

# ON THE ASYMPTOTIC FORMULA FOR GOLDBACH NUMBERS IN SHORT INTERVALS

by

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## 1. INTRODUCTION

Define a Goldbach number (G-number) to be an even number which can be written as a sum of two primes. In the following we denote by  $N$  a sufficiently large integer and let  $L = \log N$ . Let further

$$R(k) = \sum_{N < m \leq 2N} \sum_{\substack{N < l \leq 2N \\ m+l=k}} \Lambda(l)\Lambda(m)$$

be the weighted counting function of G-numbers,

$$\mathfrak{S}(k) = \begin{cases} 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p|k \\ p>2}} \left(\frac{p-1}{p-2}\right) & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

be the singular series of Goldbach's problem and

$$m(k) = \sum_{N < m \leq 2N} \sum_{\substack{N < l \leq 2N \\ m+l=k}} 1.$$

We recall that a well-known conjecture states that as  $k \rightarrow \infty$

$$R(k) \sim m(k)\mathfrak{S}(k). \tag{1}$$

In this paper we study the asymptotic formula for the average of  $R(k)$  over short intervals of type  $[n, n + H)$ . In the extreme case  $H = 1$ , Chudakov [1], van der Corput [2] and Estermann [4] proved that, as  $N \rightarrow \infty$ , (1) holds for all  $k \in [1, N]$  but

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$O(NL^{-A})$  exceptions, for every  $A > 0$ . Moreover, the same techniques prove, for  $H \leq L^D$  and  $N \rightarrow \infty$ , that

$$\sum_{k \in [n, n+H)} R(k) \sim \sum_{k \in [n, n+H)} m(k) \mathfrak{S}(k) \quad (2)$$

holds for all  $n \in (\frac{5}{2}N, \frac{7}{2}N]$  but  $O(NL^{-A})$  exceptions, for every  $A, D > 0$ .

We recall that Montgomery-Vaughan [12] improved Chudakov-van der Corput-Estermann's result proving that there exists a (small) constant  $\delta > 0$  such that  $|E(N)| \ll N^{1-\delta}$ , where  $E(N) = E \cap [1, N]$  and  $E$  is the exceptional set for Goldbach's problem. Montgomery-Vaughan's technique intrinsically does not give any information about the asymptotic formula for  $R(k)$ .

On the other hand, using the circle method and Ingham-Huxley's zero density estimate, Perelli [14] proved that (2) holds as  $n \rightarrow \infty$  uniformly for  $H \geq n^{1/6+\varepsilon}$ . Our aim here is to show, using the circle method, that the asymptotic formula (2) holds for almost all  $n \in (\frac{5}{2}N, \frac{7}{2}N]$ , uniformly for  $L^D \leq H \leq N^{1/6+\varepsilon}$ , for all  $D > 0$ .

Our result is

**Theorem.** *Let  $D, \varepsilon > 0$  be arbitrary constants and  $L^D \leq H \leq N^{1/6+\varepsilon}$ . Then, as  $N \rightarrow \infty$ , (2) holds for all  $n \in (\frac{5}{2}N, \frac{7}{2}N]$  but  $O(NL^{42+\varepsilon}H^{-2})$  exceptions.*

In fact, following the proof of the Theorem, it is easy to see that we have  $O(NL^{f(\theta)}H^{-2})$  exceptions, where

$$H = N^\theta \quad \text{and} \quad f(\theta) = \frac{24 - 18\theta}{1 - 3\theta} + \varepsilon.$$

A direct computation shows that  $f(\theta)$  is an increasing function and hence the exponent 42 in the log-factor of the Theorem follows taking  $\theta = 1/6 + \varepsilon$ .

We observe that our result, for  $\theta = 1/6 + \varepsilon$ , proves only that the number of exceptions for (2) is  $O(N^{2/3-\varepsilon})$  while, from Perelli's [14] result, we know that there are no exceptions.

We recall that Mikawa, see Lemma 4 of [10], proved a slightly weaker, in the log-factor, result without using the circle method. We finally recall that, under the assumption of the Riemann Hypothesis (RH), (2) holds uniformly for  $H \geq \infty(\log^2 n)$ , where  $f = \infty(g)$  means  $g = o(f)$ , and that, assuming further the Montgomery pair correlation conjecture, (2) holds uniformly for  $H \geq \infty(\log n)$ .

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## 2. OUTLINE OF THE METHOD

Let

$$Q = \frac{H}{L^\varepsilon}, \quad T = \frac{N}{Q}L^{2+\varepsilon} \quad \text{and} \quad K_H(n) = \sum_{k \in [n, n+H)} e(-k\alpha),$$

where  $e(x) = \exp(2\pi i x)$ . Let further  $\beta + i\gamma$  denote the generic non-trivial zero of  $\zeta(s)$ ,

$$S(\alpha) = \sum_{N < m \leq 2N} \Lambda(m) e(m\alpha), \quad T(\alpha) = \sum_{N < m \leq 2N} e(m\alpha),$$

$$T_\rho(\alpha) = \sum_{N < m \leq 2N} a_\rho(m) e(m\alpha), \quad a_\rho(m) = \int_m^{m+1} t^{\rho-1} dt.$$

Given an interval  $I = [a, b] \subset [1/2, 1]$  we define

$$\Sigma_b(\alpha) = \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\alpha), \quad \Sigma_g(\alpha) = \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} T_\rho(\alpha) + \sum_{|\gamma| > T} T_\rho(\alpha) + R(\alpha)$$

where  $R(\alpha)$  is defined by difference in the approximation

$$S(\alpha) = T(\alpha) - \Sigma_g(\alpha) - \Sigma_b(\alpha). \quad (3)$$

Subdivide now  $(-\frac{1}{2}, \frac{1}{2})$  into  $O(\log Q)$  subintervals of the following form

$$A_0 = (-\frac{1}{Q}, \frac{1}{Q}), \quad A_j = (-\frac{1}{2^j}, -\frac{1}{2^{j+1}}] \cup [\frac{1}{2^{j+1}}, \frac{1}{2^j})$$

for  $j \in [1, K]$ , where  $K = [\log Q / \log 2]$ . Hence we have

$$\begin{aligned} \sum_{k \in [n, n+H]} R(k) &= \int_{-1/2}^{1/2} S(\alpha)^2 K_H(\alpha) d\alpha = \int_{-1/Q}^{1/Q} S(\alpha)^2 K_H(\alpha) d\alpha \\ &+ \sum_{j=1}^K \int_{A_j} S(\alpha)^2 K_H(\alpha) d\alpha = \Sigma_1 + \Sigma_2, \end{aligned} \quad (4)$$

say. We will prove that

$$\Sigma_1 = \sum_{k \in [n, n+H]} m(k) \mathfrak{S}(k) + \int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha) d\alpha + o(HN), \quad (5)$$

$$\sum_{\frac{5}{2}N < n \leq \frac{7}{2}N} \left| \int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha) d\alpha \right|^2 \ll N^3 L^{f(\theta)}, \quad (6)$$

and

$$\Sigma_2 = o(HN). \quad (7)$$

We will need also that

$$\sum_{k \in [n, n+H]} m(k) \mathfrak{S}(k) \gg HN \quad (8)$$

which can be obtained immediately using  $\mathfrak{S}(2k) \gg 1$ . Since  $\varepsilon > 0$  is arbitrarily small, our Theorem follows at once from (4)-(8).

## 3. PRELIMINARY LEMMAS

In the following we will need two auxiliary lemmas.

**Lemma 1.** *Let  $N(\sigma, T)$  be the number of zeros  $\rho = \beta + i\gamma$  of the Riemann zeta-function such that  $|\gamma| \leq T$  and  $\beta \geq \sigma$ , and let  $I \subset [1/2, 1]$  be an interval. Then*

$$\int_N^{2N} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} x^\rho \frac{(1 + Q/x)^\rho - 1}{\rho} \right|^2 dx \ll Q^2 L^4 \max_{\sigma \in I} N^{2\sigma-1} N(\sigma, \frac{N}{Q}).$$

The proof of Lemma 1 is standard. It can be obtained using, *e.g.*, Saffari-Vaughan's [15] technique and hence we omit it.

**Lemma 2.** *We have, for  $|\gamma| \ll N$  and  $N$  sufficiently large, that*

$$T_\rho(\alpha) \ll N^\beta |\gamma|^{-1/2}.$$

**Proof.** We follow the line of Perelli [13] and hence we give only a brief sketch of the proof. Since

$$a_\rho(m) = \int_m^{m+1} t^{\rho-1} dt = \frac{m^\rho}{\rho} \left( \left(1 + \frac{1}{m}\right)^\rho - 1 \right),$$

and, for  $P$  sufficiently large but fixed,

$$\left(1 + \frac{1}{m}\right)^\rho - 1 = \sum_{j=1}^P \frac{\rho(\rho-1) \cdots (\rho-j+1)}{j!} \left(\frac{1}{m}\right)^j + O(N^{-11}),$$

we can write

$$T_\rho(\alpha) = T_{\rho,1}(\alpha) + \sum_{j=2}^P \frac{(\rho-1)(\rho-2) \cdots (\rho-j+1)}{j!} T_{\rho,j}(\alpha) + O(N^{\beta-10}), \quad (9)$$

where

$$T_{\rho,j}(\alpha) = \sum_{N < m \leq 2N} m^{\rho-j} e(m\alpha).$$

From Abel's inequality we have

$$|T_{\rho,j}(\alpha)| \ll N^{\beta-j} \max_{N \leq y \leq 2N} \left| \sum_{N \leq m \leq y} e^{2\pi i f_\rho(\alpha)} \right|,$$

where  $f_\rho(\alpha) = \frac{\gamma}{2\pi} \log n + \alpha n$ . We can assume that the maximum is attained at  $Y = 2N$ , and so, using van der Corput's second derivative method, see Theorem 2.2 of Graham-Kolesnik [5], we get

$$T_{\rho,j}(\alpha) \ll N^{\beta-j+1} |\gamma|^{-1/2}. \quad (10)$$

Lemma 2 now follows inserting (10) in (9).  $\square$

4. ESTIMATION OF  $\Sigma_2$ 

Letting  $S(\alpha) = T(\alpha) + R_1(\alpha)$ , where  $R_1(\alpha)$  is defined by difference, and using

$$K_H(\alpha) \ll \min(H, \frac{1}{|\alpha|}) \quad \text{for every } \alpha \in [-\frac{1}{2}, \frac{1}{2}], \quad (11)$$

we have

$$\begin{aligned} \Sigma_2 &\ll \sum_{j=1}^K \left( \int_{A_j} |T(\alpha)|^2 |K_H(\alpha)| d\alpha + \int_{A_j} |R_1(\alpha)|^2 |K_H(\alpha)| d\alpha \right) \\ &\ll \sum_{j=1}^K 2^j \left( \int_{A_j} |T(\alpha)|^2 d\alpha + \int_{A_j} |R_1(\alpha)|^2 d\alpha \right) = \Sigma_{2,1} + \Sigma_{2,2}, \end{aligned} \quad (12)$$

say. Using

$$T(\alpha) \ll \min(N, \frac{1}{|\alpha|}) \quad \text{for every } \alpha \in [-\frac{1}{2}, \frac{1}{2}], \quad (13)$$

we obtain

$$\Sigma_{2,1} \ll \sum_{j=1}^K 4^j \ll 4^K \ll Q^2 = o(HN). \quad (14)$$

By Gallagher's lemma, see, *e.g.*, Lemma 1.9 of Montgomery [11], and the Brun-Titchmarsh theorem we get

$$\Sigma_{2,2} \ll \sum_{j=1}^K 2^j \int_{-2^{-j}}^{2^{-j}} \left| \sum_{N < m \leq 2N} (\Lambda(m) - 1) e(m\alpha) \right|^2 d\alpha \ll \sum_{j=1}^K 2^{-j} (J(N, 2^j) + L^2 2^{3j}), \quad (15)$$

where  $J(N, h)$  is the Selberg integral. Inserting the estimate  $J(N, h) \ll h^2 N + hNL$  for all  $h \geq 1$ , see the Lemma in Languasco [7], in (15) we have

$$\Sigma_{2,2} \ll \sum_{j=1}^K 2^{-j} (2^{3j} L^2 + 2^{2j} N + 2^j NL) \ll L^2 Q^2 + NQ + NL \log Q = o(HN). \quad (16)$$

Hence, inserting (14) and (16) in (12), we finally have that (7) holds.

5. ESTIMATION OF  $\Sigma_1$ 

Inserting the identity

$$S(\alpha)^2 = (2S(\alpha)T(\alpha) - T(\alpha)^2) - \Sigma_g(\alpha)^2 - 2T(\alpha)\Sigma_g(\alpha) + 2S(\alpha)\Sigma_g(\alpha) + \Sigma_b(\alpha)^2$$

into the definition of  $\Sigma_1$ , we obtain

$$\Sigma_1 = \Sigma_{1,1} - \Sigma_{1,2} - \Sigma_{1,3} + \Sigma_{1,4} + \int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha) d\alpha, \quad (17)$$

where

$$\Sigma_{1,1} = \int_{-1/Q}^{1/Q} (2S(\alpha)T(\alpha) - T(\alpha)^2) K_H(\alpha) d\alpha,$$

$$\begin{aligned}\Sigma_{1,2} &= \int_{-1/Q}^{1/Q} \Sigma_g(\alpha)^2 K_H(\alpha) d\alpha, \\ \Sigma_{1,3} &= \int_{-1/Q}^{1/Q} 2T(\alpha) \Sigma_g(\alpha) K_H(\alpha) d\alpha \\ &\quad \text{and} \\ \Sigma_{1,4} &= \int_{-1/Q}^{1/Q} 2S(\alpha) \Sigma_g(\alpha) K_H(\alpha) d\alpha.\end{aligned}$$

In this section we will prove

$$\Sigma_{1,1} = \sum_{k \in [n, n+H)} m(k) \mathfrak{S}(k) + o(HN) \quad (18)$$

and

$$\Sigma_{1,2} = o(HN), \quad (19)$$

while the estimation of the mean-square of  $\int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha) d\alpha$  will be performed in the next section.

Assuming that (19) holds, the contribution of  $\Sigma_{1,3}$  and  $\Sigma_{1,4}$  can be estimated using the Cauchy-Schwarz inequality and

$$\int_{-1/Q}^{1/Q} |S(\alpha)|^2 d\alpha \ll N, \quad (20)$$

which can be proved using the same argument in the proof of Corollary 3 of Languasco-Perelli [9]. We obtain

$$\Sigma_{1,3} = o(HN) \quad \text{and} \quad \Sigma_{1,4} = o(HN). \quad (21)$$

Hence, by (17)-(19) and (21), we have that (5) holds.

Now we proceed to evaluate  $\Sigma_{1,1}$  and  $\Sigma_{1,2}$ .

#### Contribution of $\Sigma_{1,1}$

Squaring out we obtain

$$\int_{-1/2}^{1/2} T(\alpha)^2 K_H(\alpha) d\alpha = \sum_{k \in [n, n+H)} m(k)$$

and hence, using (11) and (13), we get

$$\int_{-1/Q}^{1/Q} T(\alpha)^2 K_H(\alpha) d\alpha = \int_{-1/2}^{1/2} T(\alpha)^2 K_H(\alpha) d\alpha + O(Q^2) = \sum_{k \in [n, n+H)} m(k) + o(HN). \quad (22)$$

Using the Prime Number Theorem, the Cauchy-Schwarz inequality and arguing analogously, we can write

$$\int_{-1/Q}^{1/Q} S(\alpha) T(\alpha) K_H(\alpha) d\alpha = \sum_{k \in [n, n+H)} m'(k) + o(HN), \quad (23)$$

where

$$m'(k) = \sum_{N < m \leq 2N} \Lambda(m) \sum_{\substack{N < h \leq 2N \\ m+h=k}} 1.$$

Again by the Prime Number Theorem, we get

$$\sum_{k \in [n, n+H)} m(k) = \sum_{k \in [n, n+H)} m'(k) + o(HN) \quad (24)$$

and hence, by (22)-(24), we have

$$\Sigma_{1,1} = \sum_{k \in [n, n+H)} m(k) + o(HN). \quad (25)$$

Using the Theorem of Languasco [8] and by partial summation, it is easy to prove

$$\sum_{k \in [n, n+H)} m(k) = \sum_{k \in [n, n+H)} m(k) \mathfrak{S}(k) + o(HN) \quad \text{for } H \geq L^{2/3+\varepsilon}. \quad (26)$$

Now (18) follows from (25) and (26).

### Contribution of $\Sigma_{1,2}$

Since

$$\Sigma_g(\alpha)^2 \ll \left| \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} T_\rho(\alpha) \right|^2 + \left| \sum_{|\gamma| > T} T_\rho(\alpha) \right|^2 + |R(\alpha)|^2,$$

we have

$$\Sigma_{1,2} \ll A_1 + A_2 + A_3, \quad (27)$$

where

$$A_1 = \int_{-1/Q}^{1/Q} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} T_\rho(\alpha) \right|^2 |K_H(\alpha)| d\alpha,$$

$$A_2 = \int_{-1/Q}^{1/Q} \left| \sum_{|\gamma| > T} T_\rho(\alpha) \right|^2 |K_H(\alpha)| d\alpha$$

and

$$A_3 = \int_{-1/Q}^{1/Q} |R(\alpha)|^2 |K_H(\alpha)| d\alpha.$$

Using (11) and Gallagher's lemma, we obtain

$$\begin{aligned} A_1 &\ll \frac{H}{Q^2} \left( \int_N^{2N} \left| \sum_{x < m < x+Q} \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} a_\rho(m) \right|^2 dx + \int_{N-Q}^N \left| \sum_{N < m < x+Q} \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} a_\rho(m) \right|^2 dx \right. \\ &\quad \left. + \int_{2N-Q}^{2N} \left| \sum_{x < m \leq 2N} \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} a_\rho(m) \right|^2 dx \right) = A_{1,1} + A_{1,2} + A_{1,3}, \end{aligned} \quad (28)$$

say. Interchanging summation and integration in  $A_{1,1}$ , we get

$$\begin{aligned} A_{1,1} &\ll \frac{H}{Q^2} \int_N^{2N} \left| \int_{[x]+1}^{[x+Q]} \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} t^{\rho-1} dt \right|^2 dx \\ &\ll \frac{H}{Q^2} \int_N^{2N} \left| \left( \int_x^{x+Q} - \int_x^{[x]+1} - \int_{[x+Q]}^{x+Q} \right) \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} t^{\rho-1} dt \right|^2 dx. \end{aligned} \quad (29)$$

To bound the contribution of the integral on  $[x, [x] + 1]$  in (29), we argue as follows.

Interchanging summation and integration, we get

$$\int_N^{2N} \left| \int_x^{[x]+1} \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} t^{\rho-1} dt \right|^2 dx \ll \sum_{N < n \leq 2N} \int_n^{n+1} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} x^\rho \frac{((n+1)/x)^\rho - 1}{\rho} \right|^2 dx$$

and then, using  $\frac{((n+1)/x)^\rho - 1}{\rho} \ll \min(\frac{1}{N}, \frac{1}{|\gamma|})$ , we have

$$\int_N^{2N} \left| \int_x^{[x]+1} \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} t^{\rho-1} dt \right|^2 dx \ll L^4 \max_{\sigma \notin I} N^{2\sigma-1} N(\sigma, \frac{N}{Q}). \quad (30)$$

To estimate the integral on  $[[x+Q], x+Q]$  in (29) we proceed analogously and hence we get

$$\int_N^{2N} \left| \int_{[x+Q]}^{x+Q} \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} t^{\rho-1} dt \right|^2 dx \ll L^4 \max_{\sigma \notin I} N^{2\sigma-1} N(\sigma, \frac{N}{Q}). \quad (31)$$

Now we treat the integral on  $[x, x+Q]$  in (29). Proceeding as above we obtain

$$\begin{aligned} \int_N^{2N} \left| \int_x^{x+Q} \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} t^{\rho-1} dt \right|^2 dx &\ll \int_N^{2N} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} x^\rho \frac{(1+Q/x)^\rho - 1}{\rho} \right|^2 dx \\ &\ll Q^2 L^4 \max_{\sigma \notin I} N^{2\sigma-1} N(\sigma, \frac{N}{Q}), \end{aligned} \quad (32)$$

where the last inequality follows by Lemma 1.

Choosing, in the definition of the interval  $I$ ,

$$a = \frac{1+3\theta}{2} - l \frac{\log L}{L} \quad \text{and} \quad b = \frac{5-3\theta}{6} + k \frac{\log L}{L}, \quad (33)$$

where  $l > \frac{27(1-\theta)}{2(1-3\theta)}$  and  $k$  is a sufficiently large constant, we have, using Ingham-Huxley's density estimate, see, *e.g.*, Ivić [6], and (29)-(33), that

$$A_{1,1} \ll H L^4 \max_{\sigma \notin I} N^{2\sigma-1} N(\sigma, \frac{N}{Q}) = o(HN). \quad (34)$$



Interchanging summation and integration in  $A_{1,2}$ , we get

$$A_{1,2} \ll \frac{H}{Q^2} \int_{N-Q}^N \left| \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} x^\rho c_{\rho,Q} \right|^2 dx,$$

where  $c_{\rho,Q} = ((\frac{[x+Q]}{x})^\rho - (\frac{N}{x})^\rho)/\rho$ . Splitting the summation according to  $|\gamma| \leq N/Q$  and  $N/Q \leq |\gamma| \leq T$  and using  $c_{\rho,Q} \ll \min(\frac{Q}{N}, \frac{1}{|\gamma|})$ , we obtain

$$\begin{aligned} A_{1,2} &\ll \frac{H}{Q^2} \left( \frac{Q^2}{N^2} \int_{N-Q}^N \left| \sum_{\substack{|\gamma| \leq N/Q \\ \beta \notin I}} x^\beta \right|^2 dx + \int_{N-Q}^N \left| \sum_{\substack{N/Q \leq |\gamma| \leq T \\ \beta \notin I}} \frac{x^\beta}{|\gamma|} \right|^2 dx \right) \\ &\ll HQL^4 \max_{\sigma \notin I} N^{2\sigma-2} N(\sigma, \frac{N}{Q})^2. \end{aligned}$$

Using Ingham-Huxley's density estimate, we see that the maximum is attained at  $\sigma = 1/2$  and hence we can write

$$A_{1,2} \ll HQL^4 N^{-1} \left( \frac{N}{Q} \right)^2 L^2 = \frac{HNL^6}{Q} = o(HN). \quad (35)$$

$A_{1,3}$  can be bounded following the lines of the estimation of  $A_{1,2}$ . We have

$$A_{1,3} = o(HN). \quad (36)$$

Inserting (34) and (35)-(36) in (28) we obtain

$$A_1 = o(HN). \quad (37)$$

Now we proceed to estimate  $A_2$ . By (11) we get

$$A_2 \ll H \int_{-1/Q}^{1/Q} \left| \sum_{N < m \leq 2N} \sum_{|\gamma| > T} a_\rho(m) e(m\alpha) \right|^2 d\alpha. \quad (38)$$

Using (38), Gallagher's lemma and the explicit formula for  $\psi(x)$ , see equations (9)-(10) in ch. 17 of Davenport [3], we have

$$A_2 \ll \frac{H}{Q^2} \int_{N-Q}^{2N} \frac{N^2 L^4}{T^2} dx \ll \frac{H N^3}{Q^2 T^2} L^4 = o(HN). \quad (39)$$

To bound  $A_3$  we use (11), Gallagher's lemma and the explicit formula for  $\psi(x)$ , see equation (1) in ch. 17 of Davenport [3]. Hence

$$\begin{aligned} A_3 &\ll \frac{H}{Q^2} \int_{N-Q}^{2N} \left| \sum_{\substack{x < m < x+Q \\ N < m \leq 2N}} (\Lambda(m) - 1 + \sum_{\rho} a_\rho(m)) \right|^2 dx \\ &\ll \frac{H}{Q^2} \int_{N-Q}^{2N} L^4 dx \ll \frac{H N L^4}{Q^2} = o(HN). \end{aligned} \quad (40)$$

Now (19) follows inserting (37) and (39)-(40) in (27).

6. MEAN-SQUARE ESTIMATE OF  $\Sigma_b(\alpha)^2$ 

Squaring out and using the definition of  $\Sigma_b(\alpha)$ , we get

$$\begin{aligned}
& \sum_{\frac{5}{2}N < n \leq \frac{7}{2}N} \left| \int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha) d\alpha \right|^2 \\
&= \sum_{\frac{5}{2}N < n \leq \frac{7}{2}N} \int_{-1/Q}^{1/Q} \left( \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\alpha) \right)^2 K_H(\alpha) d\alpha \int_{-1/Q}^{1/Q} \left( \sum_{\substack{|\gamma'| \leq T \\ \beta' \in I}} T_{\bar{\rho}'}(\delta) \right)^2 \overline{K_H}(\delta) d\delta \\
&\ll \int_{-1/Q}^{1/Q} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\alpha) \right|^2 \int_{-1/Q}^{1/Q} \left| \sum_{\substack{|\gamma'| \leq T \\ \beta' \in I}} T_{\bar{\rho}'}(\delta) \right|^2 \sum_{\frac{5}{2}N < n \leq \frac{7}{2}N} K_H(\alpha) \overline{K_H}(\delta) |d\delta d\alpha| = \Sigma_3,
\end{aligned} \tag{41}$$

say. Since  $K_H(\alpha) = \frac{\sin \pi H \alpha}{\sin \pi \alpha} e(\frac{1-H}{2}\alpha) e(-n\alpha)$ , we have

$$\Sigma_3 \ll H^2 \int_{-1/Q}^{1/Q} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\alpha) \right|^2 \left( \int_{-1/Q}^{1/Q} \left| \sum_{\substack{|\gamma'| \leq T \\ \beta' \in I}} T_{\bar{\rho}'}(\delta) \right|^2 K_N(\alpha - \delta) d\delta \right) d\alpha, \tag{42}$$

where  $K_N(t) = \sum_{\frac{5}{2}N < n \leq \frac{7}{2}N} e(-nt) \ll \min(N, \frac{1}{|t|})$ .

Using the latest estimate and (42), we obtain

$$\begin{aligned}
\Sigma_3 &\ll H^2 N \int_{-1/Q}^{1/Q} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\alpha) \right|^2 \left( \int_{(-\frac{1}{Q}, \frac{1}{Q}) \cap (\alpha - \frac{1}{N}, \alpha + \frac{1}{N})} \left| \sum_{\substack{|\gamma'| \leq T \\ \beta' \in I}} T_{\bar{\rho}'}(\delta) \right|^2 d\delta \right) d\alpha \\
&+ H^2 \int_{-1/Q}^{1/Q} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\alpha) \right|^2 \left( \int_{(-\frac{1}{Q}, \frac{1}{Q}) \setminus (\alpha - \frac{1}{N}, \alpha + \frac{1}{N})} \left| \sum_{\substack{|\gamma'| \leq T \\ \beta' \in I}} T_{\bar{\rho}'}(\delta) \right|^2 \frac{1}{|\alpha - \delta|} d\delta \right) d\alpha \tag{43} \\
&= \Sigma_{3,1} + \Sigma_{3,2},
\end{aligned}$$

say. Using (3) and arguing as in section 6, we get

$$\int_{-1/Q}^{1/Q} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\alpha) \right|^2 d\alpha \ll \int_{-1/Q}^{1/Q} |S(\alpha)|^2 d\alpha + O(N) \ll N, \tag{44}$$

where the latest inequality follows from (20).

Now, inserting (44) in  $\Sigma_{3,1}$ , we have

$$\begin{aligned}
\Sigma_{3,1} &\ll H^2 N^2 \left( \max_{\alpha \in (-1/Q, 1/Q)} \int_{(-\frac{1}{Q}, \frac{1}{Q}) \cap (\alpha - \frac{1}{N}, \alpha + \frac{1}{N})} \left| \sum_{\substack{|\gamma'| \leq T \\ \beta' \in I}} T_{\bar{\rho}'}(\delta) \right|^2 d\delta \right) \\
&\ll H^2 N \left( \max_{\delta \in (-1/Q, 1/Q)} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\delta) \right|^2 \right).
\end{aligned} \tag{45}$$

To bound  $\Sigma_{3,2}$ , we argue as for  $\Sigma_{3,1}$  and we can prove that the bound in (45) holds, with an extra  $L$  factor, for  $\Sigma_{3,2}$  too. Finally, by (41), (43), (45) and the above remark, we obtain

$$\sum_{\frac{5}{2}N \leq n \leq \frac{7}{2}N} \left| \int_{-1/Q}^{1/Q} \left( \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\alpha) \right)^2 K_H(\alpha) d\alpha \right|^2 \ll H^2 N L \left( \max_{\delta \in (-1/Q, 1/Q)} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\delta) \right|^2 \right). \quad (46)$$

Using Lemma 2 and a standard argument to bound sums over zeros of  $\zeta(s)$ , we have

$$\begin{aligned} \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\delta) &\ll L^2 \left( \max_{\substack{\sigma \in I \\ \sigma < 7/9}} N^\sigma \max_{|t| \leq T} N(\sigma, t) |t|^{-1/2} + \max_{\substack{\sigma \in I \\ \sigma \geq 7/9}} N^\sigma \max_{|t| \leq T} N(\sigma, t) |t|^{-1/2} \right) \\ &\ll L^2 \left( \max_{\substack{\sigma \in I \\ \sigma < 7/9}} N^\sigma N(\sigma, T) T^{-1/2} + \max_{\substack{\sigma \in I \\ \sigma \geq 7/9}} N^\sigma \right). \end{aligned} \quad (47)$$

By Ingham-Huxley's density estimate, we have that the first maximum is attained at  $\sigma = a$  and the second at  $\sigma = b$ . Hence, by (46) and (47), we see that (6) holds.

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