On the asymptotic formula for Goldbach numbers in short intervals

ON THE ASYMPTOTIC FORMULA FOR GOLDBACH NUMBERS IN SHORT INTERVALS

by

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1. Introduction

Define a Goldbach number (G-number) to be an even number which can be written as a sum of two primes. In the following we denote by N a sufficiently large integer and let $L = \log N$. Let further

$$R(k) = \sum_{N<m\le 2N} \sum_{\substack{N<l\le 2N \\ m+l=k}} \Lambda(l)\Lambda(m)$$

be the weighted counting function of G-numbers,

$$\mathfrak{S}(k) = \begin{cases} 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{\substack{p\mid k\\ p>2}}\left(\frac{p-1}{p-2}\right) & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

be the singular series of Goldbach's problem and

$$m(k) = \sum_{N<m\le 2N} \sum_{\substack{N<l\le 2N \\ m+l=k}} 1.$$

We recall that a well-known conjecture states that as $k \to \infty$

$$R(k) \sim m(k)\mathfrak{S}(k). \tag{1}$$

In this paper we study the asymptotic formula for the average of $R(k)$ over short intervals of type $[n, n+H)$. In the extreme case $H = 1$, Chudakov [1], van der Corput [2] and Estermann [4] proved that, as $N \to \infty$, (1) holds for all $k \in [1, N]$ but

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$O(NL^{-A})$ exceptions, for every $A>0$. Moreover, the same techniques prove, for $H \le L^D$ and $N \to \infty$, that

$$\sum_{k\in[n,n+H)} R(k) \sim \sum_{k\in[n,n+H)} m(k)\mathfrak{S}(k) \qquad (2)$$

holds for all $n \in (\frac{5}{2}N, \frac{7}{2}N]$ but $O(NL^{-A})$ exceptions, for every $A, D > 0$.

We recall that Montgomery-Vaughan [12] improved Chudakov-van der Corput-Estermann's result proving that there exists a (small) constant $\delta > 0$ such that $|E(N)| \ll N^{1-\delta}$, where $E(N) = E \cap [1,N]$ and E is the exceptional set for Goldbach's problem. Montgomery-Vaughan's technique intrinsically does not give any information about the asymptotic formula for $R(k)$.

On the other hand, using the circle method and Ingham-Huxley's zero density estimate, Perelli [14] proved that (2) holds as $n \to \infty$ uniformly for $H \ge n^{1/6+\varepsilon}$. Our aim here is to show, using the circle method, that the asymptotic formula (2) holds for almost all $n \in (\frac{5}{2}N, \frac{7}{2}N]$, uniformly for $L^D \le H \le N^{1/6+\varepsilon}$, for all $D>0$. Our result is

Theorem. *Let $D, \varepsilon > 0$ be arbitrary constants and $L^D \le H \le N^{1/6+\varepsilon}$. Then, as $N \to \infty$, (2) holds for all $n \in (\frac{5}{2}N, \frac{7}{2}N]$ but $O(NL^{42+\varepsilon}H^{-2})$ exceptions.*

In fact, following the proof of the Theorem, it is easy to see that we have $O(NL^{f(\theta)} H^{-2})$ exceptions, where

$$H = N^{\theta} \quad \text{and} \quad f(\theta) = \frac{24-18\theta}{1-3\theta} + \varepsilon.$$

A direct computation shows that $f(\theta)$ is an increasing function and hence the exponent 42 in the log-factor of the Theorem follows taking $\theta = 1/6 + \varepsilon$.

We observe that our result, for $\theta = 1/6+\varepsilon$, proves only that the number of exceptions for (2) is $O(N^{2/3-\varepsilon})$ while, from Perelli's [14] result, we know that there are no exceptions.

We recall that Mikawa, see Lemma 4 of [10], proved a slightly weaker, in the log-factor, result without using the circle method. We finally recall that, under the assumption of the Riemann Hypothesis (RH), (2) holds uniformly for $H \ge \infty(\log^2 n)$, where $f = \infty(g)$ means $g = o(f)$, and that, assuming further the Montgomery pair correlation conjecture, (2) holds uniformly for $H \ge \infty(\log n)$.

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2. Outline of the method

Let

$$Q = \frac{H}{L^{\varepsilon}}, \; T = \frac{N}{Q}L^{2+\varepsilon} \quad \text{and} \quad K_H(n) = \sum_{k\in[n,n+H)} e(-k\alpha),$$

where $e(x) = \exp(2\pi i x)$. Let further $\beta + i\gamma$ denote the generic non-trivial zero of $\zeta(s)$,

$$S(\alpha) = \sum_{N<m\le 2N} \Lambda(m)e(m\alpha),\ T(\alpha) = \sum_{N<m\le 2N} e(m\alpha),$$

$$T_\rho(\alpha) = \sum_{N<m\le 2N} a_\rho(m)e(m\alpha),\ a_\rho(m) = \int_m^{m+1} t^{\rho-1}dt.$$

Given an interval $I = [a,b] \subset [1/2,1]$ we define

$$\Sigma_b(\alpha) = \sum_{\substack{|\gamma|\le T\\ \beta\in I}} T_\rho(\alpha), \quad \Sigma_g(\alpha) = \sum_{\substack{|\gamma|\le T\\ \beta\notin I}} T_\rho(\alpha) + \sum_{|\gamma|>T} T_\rho(\alpha) + R(\alpha)$$

where $R(\alpha)$ is defined by difference in the approximation

$$S(\alpha) = T(\alpha) - \Sigma_g(\alpha) - \Sigma_b(\alpha). \tag{3}$$

Subdivide now $(-\frac{1}{2},\frac{1}{2})$ into $O(\log Q)$ subintervals of the following form

$$A_0 = (-\frac{1}{Q},\frac{1}{Q})\ , A_j = (-\frac{1}{2^j},-\frac{1}{2^{j+1}}] \cup [\frac{1}{2^{j+1}},\frac{1}{2^j})$$

for $j \in [1,K]$, where $K = [\log Q/\log 2]$. Hence we have

$$\begin{aligned}\sum_{k\in[n,n+H)} R(k) &= \int_{-1/2}^{1/2} S(\alpha)^2 K_H(\alpha)d\alpha = \int_{-1/Q}^{1/Q} S(\alpha)^2 K_H(\alpha)d\alpha\\ &+ \sum_{j=1}^{K}\int_{A_j} S(\alpha)^2 K_H(\alpha)d\alpha = \Sigma_1 + \Sigma_2,\end{aligned} \tag{4}$$

say. We will prove that

$$\Sigma_1 = \sum_{k\in[n,n+H)} m(k)\mathfrak{S}(k) + \int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha)d\alpha + o(HN), \tag{5}$$

$$\sum_{\frac{5}{2}N<n\le\frac{7}{2}N} |\int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha)d\alpha|^2 \ll N^3 L^{f(\theta)}, \tag{6}$$

and

$$\Sigma_2 = o(HN). \tag{7}$$

We will need also that

$$\sum_{k\in[n,n+H)} m(k)\mathfrak{S}(k) \gg HN \tag{8}$$

which can be obtained immediately using $\mathfrak{S}(2k) \gg 1$. Since $\varepsilon > 0$ is arbitrarily small, our Theorem follows at once from (4)-(8).

3. Preliminary Lemmas

In the following we will need two auxiliary lemmas.

Lemma 1. *Let $N(\sigma, T)$ be the number of zeros $\rho = \beta + i\gamma$ of the Riemann zeta-function such that $|\gamma| \le T$ and $\beta \ge \sigma$, and let $I \subset [1/2, 1]$ be an interval. Then*

$$\int_N^{2N} |\sum_{\substack{|\gamma|\le T\\ \beta\in I}} x^\rho \frac{(1+Q/x)^\rho - 1}{\rho}|^2 dx \ll Q^2 L^4 \max_{\sigma\in I} N^{2\sigma-1} N(\sigma, \frac{N}{Q}).$$

The proof of Lemma 1 is standard. It can be obtained using, *e.g.*, Saffari-Vaughan's [15] technique and hence we omit it.

Lemma 2. *We have, for $|\gamma| \ll N$ and N sufficiently large, that*

$$T_\rho(\alpha) \ll N^\beta |\gamma|^{-1/2}.$$

Proof. We follow the line of Perelli [13] and hence we give only a brief sketch of the proof. Since

$$a_\rho(m) = \int_m^{m+1} t^{\rho-1} dt = \frac{m^\rho}{\rho}((1+\frac{1}{m})^\rho - 1),$$

and, for P sufficiently large but fixed,

$$(1+\frac{1}{m})^\rho - 1 = \sum_{j=1}^{P} \frac{\rho(\rho-1)\cdots(\rho-j+1)}{j!}(\frac{1}{m})^j + O(N^{-11}),$$

we can write

$$T_\rho(\alpha) = T_{\rho,1}(\alpha) + \sum_{j=2}^{P} \frac{(\rho-1)(\rho-2)\cdots(\rho-j+1)}{j!} T_{\rho,j}(\alpha) + O(N^{\beta-10}), \tag{9}$$

where

$$T_{\rho,j}(\alpha) = \sum_{N<m\le 2N} m^{\rho-j} e(m\alpha).$$

From Abel's inequality we have

$$|T_{\rho,j}(\alpha)| \ll N^{\beta-j} \max_{N\le y\le 2N} |\sum_{N\le m\le y} e^{2\pi i f_\rho(\alpha)}|,$$

where $f_\rho(\alpha) = \frac{\gamma}{2\pi}\log n + \alpha n$. We can assume that the maximum is attained at $Y = 2N$, and so, using van der Corput's second derivative method, see Theorem 2.2 of Graham-Kolesnik [5], we get

$$T_{\rho,j}(\alpha) \ll N^{\beta-j+1}|\gamma|^{-1/2}. \tag{10}$$

Lemma 2 now follows inserting (10) in (9). □

4. Estimation of Σ_2

Letting $S(\alpha) = T(\alpha) + R_1(\alpha)$, where $R_1(\alpha)$ is defined by difference, and using

$$K_H(\alpha) \ll \min(H, \frac{1}{|\alpha|}) \quad \text{for every} \quad \alpha \in [-\frac{1}{2}, \frac{1}{2}], \tag{11}$$

we have

$$\begin{aligned}\Sigma_2 &\ll \sum_{j=1}^{K} \Big(\int_{A_j} |T(\alpha)|^2 |K_H(\alpha)| d\alpha + \int_{A_j} |R_1(\alpha)|^2 |K_H(\alpha)| d\alpha \Big) \\ &\ll \sum_{j=1}^{K} 2^j \Big(\int_{A_j} |T(\alpha)|^2 d\alpha + \int_{A_j} |R_1(\alpha)|^2 d\alpha \Big) = \Sigma_{2,1} + \Sigma_{2,2},\end{aligned} \tag{12}$$

say. Using

$$T(\alpha) \ll \min(N, \frac{1}{|\alpha|}) \quad \text{for every} \quad \alpha \in [-\frac{1}{2}, \frac{1}{2}], \tag{13}$$

we obtain

$$\Sigma_{2,1} \ll \sum_{j=1}^{K} 4^j \ll 4^K \ll Q^2 = o(HN). \tag{14}$$

By Gallagher's lemma, see, *e.g.*, Lemma 1.9 of Montgomery [11], and the Brun-Titchmarsh theorem we get

$$\Sigma_{2,2} \ll \sum_{j=1}^{K} 2^j \int_{-2^{-j}}^{2^{-j}} \Big| \sum_{N < m \le 2N} (\Lambda(m) - 1) e(m\alpha) \Big|^2 d\alpha \ll \sum_{j=1}^{K} 2^{-j} (J(N, 2^j) + L^2 2^{3j}), \tag{15}$$

where $J(N,h)$ is the Selberg integral. Inserting the estimate $J(N,h) \ll h^2 N + hNL$ for all $h \ge 1$, see the Lemma in Languasco [7], in (15) we have

$$\Sigma_{2,2} \ll \sum_{j=1}^{K} 2^{-j} \big(2^{3j} L^2 + 2^{2j} N + 2^j NL \big) \ll L^2 Q^2 + NQ + NL \log Q = o(HN). \tag{16}$$

Hence, inserting (14) and (16) in (12), we finally have that (7) holds.

5. Estimation of Σ_1

Inserting the identity

$$S(\alpha)^2 = \big(2S(\alpha)T(\alpha) - T(\alpha)^2 \big) - \Sigma_g(\alpha)^2 - 2T(\alpha)\Sigma_g(\alpha) + 2S(\alpha)\Sigma_g(\alpha) + \Sigma_b(\alpha)^2$$

into the definition of Σ_1, we obtain

$$\Sigma_1 = \Sigma_{1,1} - \Sigma_{1,2} - \Sigma_{1,3} + \Sigma_{1,4} + \int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha) d\alpha, \tag{17}$$

where

$$\Sigma_{1,1} = \int_{-1/Q}^{1/Q} (2S(\alpha)T(\alpha) - T(\alpha)^2) K_H(\alpha) d\alpha,$$

$$\Sigma_{1,2} = \int_{-1/Q}^{1/Q} \Sigma_g(\alpha)^2 K_H(\alpha) d\alpha,$$

$$\Sigma_{1,3} = \int_{-1/Q}^{1/Q} 2T(\alpha)\Sigma_g(\alpha) K_H(\alpha) d\alpha$$

and

$$\Sigma_{1,4} = \int_{-1/Q}^{1/Q} 2S(\alpha)\Sigma_g(\alpha) K_H(\alpha) d\alpha.$$

In this section we will prove

$$\Sigma_{1,1} = \sum_{k\in[n,n+H)} m(k)\mathfrak{S}(k) + o(HN) \tag{18}$$

and

$$\Sigma_{1,2} = o(HN), \tag{19}$$

while the estimation of the mean-square of $\int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha) d\alpha$ will be performed in the next section.

Assuming that (19) holds, the contribution of $\Sigma_{1,3}$ and $\Sigma_{1,4}$ can be estimated using the Cauchy-Schwarz inequality and

$$\int_{-1/Q}^{1/Q} |S(\alpha)|^2 d\alpha \ll N, \tag{20}$$

which can be proved using the same argument in the proof of Corollary 3 of Languasco-Perelli [9]. We obtain

$$\Sigma_{1,3} = o(HN) \quad \text{and} \quad \Sigma_{1,4} = o(HN). \tag{21}$$

Hence, by (17)-(19) and (21), we have that (5) holds.

Now we proceed to evaluate $\Sigma_{1,1}$ and $\Sigma_{1,2}$.

Contribution of $\Sigma_{1,1}$

Squaring out we obtain

$$\int_{-1/2}^{1/2} T(\alpha)^2 K_H(\alpha) d\alpha = \sum_{k\in[n,n+H)} m(k)$$

and hence, using (11) and (13), we get

$$\int_{-1/Q}^{1/Q} T(\alpha)^2 K_H(\alpha) d\alpha = \int_{-1/2}^{1/2} T(\alpha)^2 K_H(\alpha) d\alpha + O(Q^2) = \sum_{k\in[n,n+H)} m(k) + o(HN). \tag{22}$$

Using the Prime Number Theorem, the Cauchy-Schwarz inequality and arguing analogously, we can write

$$\int_{-1/Q}^{1/Q} S(\alpha)T(\alpha) K_H(\alpha) d\alpha = \sum_{k\in[n,n+H)} m'(k) + o(HN), \tag{23}$$

where

$$m'(k) = \sum_{N<m\le 2N} \Lambda(m) \sum_{\substack{N<h\le 2N\\ m+h=k}} 1.$$

Again by the Prime Number Theorem, we get

$$\sum_{k\in[n,n+H)} m(k) = \sum_{k\in[n,n+H)} m'(k) + o(HN) \tag{24}$$

and hence, by (22)-(24), we have

$$\Sigma_{1,1} = \sum_{k\in[n,n+H)} m(k) + o(HN). \tag{25}$$

Using the Theorem of Languasco [8] and by partial summation, it is easy to prove

$$\sum_{k\in[n,n+H)} m(k) = \sum_{k\in[n,n+H)} m(k)\mathfrak{S}(k) + o(HN) \quad \text{for} \quad H \ge L^{2/3+\varepsilon}. \tag{26}$$

Now (18) follows from (25) and (26).

Contribution of $\Sigma_{1,2}$

Since

$$\Sigma_g(\alpha)^2 \ll |\sum_{\substack{|\gamma|\le T\\ \beta\notin I}} T_\rho(\alpha)|^2 + |\sum_{|\gamma|>T} T_\rho(\alpha)|^2 + |R(\alpha)|^2,$$

we have

$$\Sigma_{1,2} \ll A_1 + A_2 + A_3, \tag{27}$$

where

$$A_1 = \int_{-1/Q}^{1/Q} |\sum_{\substack{|\gamma|\le T\\ \beta\notin I}} T_\rho(\alpha)|^2 |K_H(\alpha)| d\alpha,$$

$$A_2 = \int_{-1/Q}^{1/Q} |\sum_{|\gamma|>T} T_\rho(\alpha)|^2 |K_H(\alpha)| d\alpha$$

and

$$A_3 = \int_{-1/Q}^{1/Q} |R(\alpha)|^2 |K_H(\alpha)| d\alpha.$$

Using (11) and Gallagher's lemma, we obtain

$$A_1 \ll \frac{H}{Q^2}\Big(\int_{N}^{2N} |\sum_{x<m<x+Q} \sum_{\substack{|\gamma|\le T\\ \beta\notin I}} a_\rho(m)|^2 dx + \int_{N-Q}^{N} |\sum_{N<m<x+Q} \sum_{\substack{|\gamma|\le T\\ \beta\notin I}} a_\rho(m)|^2 dx$$

$$+ \int_{2N-Q}^{2N} |\sum_{x<m\le 2N} \sum_{\substack{|\gamma|\le T\\ \beta\notin I}} a_\rho(m)|^2 dx\Big) = A_{1,1} + A_{1,2} + A_{1,3}, \tag{28}$$

say. Interchanging summation and integration in $A_{1,1}$, we get

$$\begin{aligned} A_{1,1} &\ll \frac{H}{Q^2}\int_N^{2N} \Big| \int_{[x]+1}^{[x+Q]} \sum_{\substack{|\gamma|\le T\\ \beta\notin I}} t^{\rho-1}dt\Big|^2 dx \\ &\ll \frac{H}{Q^2}\int_N^{2N} \Big|\Big(\int_x^{x+Q} - \int_x^{[x]+1} - \int_{[x+Q]}^{x+Q}\Big) \sum_{\substack{|\gamma|\le T\\ \beta\notin I}} t^{\rho-1}dt\Big|^2 dx. \end{aligned} \tag{29}$$

To bound the contribution of the integral on $[x,[x]+1]$ in (29), we argue as follows. Interchanging summation and integration, we get

$$\int_N^{2N} \Big| \int_x^{[x]+1} \sum_{\substack{|\gamma|\le T\\ \beta\notin I}} t^{\rho-1}dt\Big|^2 dx \ll \sum_{N<n\le 2N}\int_n^{n+1} \Big|\sum_{\substack{|\gamma|\le T\\ \beta\notin I}} x^\rho \frac{((n+1)/x)^\rho - 1}{\rho}\Big|^2 dx$$

and then, using $\frac{((n+1)/x)^\rho-1}{\rho} \ll \min(\frac{1}{N},\frac{1}{|\gamma|})$, we have

$$\int_N^{2N} \Big| \int_x^{[x]+1} \sum_{\substack{|\gamma|\le T\\ \beta\notin I}} t^{\rho-1}dt\Big|^2 dx \ll L^4 \max_{\sigma\notin I} N^{2\sigma-1}N(\sigma,\frac{N}{Q}). \tag{30}$$

To estimate the integral on $[[x+Q],x+Q]$ in (29) we proceed analogously and hence we get

$$\int_N^{2N} \Big| \int_{[x+Q]}^{x+Q} \sum_{\substack{|\gamma|\le T\\ \beta\notin I}} t^{\rho-1}dt\Big|^2 dx \ll L^4 \max_{\sigma\notin I} N^{2\sigma-1}N(\sigma,\frac{N}{Q}). \tag{31}$$

Now we treat the integral on $[x,x+Q]$ in (29). Proceeding as above we obtain

$$\begin{aligned} \int_N^{2N} \Big| \int_x^{x+Q} \sum_{\substack{|\gamma|\le T\\ \beta\notin I}} t^{\rho-1}dt\Big|^2 dx &\ll \int_N^{2N} \Big|\sum_{\substack{|\gamma|\le T\\ \beta\notin I}} x^\rho \frac{(1+Q/x)^\rho - 1}{\rho}\Big|^2 dx \\ &\ll Q^2L^4 \max_{\sigma\notin I} N^{2\sigma-1}N(\sigma,\frac{N}{Q}), \end{aligned} \tag{32}$$

where the last inequality follows by Lemma 1.

Choosing, in the definition of the interval I,

$$a = \frac{1+3\theta}{2} - l\frac{\log L}{L} \quad \text{and} \quad b = \frac{5-3\theta}{6} + k\frac{\log L}{L}, \tag{33}$$

where $l > \frac{27(1-\theta)}{2(1-3\theta)}$ and k is a sufficiently large constant, we have, using Ingham-Huxley's density estimate, see, *e.g.*, Ivić [6], and (29)-(33), that

$$A_{1,1} \ll HL^4 \max_{\sigma\notin I} N^{2\sigma-1}N(\sigma,\frac{N}{Q}) = o(HN). \tag{34}$$

Interchanging summation and integration in $A_{1,2}$, we get

$$A_{1,2} \ll \frac{H}{Q^2} \int_{N-Q}^{N} |\sum_{\substack{|\gamma|\leq T \\ \beta\notin I}} x^\rho c_{\rho,Q}|^2 dx,$$

where $c_{\rho,Q} = ((\frac{[x+Q]}{x})^\rho - (\frac{N}{x})^\rho)/\rho$. Splitting the summation according to $|\gamma| \leq N/Q$ and $N/Q \leq |\gamma| \leq T$ and using $c_{\rho,Q} \ll \min(\frac{Q}{N}, \frac{1}{|\gamma|})$, we obtain

$$\begin{aligned} A_{1,2} &\ll \frac{H}{Q^2}(\frac{Q^2}{N^2}\int_{N-Q}^{N} |\sum_{\substack{|\gamma|\leq N/Q \\ \beta\notin I}} x^\beta|^2 dx + \int_{N-Q}^{N} |\sum_{\substack{N/Q\leq|\gamma|\leq T \\ \beta\notin I}} \frac{x^\beta}{|\gamma|}|^2 dx) \\ &\ll HQL^4 \max_{\sigma\notin I} N^{2\sigma-2} N(\sigma, \frac{N}{Q})^2. \end{aligned}$$

Using Ingham-Huxley's density estimate, we see that the maximum is attained at $\sigma = 1/2$ and hence we can write

$$A_{1,2} \ll HQL^4 N^{-1}(\frac{N}{Q})^2 L^2 = \frac{HNL^6}{Q} = o(HN). \tag{35}$$

$A_{1,3}$ can be bounded following the lines of the estimation of $A_{1,2}$. We have

$$A_{1,3} = o(HN). \tag{36}$$

Inserting (34) and (35)-(36) in (28) we obtain

$$A_1 = o(HN). \tag{37}$$

Now we proceed to estimate A_2. By (11) we get

$$A_2 \ll H \int_{-1/Q}^{1/Q} |\sum_{N<m\leq 2N} \sum_{|\gamma|>T} a_\rho(m) e(m\alpha)|^2 d\alpha. \tag{38}$$

Using (38), Gallagher's lemma and the explicit formula for $\psi(x)$, see equations (9)-(10) in ch. 17 of Davenport [3], we have

$$A_2 \ll \frac{H}{Q^2} \int_{N-Q}^{2N} \frac{N^2 L^4}{T^2} dx \ll \frac{HN^3}{Q^2 T^2} L^4 = o(HN). \tag{39}$$

To bound A_3 we use (11), Gallagher's lemma and the explicit formula for $\psi(x)$, see equation (1) in ch. 17 of Davenport [3]. Hence

$$\begin{aligned} A_3 &\ll \frac{H}{Q^2} \int_{N-Q}^{2N} |\sum_{\substack{x<m<x+Q \\ N<m\leq 2N}} (\Lambda(m) - 1 + \sum_{\rho} a_\rho(m))|^2 dx \\ &\ll \frac{H}{Q^2} \int_{N-Q}^{2N} L^4 dx \ll \frac{HNL^4}{Q^2} = o(HN). \end{aligned} \tag{40}$$

Now (19) follows inserting (37) and (39)-(40) in (27).

6. Mean-square estimate of $\Sigma_b(\alpha)^2$

Squaring out and using the definition of $\Sigma_b(\alpha)$, we get

$$\sum_{\frac{5}{2}N<n\le\frac{7}{2}N} \Big|\int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha)d\alpha\Big|^2$$

$$= \sum_{\frac{5}{2}N<n\le\frac{7}{2}N} \int_{-1/Q}^{1/Q} \Big(\sum_{\substack{|\gamma|\le T\\ \beta\in I}} T_{\rho}(\alpha)\Big)^2 K_H(\alpha)d\alpha \int_{-1/Q}^{1/Q} \Big(\sum_{\substack{|\gamma'|\le T\\ \beta'\in I}} T_{\overline{\rho}'}(\delta)\Big)^2 \overline{K_H}(\delta)d\delta \tag{41}$$

$$\ll \int_{-1/Q}^{1/Q} \Big|\sum_{\substack{|\gamma|\le T\\ \beta\in I}} T_{\rho}(\alpha)\Big|^2 \int_{-1/Q}^{1/Q} \Big|\sum_{\substack{|\gamma'|\le T\\ \beta'\in I}} T_{\overline{\rho}'}(\delta)\Big|^2 \Big|\sum_{\frac{5}{2}N<n\le\frac{7}{2}N} K_H(\alpha)\overline{K_H}(\delta)\Big| d\delta d\alpha = \Sigma_3,$$

say. Since $K_H(\alpha) = \frac{\sin \pi H\alpha}{\sin \pi\alpha} e(\frac{1-H}{2}\alpha)e(-n\alpha)$, we have

$$\Sigma_3 \ll H^2 \int_{-1/Q}^{1/Q} \Big|\sum_{\substack{|\gamma|\le T\\ \beta\in I}} T_{\rho}(\alpha)\Big|^2 \Big(\int_{-1/Q}^{1/Q} \Big|\sum_{\substack{|\gamma'|\le T\\ \beta'\in I}} T_{\overline{\rho}'}(\delta)\Big|^2 K_N(\alpha-\delta)d\delta\Big) d\alpha, \tag{42}$$

where $K_N(t) = \sum_{\frac{5}{2}N<n\le\frac{7}{2}N} e(-nt) \ll \min(N, \frac{1}{|t|})$.

Using the latest estimate and (42), we obtain

$$\Sigma_3 \ll H^2 N \int_{-1/Q}^{1/Q} \Big|\sum_{\substack{|\gamma|\le T\\ \beta\in I}} T_{\rho}(\alpha)\Big|^2 \Big(\int_{(-\frac{1}{Q},\frac{1}{Q})\cap(\alpha-\frac{1}{N},\alpha+\frac{1}{N})} \Big|\sum_{\substack{|\gamma'|\le T\\ \beta'\in I}} T_{\overline{\rho}'}(\delta)\Big|^2 d\delta\Big) d\alpha$$

$$+ H^2 \int_{-1/Q}^{1/Q} \Big|\sum_{\substack{|\gamma|\le T\\ \beta\in I}} T_{\rho}(\alpha)\Big|^2 \Big(\int_{(-\frac{1}{Q},\frac{1}{Q})\setminus(\alpha-\frac{1}{N},\alpha+\frac{1}{N})} \Big|\sum_{\substack{|\gamma'|\le T\\ \beta'\in I}} T_{\overline{\rho}'}(\delta)\Big|^2 \frac{1}{|\alpha-\delta|} d\delta\Big) d\alpha \tag{43}$$

$$= \Sigma_{3,1} + \Sigma_{3,2},$$

say. Using (3) and arguing as in section 6, we get

$$\int_{-1/Q}^{1/Q} \Big|\sum_{\substack{|\gamma|\le T\\ \beta\in I}} T_{\rho}(\alpha)\Big|^2 d\alpha \ll \int_{-1/Q}^{1/Q} |S(\alpha)|^2 d\alpha + O(N) \ll N, \tag{44}$$

where the latest inequality follows from (20).

Now, inserting (44) in $\Sigma_{3,1}$, we have

$$\Sigma_{3,1} \ll H^2 N^2 \Big(\max_{\alpha\in(-1/Q,1/Q)} \int_{(-\frac{1}{Q},\frac{1}{Q})\cap(\alpha-\frac{1}{N},\alpha+\frac{1}{N})} \Big|\sum_{\substack{|\gamma'|\le T\\ \beta'\in I}} T_{\overline{\rho}'}(\delta)\Big|^2 d\delta\Big)$$

$$\ll H^2 N \Big(\max_{\delta\in(-1/Q,1/Q)} \Big|\sum_{\substack{|\gamma|\le T\\ \beta\in I}} T_{\rho}(\cdot)\Big|^2\Big). \tag{45}$$

To bound $\Sigma_{3,2}$, we argue as for $\Sigma_{3,1}$ and we can prove that the bound in (45) holds, with an extra L factor, for $\Sigma_{3,2}$ too. Finally, by (41), (43), (45) and the above remark, we obtain

$$\sum_{\frac{5}{2}N\leq n\leq\frac{7}{2}N} \Big|\int_{-1/Q}^{1/Q}\Big(\sum_{\substack{|\gamma|\leq T\\ \beta\in I}} T_\rho(\alpha)\Big)^2 K_H(\alpha)d\alpha\Big|^2 \ll H^2NL\Big(\max_{\delta\in(-1/Q,1/Q)}\Big|\sum_{\substack{|\gamma|\leq T\\ \beta\in I}} T_\rho(\delta)\Big|^2\Big). \tag{46}$$

Using Lemma 2 and a standard argument to bound sums over zeros of $\zeta(s)$, we have

$$\begin{aligned}\sum_{\substack{|\gamma|\leq T\\ \beta\in I}} T_\rho(\delta) &\ll L^2\Big(\max_{\substack{\sigma\in I\\ \sigma<7/9}} N^\sigma \max_{|t|\leq T} N(\sigma,t)|t|^{-1/2} + \max_{\substack{\sigma\in I\\ \sigma\geq 7/9}} N^\sigma \max_{|t|\leq T} N(\sigma,t)|t|^{-1/2}\Big)\\ &\ll L^2\Big(\max_{\substack{\sigma\in I\\ \sigma<7/9}} N^\sigma N(\sigma,T)T^{-1/2} + \max_{\substack{\sigma\in I\\ \sigma\geq 7/9}} N^\sigma\Big).\end{aligned} \tag{47}$$

By Ingham-Huxley's density estimate, we have that the first maximum is attained at $\sigma=a$ and the second at $\sigma=b$. Hence, by (46) and (47), we see that (6) holds.

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