{1/Q} \sum_{|\gamma| \leq T} T_\rho(\alpha) K_H(\alpha) \, d\alpha \right| \left| \int_{-Q}^{1/Q} \sum_{|\gamma'| \leq T} T_\rho'(\delta) K_H(\delta) \, d\delta \right|
\]

\[
\ll \int_{-1/Q}^{1/Q} \left| \sum_{|\gamma| \leq T} T_\rho(\alpha) \right|^2 \left( \int_{-1/Q}^{1/Q} \left| \sum_{|\gamma'| \leq T} T_\rho'(\delta) \right|^2 K_N(\alpha - \delta) \, d\delta \right) d\alpha
\]

where \( K_N(t) = \sum_{\frac{1}{2} N < \alpha \leq \frac{3}{2} N} e(-nt) \ll (N, \frac{1}{|t|}) \).

Using the latest estimate and (42), we obtain

\[
\Sigma_3 \ll H^2 N \int_{-1/Q}^{1/Q} \left| \sum_{|\gamma| \leq T} T_\rho(\alpha) \right|^2 \left( \int_{-Q}^{1/Q} \left| \sum_{|\gamma'| \leq T} T_\rho'(\delta) \right|^2 \frac{1}{|\alpha - \delta|} \, d\delta \right) d\alpha
\]

\[
+ H^2 \int_{-1/Q}^{1/Q} \left| \sum_{|\gamma| \leq T} T_\rho(\alpha) \right|^2 \left( \int_{-Q}^{1/Q} \left| \sum_{|\gamma'| \leq T} T_\rho'(\delta) \right|^2 \frac{1}{|\alpha - \delta|} \, d\delta \right) d\alpha
\]

\[
= \Sigma_{3,1} + \Sigma_{3,2},
\]

say. Using (3) and arguing as in section 6, we get

\[
\int_{-1/Q}^{1/Q} \left| \sum_{|\gamma| \leq T} T_\rho(\alpha) \right|^2 \, d\alpha \ll \int_{-1/Q}^{1/Q} |S(\alpha)|^2 \, d\alpha + O(N) \ll N,
\]

where the latest inequality follows from (20).

Now, inserting (44) in \( \Sigma_{3,1} \), we have

\[
\Sigma_{3,1} \ll H^2 N \left( \max_{\alpha \in (-1/Q,1/Q)} \int_{-Q}^{1/Q} \left| \sum_{|\gamma| \leq T} T_\rho(\alpha) \right|^2 \, d\delta \right)
\]

\[
\ll H^2 N \left( \max_{\delta \in (-1/Q,1/Q)} \left| \sum_{|\gamma| \leq T} T_\rho(\delta) \right|^2 \right).
\]
To bound $\Sigma_{3,2}$, we argue as for $\Sigma_{3,1}$ and we can prove that the bound in (45) holds, with an extra $L$ factor, for $\Sigma_{3,2}$ too. Finally, by (41), (43), (45) and the above remark, we obtain

$$\sum_{\frac{3}{2}N \leq n \leq 2N} \left| \int_{-1/Q}^{1/Q} \left( \sum_{\gamma \in \beta \in I} T_\rho(\alpha) \right)^2 K_H(\alpha) \right|^2 \ll H^2 NL \left( \max_{\delta \in (-1/Q,1/Q)} \left| \sum_{|\gamma| \leq T} T_\rho(\delta)^2 \right| \right). \quad (46)$$

Using Lemma 2 and a standard argument to bound sums over zeros of $\zeta(s)$, we have

$$\sum_{|\gamma| \leq T} T_\rho(\delta) \ll L^2 \left( \max_{\sigma < 7/9} N^\sigma \max_{|t| \leq T} N(\sigma, t) |t|^{-1/2} \right. \left. + \max_{\sigma \geq 7/9} N^\sigma \max_{|t| \leq T} N(\sigma, t) |t|^{-1/2} \right)$$

$$\ll L^2 \left( \max_{\sigma < 7/9} N^\sigma N(\sigma, T) T^{-1/2} + \max_{\sigma \geq 7/9} N^\sigma \right). \quad (47)$$

By Ingham-Huxley’s density estimate, we have that the first maximum is attained at $\sigma = a$ and the second at $\sigma = b$. Hence, by (46) and (47), we see that (6) holds.

REFERENCES


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