

Robustness in a posteriori error analysis for FEM flow models

*Original*

Robustness in a posteriori error analysis for FEM flow models / Berrone, Stefano. - In: NUMERISCHE MATHEMATIK. - ISSN 0029-599X. - 91:(2002), pp. 389-422.

*Availability:*

This version is available at: 11583/1397685 since:

*Publisher:*

*Published*

DOI:

*Terms of use:*

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

*Publisher copyright*

(Article begins on next page)

# Robustness in *a posteriori* error analysis for FEM flow models

**Stefano Berrone**

Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy, e-mail: sberrone@calvino.polito.it

**Abstract** We derive a residual-based *a posteriori* error estimator for a stabilized finite element discretization of certain incompressible Oseen-like equations. We focus our attention on the behaviour of the effectivity index and we carry on a numerical study of its sensitiveness to the problem and mesh parameters. We also consider a scalar reaction-convection-diffusion problem and a divergence-free projection problem in order to investigate the effects on the robustness of our *a posteriori* error estimator of the reaction-convection-diffusion phenomena and, separately, of the incompressibility constraint.

**Key words** *A posteriori* error estimators – incompressible flows – bounds on the effectivity index.

*Mathematics Subject Classification (1991):* 65N30, 65N15, 65N50, 76D05, 76M10.

## 1 Introduction

The use of adaptive methods for the numerical discretization of flow models is a subject of strong interest from both a theoretical and an applicative point of view. From the pioneering work of Babuška and Rheinboldt [2], many important problems have been solved and interesting results have been achieved. Many other questions, concerning, e.g., saddle point problems and singularly perturbed problems with parameter becoming very small or very large, are still open. One of them is to find robust *a posteriori* error estimates. The robustness of

an *a posteriori* error estimate concerns the upper and lower bounds for the effectivity index defined as the ratio between the error estimator and the true error. The ideal situation is when the effectivity index is uniformly bounded from above and from below with respect to any mesh-size and any parameter of the problem. Such a strong robustness implies that one can easily build an adaptive algorithm which guarantees reliability, by controlling the error in the solution from above, and efficiency, by controlling the error from below.

In [3], we considered the stationary Oseen equations and we obtained a uniform lower bound for the inverse of the effectivity index and an upper bound which grows linearly with the Reynolds number.

Here, we consider the generalized stationary Oseen equations obtained by adding a zero-order term in the velocity to the momentum equation. This model has already been considered by several authors, e.g. [13], [15]. A zero-order term can be produced by a semi-discretization in time. Another source is a shift of the operator to efficiently deal with the non-linearity of the momentum equation of the Navier-Stokes problem by a Newton-like method. A third motivation is a shift of the spectrum of the operator in the numerical computation of eigenvalues and eigenfunctions.

For this model, we analytically derive an error estimator and we report many numerical tests. Our goal is to carefully study the dependence of the bounds for the effectivity index from above and from below on all the parameters of the problem (the physical as well as the mesh parameters). We carefully control the coefficients appearing in each inequality and our final estimates can be considered as sharp as possible. Sharpness is proved by the fact that our numerical tests essentially confirm the predicted theoretical behaviour of the effectivity index.

Furthermore, we highlight the effect of the different physical phenomena modeled by the Oseen equation upon the robustness of the *a posteriori* error estimate. One of these phenomena is the mixing of diffusion and transport. Therefore we compare our techniques with those presented in [20] for a scalar reaction-convection-diffusion model. From this comparison, we see that our techniques leads to estimates as sharp as those in [20] which can be considered the *state of the art* for residual based *a posteriori* errors estimates for reaction-convection-diffusion models. Another important phenomenon is the incompressibility of the flow. To study its effect upon robustness we consider a reduced model obtained from the Oseen model by neglecting the diffusion and convection terms. This leads to a divergence-free projection model [4], useful also in linear elasticity theory [5],

for which we derive several *a posteriori* error estimators. Then, we characterize those norms for the true error that yield a robust error estimator.

## 2 Linear incompressible flow model

### 2.1 The continuous problem

We consider the following steady-state, Oseen-like problem:

$$-\frac{1}{Re} \Delta u + (a \cdot \nabla) u + z u + \nabla p = f \quad \text{in } \Omega, \quad (2.1)$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega, \quad (2.2)$$

$$u = 0 \quad \text{on } \partial\Omega, \quad (2.3)$$

where:  $Re$  is the Reynolds number;  $z \geq 0$  is a constant in the whole domain;  $\Omega$  is a bounded Lipschitz continuous domain in  $\mathbb{R}^2$ ;  $a \in [H^1(\Omega)]^2 \cap [L^\infty(\Omega)]^2$  with  $\nabla \cdot a = 0$  in  $\bar{\Omega}$ ;  $f \in [L^2(\Omega)]^2$ .

Let us first derive a weak formulation of problem (2.1)-(2.3). The functional spaces we deal with are the usual Sobolev spaces  $H^1(\Omega)$ ,  $H_0^1(\Omega)$  and Lebesgue spaces  $L^2(\Omega)$ ,  $L_0^2(\Omega)$ . Moreover we set  $V \stackrel{def}{=} [H_0^1(\Omega)]^2$  and  $Q \stackrel{def}{=} L_0^2(\Omega)$ . The weak formulation of the problem is: *Find*  $[u, p] \in V \times Q$  *such that*  $\forall [v, q] \in V \times Q$

$$\frac{1}{Re} (\nabla u, \nabla v) + ((a \cdot \nabla) u, v) + z (u, v) - (p, \nabla \cdot v) = (f, v), \quad (2.4)$$

$$(q, \nabla \cdot u) = 0, \quad (2.5)$$

where  $(\cdot, \cdot)$  denotes the usual inner product in  $L^2(\Omega)$  or in  $[L^2(\Omega)]^2$ . As usual  $\|\cdot\|_0$  denotes the  $L^2$ -norm,  $\|\cdot\|_1$  the  $H^1$ -norm and  $|\cdot|_1$  the  $H^1$ -seminorm. We define our energy norm for the velocity on some  $\omega \subseteq \Omega$  in the following manner:

$$\|u\|_\omega^2 \stackrel{def}{=} \frac{1}{Re} |u|_{1,\omega}^2 + z \|u\|_{0,\omega}^2. \quad (2.6)$$

Existence and uniqueness of the solution for all positive  $Re$  follows from the classical coercivity and *inf* – *sup* inequality:

$$\inf_{q \in Q \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{(q, \nabla \cdot v)}{\|q\|_0 |v|_1} \geq \beta \quad (2.7)$$

(see, e.g., [10], [13], [14]).

## 2.2 The discrete problem

In order to discretize problem (2.1)-(2.3), we assume  $\Omega$  to be a polygonal domain and we introduce a regular family of partitions  $\{\mathcal{T}_h\}_h$  of  $\overline{\Omega}$  into triangles which satisfies the usual conformity and minimal-angle conditions [6]. It is useful to introduce the diameter  $h_T$  of the element  $T \in \mathcal{T}_h$ . Then, the parameter  $h$  of the family  $\{\mathcal{T}_h\}_h$  is  $h = \max_{T \in \mathcal{T}_h} h_T$ .

In what follows, we are going to use continuous finite elements for the velocity and the pressure:

$$\mathbf{V}_h \stackrel{\text{def}}{=} \left\{ v_h \in \mathbf{V} \cap [\mathcal{C}^0(\overline{\Omega})]^2 : v_h|_T \in [\mathbb{P}_k(T)]^2, \forall T \in \mathcal{T}_h \right\}, \quad (2.8)$$

$$\mathbf{Q}_h \stackrel{\text{def}}{=} \left\{ q_h \in \mathbf{Q} \cap \mathcal{C}^0(\overline{\Omega}) : q_h|_T \in \mathbb{P}_l(T), \forall T \in \mathcal{T}_h \right\}, \quad (2.9)$$

where  $\mathbb{P}_i(T)$  is the space of polynomials of degree  $i \geq 1$  on the element  $T \in \mathcal{T}_h$ . In the discretization of the problem, we also consider approximations of the data  $a, f$  by some projections  $\Pi_T a, \Pi_T f$ , whose definition will be given later on.

With an arbitrary choice of  $k$  and  $l$  these spaces need not satisfy the discrete *inf-sup* condition for the bilinear form  $(p_h, \nabla \cdot v_h)$  [4], [10]. However, this may be avoided by resorting to a consistently modified approximation of the problem known as the *Streamline Upwind/Petrov Galerkin (SUPG)* method [8], [9]: *Find*  $[u_h, p_h] \in \mathbf{V}_h \times \mathbf{Q}_h$  such that  $\forall [v_h, q_h] \in \mathbf{V}_h \times \mathbf{Q}_h$

$$\begin{aligned} & \frac{1}{Re} (\nabla u_h, \nabla v_h) + ((\Pi_T a \cdot \nabla) u_h, v_h) + z (u_h, v_h) - (p_h, \nabla \cdot v_h) + \\ & + \sum_{T \in \mathcal{T}_h} \tau_T \left( -\frac{1}{Re} \Delta u_h + (\Pi_T a \cdot \nabla) u_h + z u_h + \nabla p_h, (\Pi_T a \cdot \nabla) v_h \right)_T \\ & + \sum_{T \in \mathcal{T}_h} \delta_T (\nabla \cdot u_h, \nabla \cdot v_h) = (\Pi_T f, v_h) \\ & + \sum_{T \in \mathcal{T}_h} \tau_T (\Pi_T f, (\Pi_T a \cdot \nabla) v_h)_T, \quad (2.10) \end{aligned}$$

$$\begin{aligned} & (q_h, \nabla \cdot u_h) + \\ & + \sum_{T \in \mathcal{T}_h} \tau_T \left( -\frac{1}{Re} \Delta u_h + (\Pi_T a \cdot \nabla) u_h + z u_h + \nabla p_h, \nabla q_h \right)_T = \\ & = \sum_{T \in \mathcal{T}_h} \tau_T (\Pi_T f, \nabla q_h)_T, \quad (2.11) \end{aligned}$$

The parameters  $\tau_T$  and  $\delta_T$  depend on the local conditions of the flow in each element, i.e., following [8]:

$\tau_T \stackrel{\text{def}}{=} m_k \frac{h_T^2}{8} Re$  and  $\delta_T \stackrel{\text{def}}{=} \lambda m_k \frac{\| \Pi_T a \|_{\infty, T} h_T^2 Re}{4}$  if  $Re_T < 1$ , whereas  $\tau_T \stackrel{\text{def}}{=} \frac{h_T}{2 \| \Pi_T a \|_{\infty, T}}$  and  $\delta_T \stackrel{\text{def}}{=} \lambda \| \Pi_T a \|_{\infty, T} h_T$  if  $Re_T \geq 1$ . Here  $Re_T \stackrel{\text{def}}{=} m_k \frac{\| \Pi_T a \|_{\infty, T} h_T Re}{4}$  and  $m_k \stackrel{\text{def}}{=} \min \left\{ \frac{1}{3}, \frac{2}{C_*} \right\}$ ,  $C_*$  being the constant of the inverse inequality [11]:  $h_T^2 \| \Delta v_h \|_{0, T}^2 \leq C_* \| \nabla v_h \|_{0, T}^2$ ,  $\forall v_h \in V_h$ . For linear elements, obviously,  $m_k = \frac{1}{3}$ . We take  $\lambda$  either 1 or 0, respectively if we want consider the  $\delta$ -terms or not.

Throughout the paper, we often use the following notations:

**Notation 1** For each  $\xi, \eta > 0$ :  $\xi \lesssim \eta \iff \exists C > 0 : \xi \leq C \eta$ ;  $\xi \asymp \eta \iff \xi \lesssim \eta$  and  $\eta \lesssim \xi$ . Without further specification, we intend the constant  $C$  independent of the mesh-size and the Reynolds number. Moreover for each  $\xi \geq 0, \eta > 0$ :  $\xi \preceq \eta \iff \exists C_1 \geq 0 : \xi \leq C_1 \eta$  with a constant  $C_1$  at most of the order of magnitude of the unity.

*Remark 1* We assume that problem (2.1)-(2.3) has been written in non-dimensional variables. This implies  $|\Omega| \preceq 1$  so that  $h_T \preceq 1$ ,  $\forall T \in \mathcal{T}_h$ ; moreover,  $\| a \|_{\infty, \omega_T} \preceq 1$  and  $\| \Pi_T a \|_{\infty, \omega_T} \preceq 1$ ,  $\forall T \in \mathcal{T}_h$ .

### 3 A residual-based error estimator

In this section, we derive a residual-based error estimator for our model problem following Verfürth's works [16], [17], [18], [19], [20]. Particularly, we shall derive a global upper bound and a local lower bound for the error measured in an energy-like norm. At first, we introduce some notation which will be used for the construction of the estimator.

#### 3.1 Definitions and general results

For any  $T \in \mathcal{T}_h$  we denote by  $\mathcal{E}(T)$  the set of its edges; we denote by  $\mathcal{E}_h \stackrel{\text{def}}{=} \bigcup_{T \in \mathcal{T}_h} \mathcal{E}(T)$  the set of all edges of the triangulation. Moreover, we define  $\mathcal{E}_{h, \Omega} \stackrel{\text{def}}{=} \{ E \in \mathcal{E}_h : E \not\subset \partial \Omega \}$ . For each triangle  $T \in \mathcal{T}_h$  and for each side  $E \in \mathcal{E}_h$  we define:  $\omega_T = \bigcup_{\{T' : \mathcal{E}(T) \cap \mathcal{E}(T') \neq \emptyset\}} T'$ ,  $\omega_E = \bigcup_{\{T' : E \in \mathcal{E}(T')\}} T'$ ,  $\tilde{\omega}_T = \bigcup_{\{T' : \partial T \cap \partial T' \neq \emptyset\}} T'$ ,  $\tilde{\omega}_E = \bigcup_{\{T' : E \cap \partial T' \neq \emptyset\}} T'$ .

Note that the sets  $\omega_T$  and  $\omega_E$  are unions of triangles that share at least one edge with  $T$  or  $E$  respectively, whereas the sets  $\tilde{\omega}_T$  and  $\tilde{\omega}_E$  are unions of triangles that share at least one point with  $T$  or  $E$ .

For each edge  $E \in \mathcal{E}_h$  we consider a unit vector  $\hat{n}_E$  such that  $\hat{n}_E$  is orthogonal to  $E$ . Given any  $E \in \mathcal{E}_{h, \Omega}$  and any  $\varphi \in L^2(\omega_E)$  with

$\varphi|_{T'} \in \mathcal{C}^0(T') \forall T' \in \omega_E$ , we denote by  $[\varphi]_E$  the jump of  $\varphi$  across  $E$  along the orientation of  $\hat{n}_E$ .

If the minimal angle of the family  $\{\mathcal{T}_h\}_h$  is bounded away from zero, there exist constants only dependent on the smallest angle in the triangulation such that:  $|T| \asymp h_T^2, \forall T \in \mathcal{T}_h, h_T \asymp h_E, \forall E \in \mathcal{E}(T), |T| \asymp h_E^2, \forall T \in \omega_E$ . Let us denote by  $\hat{T}$  the reference triangle, by  $\hat{E}$  the reference edge, i.e. the edge of  $\hat{T}$  between the vertices 0 and 1. Moreover, let  $\hat{b}_{\hat{T}}(\hat{x}, \hat{y})$  be the usual *reference triangle bubble function* and let  $\hat{b}_{\hat{E}}$  be the usual *reference edge bubble function* [16], [17], [18].

Let  $F_T : \hat{T} \rightarrow T$  be the invertible affine mapping that maps the reference triangle  $\hat{T}$  onto the triangle  $T$ . Then we define the *triangle bubble function*  $b_T$  by:  $b_T \stackrel{def}{=} \hat{b}_{\hat{T}} \circ F_T^{-1}$ . Given any  $E \in \mathcal{E}_{h,\Omega}$  with  $\omega_E = T_{\downarrow} \cup T_{\uparrow}$ , let us enumerate the vertices of  $T_{\downarrow}$  and  $T_{\uparrow}$  counterclockwise in such a way that the vertices of  $E$  are numbered first. Then we define the *edge bubble function*  $b_E$  by patching the two bubble functions:  $b_{E,T_{\downarrow}} \stackrel{def}{=} \hat{b}_{\hat{E}} \circ F_{T_{\downarrow}}^{-1}$  and  $b_{E,T_{\uparrow}} \stackrel{def}{=} \hat{b}_{\hat{E}} \circ F_{T_{\uparrow}}^{-1}$ , each one being not zero only inside  $T_{\downarrow}$  and  $T_{\uparrow}$ , respectively. Moreover, for the reference edge  $\hat{E}$  we define the extension operator  $\hat{\mathcal{P}}_{\hat{E}} : P_i(\hat{E}) \rightarrow P_i(\hat{T})$  which extends a polynomial of degree  $i$  defined on the edge  $\hat{E}$  to a polynomial of the same degree defined on  $\hat{T}$  with constant values along lines orthogonal to the edge  $\hat{E}$ . Then, we define the extension operator  $\mathcal{P}_E : P_i(E) \rightarrow P_i(\omega_E)$  which extends a polynomial of degree  $i$  defined on the edge  $E$  to a piecewise polynomial of the same degree defined on  $\omega_E$  by patching the two operators:  $\mathcal{P}_E(\cdot)|_{T_{\downarrow}} \stackrel{def}{=} \hat{\mathcal{P}}_{\hat{E}}(\cdot \circ F_{T_{\downarrow}}|_{\hat{E}}) \circ F_{T_{\downarrow}}^{-1}|_E$  and  $\mathcal{P}_E(\cdot)|_{T_{\uparrow}} \stackrel{def}{=} \hat{\mathcal{P}}_{\hat{E}}(\cdot \circ F_{T_{\uparrow}}|_{\hat{E}}) \circ F_{T_{\uparrow}}^{-1}|_E$ .

Besides, we denote by  $I_h : V \rightarrow V_h$  the *quasi-interpolation operator of Clément* [7] which satisfies the following approximation properties [6]:

**Lemma 1** *Let  $T \in \mathcal{T}_h$  and  $E \in \mathcal{E}_h$  be arbitrary, then*

$$|v - I_h v|_{l,T} \lesssim h_T^{k-l} |v|_{k,\tilde{\omega}_T}, \quad 0 \leq l \leq k \leq 2, \quad \forall v \in H^k(\tilde{\omega}_T), \quad (3.1)$$

$$\|v - I_h v\|_{0,E} \lesssim h_E^{k-\frac{1}{2}} |v|_{k,\tilde{\omega}_E}, \quad 1 \leq k \leq 2, \quad \forall v \in H^k(\tilde{\omega}_E), \quad (3.2)$$

$$|I_h v|_{k,T} \lesssim |v|_{k,\tilde{\omega}_T}, \quad 1 \leq k \leq 2, \quad \forall v \in H^k(\tilde{\omega}_T), \quad (3.3)$$

where the constants depend only on the smallest angle in the triangulation.

**Definition 1** *We define*

$$\alpha_{1,S} \stackrel{def}{=} \min \left\{ \sqrt{Re} h_S, \frac{1}{\sqrt{z}} \right\}, \quad S = T \in \mathcal{T}_h \text{ or } S = E \in \mathcal{E}_h, \quad (3.4)$$

$$\alpha_1 \stackrel{\text{def}}{=} \min \left\{ \sqrt{Re} h, \frac{1}{\sqrt{z}} \right\} = \max_{T \in \mathcal{T}_h} \alpha_{1,T}, \quad (3.5)$$

$$\alpha_2 \stackrel{\text{def}}{=} \frac{1}{\sqrt{Re}} + \left( \|a\|_{\infty, \Omega} + z \right) \min \left\{ \sqrt{Re}, \frac{1}{\sqrt{z}} \right\}. \quad (3.6)$$

**Lemma 2** *Let  $T \in \mathcal{T}_h$  be arbitrary, then*

$$\|v - I_h v\|_{0,T}^2 \lesssim \alpha_{1,T}^2 \|v\|_{\tilde{\omega}_T}^2, \quad \forall v \in [\mathbf{H}^1(\tilde{\omega}_T)]^2. \quad (3.7)$$

*Proof.* The proof is a consequence of inequality (3.1) of Lemma 1 and definition (2.6).  $\square$

**Lemma 3** *Let  $E \in \mathcal{E}_h$  be arbitrary, then*

$$\|v - I_h v\|_{0,E}^2 \lesssim \sqrt{Re} \alpha_{1,E} \|v\|_{\tilde{\omega}_T}^2, \quad \forall v \in [\mathbf{H}^1(\tilde{\omega}_T)]^2. \quad (3.8)$$

*Proof.* The proof follows from Lemma 3.1 in [20] and Lemma 1.  $\square$

**Definition 2** *We define the following useful notation*

$$\begin{aligned} \mathcal{Y} &\stackrel{\text{def}}{=} u_h - u, & \Psi &\stackrel{\text{def}}{=} p_h - p, \\ R_{T,h} &\stackrel{\text{def}}{=} -\frac{1}{Re} \Delta u_h + (\Pi_T a \cdot \nabla) u_h + z u_h + \nabla p_h - \Pi_T f \Big|_T, \\ J_{E,h} &\stackrel{\text{def}}{=} \left[ \hat{n}_E \cdot \left( \frac{1}{Re} \nabla u_h - p_h \mathbf{I} \right) \right]_E. \end{aligned}$$

### 3.2 Global upper bound

We deal separately with the velocity error  $\mathcal{Y}$  and the pressure error  $\Psi$  to derive a global upper bound for the error.

**Lemma 4** *Let  $T \in \mathcal{T}_h$  be arbitrary. The following inequality holds*

$$\|(\Pi_T a \cdot \nabla)(I_h \mathcal{Y})\|_{0,T} \lesssim \|\Pi_T a\|_{\infty, T} \frac{\alpha_{1,T}}{h_T} \|\mathcal{Y}\|_{\tilde{\omega}_T}.$$

*Proof.* We use Lemma 1 and the local inverse inequality

$$\begin{aligned} \|\nabla(I_h \mathcal{Y})\|_{0,T} &\leq \|\nabla(\mathcal{Y} - I_h \mathcal{Y})\|_{0,T} + \|\nabla \mathcal{Y}\|_{0,T} \lesssim \sqrt{Re} \|\mathcal{Y}\|_{\tilde{\omega}_T}, \\ \|\nabla(I_h \mathcal{Y})\|_{0,T} &\lesssim h_T^{-1} \|I_h \mathcal{Y}\|_{0,T} \lesssim h_T^{-1} \|\mathcal{Y}\|_{0, \tilde{\omega}_T} \leq \frac{1}{h_T z} \|\mathcal{Y}\|_{\tilde{\omega}_T} \end{aligned}$$

to conclude that

$$\|\nabla(I_h \mathcal{Y})\|_{0,T} \lesssim \frac{\alpha_{1,T}}{h_T} \|\mathcal{Y}\|_{\tilde{\omega}_T}.$$

Then the thesis comes immediately.  $\square$

**Proposition 1** *There exists a positive constant  $C_{\mathcal{Y}}$  such that, for each  $k_1 > 0$ , the following upper bound for the velocity error holds*

$$\begin{aligned} \|\mathcal{Y}\|_{\Omega} &\leq \frac{1}{\sqrt{k_1}} \|\Psi\|_0 + \sqrt{k_1} \|\nabla \cdot u_h\|_0 \\ + C_{\mathcal{Y}} &\left( \sqrt{\sum_{T \in \mathcal{T}_h} \alpha_{1,T}^2 \left(1 + \frac{\tau_T^2 \|\Pi_T a\|_{\infty,T}^2}{h_T^2}\right)} \|R_{T,h}\|_{0,T}^2 \right. \\ &\quad \left. + \sqrt{\sum_{T \in \mathcal{T}_h} \alpha_{1,T}^2 \frac{\delta_T^2}{h_T^2}} \|\nabla \cdot u_h\|_{0,T}^2 \right. \\ &\quad \left. + \alpha_1 \|\Pi_T a - a\|_{\infty} |u_h|_1 + \alpha_1 \|\Pi_T f - f\|_0 \right). \end{aligned} \quad (3.9)$$

*Proof.* From the continuous momentum equation (2.1) we get:

$$\begin{aligned} \frac{1}{Re} (\nabla \mathcal{Y}, \nabla v) + ((a \cdot \nabla) \mathcal{Y}, v) + z(\mathcal{Y}, v) - (\Psi, \nabla \cdot v) &= \frac{1}{Re} (\nabla u_h, \nabla v) \\ + ((a \cdot \nabla) u_h, v) + z(u_h, v) - (p_h, \nabla \cdot v) - (f, v), \quad \forall v \in V. \end{aligned} \quad (3.10)$$

Now we take  $v = \mathcal{Y}$  as a test function in this equation and we add to it equation (2.10) with  $v_h = I_h \mathcal{Y}$  as a test function. We apply repeatedly the Cauchy-Schwarz inequality and we use the inequalities of Lemmas 2, 3, 4, definition (2.6) and Young's inequality to obtain

$$\begin{aligned} \|\mathcal{Y}\|_{\Omega}^2 - \frac{1}{2k_1} \|\Psi\|_0^2 - \frac{k_1}{2} \|\nabla \cdot \mathcal{Y}\|_0^2 &\leq C_{\mathcal{Y}}^* \left( \sum_{T \in \mathcal{T}_h} \frac{1}{2k_2} \alpha_{1,T}^2 \|R_{T,h}\|_{0,T}^2 \right. \\ &\quad \left. + \sum_{T \in \mathcal{T}_h} \frac{k_2}{2} \|\mathcal{Y}\|_{\tilde{\omega}_T}^2 + \sum_{E \in \mathcal{E}_{h,\Omega}} \frac{1}{2k_3} \alpha_{1,E} \sqrt{Re} \|J_{E,h}\|_{0,E}^2 \right. \\ + \sum_{E \in \mathcal{E}_{h,\Omega}} \frac{k_3}{2} \|\mathcal{Y}\|_{\tilde{\omega}_E}^2 &+ \sum_{T \in \mathcal{T}_h} \frac{1}{2k_4} \left( \tau_T \|\Pi_T a\|_{\infty,T} \frac{\alpha_{1,T}}{h_T} \|R_{T,h}\|_{0,T} \right)^2 \\ &\quad \left. + \sum_{T \in \mathcal{T}_h} \frac{k_4}{2} \|\mathcal{Y}\|_{\tilde{\omega}_T}^2 + \sum_{T \in \mathcal{T}_h} \frac{1}{2k_5} \left( \delta_T \frac{\alpha_{1,T}}{h_T} \|\nabla \cdot u_h\|_{0,T} \right)^2 \right. \\ + \sum_{T \in \mathcal{T}_h} \frac{k_5}{2} \|\mathcal{Y}\|_{\tilde{\omega}_T}^2 &+ \frac{1}{2k_6} \alpha_1^2 \|\Pi_T a - a\|_{\infty}^2 |u_h|_1^2 + \frac{k_6}{2} \|\mathcal{Y}\|_{\Omega}^2 \\ &\quad \left. + \frac{1}{2k_7} \alpha_1^2 \|\Pi_T f - f\|_0^2 + \frac{k_7}{2} \|\mathcal{Y}\|_{\Omega}^2 \right). \end{aligned}$$

The thesis follows choosing the constants  $k_2, k_3, k_4, k_5, k_6, k_7$  small enough and recalling that  $\nabla \cdot u = 0$ .  $\square$

**Proposition 2** *There exists a positive constant  $C_\Psi$  such that the following upper bound for the pressure error holds*

$$\begin{aligned} \beta \|\Psi\|_0 \leq C_\Psi & \left\{ \alpha_2 \|\Upsilon\|_\Omega + \sqrt{\sum_{T \in \mathcal{T}_h} h_T^2 \left(1 + \frac{\tau_T^2 \|\Pi_T a\|_{\infty, T}^2}{h_T^2}\right)} \|R_{T,h}\|_{0,T}^2 \right. \\ & + \sqrt{\sum_{T \in \mathcal{T}_h} h_T^2 \frac{\delta_T^2}{h_T^2} \|\nabla \cdot u_h\|_{0,T}^2} + \sqrt{\sum_{E \in \mathcal{E}_{h,\Omega}} h_E \|J_{E,h}\|_{0,E}^2} \\ & \left. + \|\Pi_T a - a\|_\infty |u_h|_1 + \|\Pi_T f - f\|_0 \right\}. \quad (3.11) \end{aligned}$$

*Proof.* From the continuous *inf-sup* condition and equation (3.10) we get

$$\begin{aligned} \beta \|\Psi\|_0 \leq \sup_{v \in V \setminus \{0\}} \frac{(\Psi, \nabla \cdot v)}{|v|_1} &= \sup_{v \in V \setminus \{0\}} \frac{1}{|v|_1} \left\{ \frac{1}{Re} (\nabla \Upsilon, \nabla v) \right. \\ &+ ((a \cdot \nabla) \Upsilon, v) + z(\Upsilon, v) - \frac{1}{Re} (\nabla u_h, \nabla v) - ((a \cdot \nabla) u_h, v) \\ &\left. - z(u_h, v) + (p_h, \nabla \cdot v) + (f, v) \right\}. \end{aligned}$$

Now we bound the supremum of the expression in brackets by the sum of the suprema of the first, the second and the remaining terms; next, we integrate by parts the term  $-\frac{1}{Re} (\nabla u_h, \nabla v)$  and we add the discrete version of the momentum equation as before. Finally, we apply Poincaré-Friedrichs inequality, Cauchy-Schwarz inequality, Lemma 1 and we get

$$\begin{aligned} & \beta \|\Psi\|_0 - \frac{1}{Re} |\Upsilon|_1 - (\|a\|_\infty + z) \|\Upsilon\|_0 \leq \\ & \leq C_\Psi \sup_{v \in V \setminus \{0\}} \frac{1}{|v|_1} \left\{ \sqrt{\sum_{T \in \mathcal{T}_h} h_T^2 \|R_{T,h}\|_{0,T}^2} \sqrt{\sum_{T \in \mathcal{T}_h} |v|_{1,\tilde{\omega}_T}^2} \right. \\ & \quad + \sqrt{\sum_{E \in \mathcal{E}_{h,\Omega}} h_E \|J_{E,h}\|_{0,E}^2} \sqrt{\sum_{E \in \mathcal{E}_{h,\Omega}} |v|_{1,\tilde{\omega}_E}^2} \\ & \quad + \sqrt{\sum_{T \in \mathcal{T}_h} \tau_T^2 \|R_{T,h}\|_{0,T}^2 \|\Pi_T a\|_{\infty, T}^2} \sqrt{\sum_{T \in \mathcal{T}_h} |v|_{1,\tilde{\omega}_T}^2} \\ & \quad + \sqrt{\sum_{T \in \mathcal{T}_h} \delta_T^2 \|\nabla \cdot u_h\|_{0,T}^2} \sqrt{\sum_{T \in \mathcal{T}_h} |v|_{1,\tilde{\omega}_T}^2} \\ & \quad \left. + \|\Pi_T a - a\|_\infty |u_h|_1 |v|_1 + \|\Pi_T f - f\|_0 |v|_1 \right\}. \end{aligned}$$

Then, it is easy to get (3.11).  $\square$

With a suitable choice of  $k_1$ , expressions (3.9) and (3.11) may be merged to get independent upper bounds for  $\mathcal{Y}$  and  $\Psi$ .

**Proposition 3** *The following upper bounds hold*

$$\begin{aligned} \|\mathcal{Y}\|_{\Omega} &\lesssim \sqrt{\sum_{T \in \mathcal{T}_h} \left( \frac{h_T^2}{\alpha_2^2} + \alpha_{1,T}^2 \right) \left( 1 + \frac{\tau_T^2 \|\Pi_T a\|_{\infty,T}^2}{h_T^2} \right) \|R_{T,h}\|_{0,T}^2} \\ &\quad + \sqrt{\sum_{T \in \mathcal{T}_h} \left( \frac{h_T^2}{\alpha_2^2} + \alpha_{1,T}^2 \right) \frac{\delta_T^2}{h_T^2} \|\nabla \cdot u_h\|_{0,T}^2 + \alpha_2 \|\nabla \cdot u_h\|_0} \\ &\quad + \sqrt{\sum_{E \in \mathcal{E}_{h,\Omega}} \left( \frac{h_E}{\alpha_2^2} + \alpha_{1,E} \sqrt{Re} \right) \|J_{E,h}\|_{0,E}^2} \\ &\quad + \left( \frac{1}{\alpha_2} + \alpha_1 \right) (\|\Pi_T a - a\|_{\infty} |u_h|_1 + \|\Pi_T f - f\|_0) \quad (3.12) \end{aligned}$$

and

$$\begin{aligned} \|\Psi\|_0 &\lesssim \alpha_2 \left\{ \sqrt{\sum_{T \in \mathcal{T}_h} \left( \frac{h_T^2}{\alpha_2^2} + \alpha_{1,T}^2 \right) \left( 1 + \frac{\tau_T^2 \|\Pi_T a\|_{\infty,T}^2}{h_T^2} \right) \|R_{T,h}\|_{0,T}^2} \right. \\ &\quad + \sqrt{\sum_{T \in \mathcal{T}_h} \left( \frac{h_T^2}{\alpha_2^2} + \alpha_{1,T}^2 \right) \frac{\delta_T^2}{h_T^2} \|\nabla \cdot u_h\|_{0,T}^2 + \alpha_2 \|\nabla \cdot u_h\|_0} \\ &\quad + \sqrt{\sum_{E \in \mathcal{E}_{h,\Omega}} \left( \frac{h_E}{\alpha_2^2} + \alpha_{1,E} \sqrt{Re} \right) \|J_{E,h}\|_{0,E}^2} \\ &\quad \left. + \left( \frac{1}{\alpha_2} + \alpha_1 \right) (\|\Pi_T a - a\|_{\infty} |u_h|_1 + \|\Pi_T f - f\|_0) \right\}. \quad (3.13) \end{aligned}$$

### 3.3 Local lower bound

**3.3.1 Residual of the momentum equation** Now we consider an arbitrary triangle  $T \in \mathcal{T}_h$  and we show how the residual of the momentum equation can bound the error from below on  $T$ . Let us define:  $w_T \stackrel{def}{=} R_{T,h} b_T$ , where  $b_T$  is the *triangle bubble function*. We will also apply the results collected in the following lemmas:

**Lemma 5** For any  $T \in \mathcal{T}_h$  we have

$$\|R_{T,h}\|_{0,T}^2 \asymp (R_{T,h}, w_T)_T, \quad (3.14)$$

$$\|w_T\|_{0,T} \leq \|R_{T,h}\|_{0,T}, \quad (3.15)$$

$$\|\nabla w_T\|_{0,T} \asymp h_T^{-1} \|w_T\|_{0,T} \leq h_T^{-1} \|R_{T,h}\|_{0,T}. \quad (3.16)$$

*Proof.* The proof of these inequalities follows the guidelines of [18], [19], [20].  $\square$

**Lemma 6** The following bound holds

$$|((a \cdot \nabla) \mathcal{Y}, w_T)_T| \lesssim \|a\|_{\infty,T} \frac{\alpha_{1,T}}{h_T} \|\mathcal{Y}\|_T \|R_{T,h}\|_{0,T}. \quad (3.17)$$

*Proof.* We observe that

$$((a \cdot \nabla) \mathcal{Y}, w_T)_T = -(\mathcal{Y}, (a \cdot \nabla) w_T)_T$$

and

$$\begin{aligned} |((a \cdot \nabla) \mathcal{Y}, w_T)_T| &\leq \|a\|_{\infty,T} \sqrt{Re} \|\mathcal{Y}\|_T \|R_{T,h}\|_{0,T}, \\ |(\mathcal{Y}, (a \cdot \nabla) w_T)_T| &\leq \frac{1}{\sqrt{z}} \|\mathcal{Y}\|_T \|a\|_{\infty,T} h_T^{-1} \|R_{T,h}\|_{0,T} \end{aligned}$$

so the thesis follows recalling Definition 1.  $\square$

**Proposition 4** The following lower bound on each element  $T$  holds

$$\begin{aligned} h_T \|R_{T,h}\|_{0,T} &\gtrsim \left( \frac{1}{\sqrt{Re}} + \|a\|_{\infty,T} \alpha_{1,T} + h_T \sqrt{z} \right) \|\mathcal{Y}\|_T \\ &+ \|\Psi\|_{0,T} + h_T \left( \|\Pi_T a - a\|_{\infty,T} \|u_h\|_{1,T} + \|\Pi_T f - f\|_{0,T} \right). \end{aligned} \quad (3.18)$$

*Proof.* We have

$$\begin{aligned} (R_{T,h}, w_T)_T &= -\frac{1}{Re} (\Delta u_h, w_T)_T + ((\Pi_T a \cdot \nabla) u_h, w_T)_T \\ &+ z (u_h, w_T)_T + (\nabla p_h, w_T)_T - (f, w_T)_T - (\Pi_T f - f, w_T)_T \end{aligned}$$

Integrating by parts the second order term and subtracting the continuous momentum equation (2.4), we get

$$\begin{aligned} (R_{T,h}, w_T)_T &= \frac{1}{Re} (\nabla \mathcal{Y}, \nabla w_T)_T + ((a \cdot \nabla) \mathcal{Y}, w_T)_T + z (\mathcal{Y}, w_T)_T \\ &- (\Psi, \nabla \cdot w_T)_T + (((\Pi_T a - a) \cdot \nabla) u_h, w_T)_T - (\Pi_T f - f, w_T)_T. \end{aligned}$$

Next, we introduce the previous bound (3.17) and we apply the Cauchy-Schwarz inequality and inequalities of Lemma 5. At last, we obtain (3.18)  $\square$

*3.3.2 Inter-element jumps* Next, we show how the jumps  $J_{E,h}$  bound the error from below. We consider an arbitrary  $E \in \mathcal{E}_{h,\Omega}$  and we define:  $w_E \stackrel{\text{def}}{=} \mathcal{P}_E(J_{E,h})b_E$ , where  $b_E$  is an *edge bubble function* and  $\mathcal{P}_E(\cdot)$  is the extension operator. Let  $T'$  denote any triangle belonging to  $\omega_E$ .

**Lemma 7** *For any  $E \in \mathcal{E}_{h,\Omega}$  we have*

$$\|J_{E,h}\|_{0,E}^2 \asymp (J_{E,h}, w_E)_E, \quad (3.19)$$

$$\|w_E\|_{0,T'} \lesssim \sqrt{h_E} \|J_{E,h}\|_{0,E}, \quad (3.20)$$

$$\|\nabla w_E\|_{0,T'} \lesssim h_T^{-1} \|w_E\|_{0,T'} \lesssim h_E^{-\frac{1}{2}} \|J_{E,h}\|_{0,E}. \quad (3.21)$$

*Proof.* The proof of these inequalities follows the guidelines of [18], [19], [20].  $\square$

**Lemma 8** *The following bound holds*

$$\left| ((a \cdot \nabla) \mathcal{Y}, w_E)_{\omega_E} \right| \lesssim \sum_{T' \subset \omega_E} \|a\|_{\infty, T'} \frac{\alpha_{1, T'}}{\sqrt{h_E}} \|\mathcal{Y}\|_{T'} \|J_{E,h}\|_{0,E}. \quad (3.22)$$

*Proof.* We start from the identity

$$((a \cdot \nabla) \mathcal{Y}, w_E)_{\omega_E} = -(\mathcal{Y}, (a \cdot \nabla) w_E)_{\omega_E}$$

so that using (3.20) and (3.21) we have

$$\begin{aligned} \left| ((a \cdot \nabla) \mathcal{Y}, w_E)_{\omega_E} \right| &\lesssim \sum_{T' \subset \omega_E} \|a\|_{\infty, T'} \sqrt{Re} \|\mathcal{Y}\|_{T'} \sqrt{h_E} \|J_{E,h}\|_{0,E}, \\ \left| (\mathcal{Y}, (a \cdot \nabla) w_E)_{\omega_E} \right| &\lesssim \sum_{T' \subset \omega_E} \frac{1}{\sqrt{z}} \|\mathcal{Y}\|_{T'} \|a\|_{\infty, T'} h_E^{-\frac{1}{2}} \|J_{E,h}\|_{0,E} \end{aligned}$$

so the thesis follows recalling Definition 1.  $\square$

**Proposition 5** *The following lower bound on each internal edge  $E \in \mathcal{E}_h$  holds*

$$\begin{aligned} \sqrt{h_E} \|J_{E,h}\|_{0,E} &\lesssim \sum_{T' \subset \omega_E} \left[ \left( \frac{1}{\sqrt{Re}} + \|a\|_{\infty, T'} \alpha_{1, T'} + h_E \sqrt{z} \right) \|\mathcal{Y}\|_{T'} \right. \\ &\quad \left. + \|\Psi\|_{0, T'} + h_E \left( \|\Pi_T a - a\|_{\infty, T'} |u_h|_{1, T'} + \|\Pi_T f - f\|_{0, T'} \right) \right]. \quad (3.23) \end{aligned}$$

*Proof.* We integrate  $J_{E,h}$  against  $w_E$  on  $E$  and we apply the divergence theorem. Then, we subtract the continuous momentum equation (2.4). We get

$$\begin{aligned}
(J_{E,h}, w_E)_E &= \sum_{T' \subset \omega_E} \int_{T'} \nabla \cdot \left[ \left( \frac{1}{Re} \nabla u_h - p_h \mathbf{I} \right) w_E \right] d\Omega \\
&= \frac{1}{Re} (\nabla \mathcal{Y}, \nabla w_E)_{\omega_E} + z (\mathcal{Y}, w_E)_{\omega_E} - (\Psi, \nabla \cdot w_E)_{\omega_E} \\
&\quad - (R_{T,h}, w_E)_{\omega_E} + (((\Pi_T a - a) \cdot \nabla) u_h, w_E)_{\omega_E} \\
&\quad + ((a \cdot \nabla) \mathcal{Y}, w_E)_{\omega_E} - (\Pi_T f - f, w_E)_{\omega_E}. \tag{3.24}
\end{aligned}$$

We apply the Cauchy-Schwarz inequality, inequalities (3.20), (3.21) and (3.22) on (3.24). Then, using (3.19) and relation (3.18) with  $h_T \asymp h_E$ , we get the thesis.  $\square$

*3.3.3 Residual of the continuity equation* Finally, we consider again an arbitrary  $T \in \mathcal{T}_h$  and we show how the residual of the continuity equation bounds from below the error for the velocity on each triangle  $T$ . Let us define  $w_T \stackrel{def}{=} [\nabla \cdot u_h] b_T$ .

**Proposition 6** *The following lower bound on each element  $T$  holds*

$$\|\nabla \cdot u_h\|_{0,T} \lesssim \frac{\alpha_{1,T}}{h_T} \|\mathcal{Y}\|_T. \tag{3.25}$$

*Proof.* As in the previous cases, we have

$$\|\nabla \cdot u_h\|_{0,T}^2 \lesssim \sqrt{Re} \|\mathcal{Y}\|_T \|\nabla \cdot u_h\|_{0,T}$$

or

$$\|\nabla \cdot u_h\|_{0,T}^2 \lesssim (\nabla \cdot \mathcal{Y}, w_T)_T \lesssim \frac{1}{\sqrt{z}h_T} \|\mathcal{Y}\|_T \|\nabla \cdot u_h\|_{0,T};$$

and this yields the thesis.  $\square$

### 3.4 Final results

Estimates (3.12) and (3.13) and inequalities

$$\frac{\tau_T^2 \|\Pi_T a\|_{\infty,T}^2}{h_T^2} \leq \frac{1}{4}, \quad \frac{\delta_T^2}{h_T^2} \leq \lambda^2 \|\Pi_T a\|_{\infty,T}^2 \preceq 1$$

suggest the definition of the following *a posteriori* error estimator on the element  $T$ :

**Definition 3**

$$\begin{aligned}
& \eta_{R,T}^2 \stackrel{def}{=} \left( \frac{h_T^2}{\alpha_2^2} + \alpha_{1,T}^2 \right) \| R_{T,h} \|_{0,T}^2 \\
& + \left[ \alpha_2^2 + \left( \frac{h_T^2}{\alpha_2^2} + \alpha_{1,T}^2 \right) \lambda^2 \| \Pi_T a \|_{\infty,T}^2 \right] \| \nabla \cdot u_h \|_{0,T}^2 \\
& + \frac{1}{2} \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,\Omega}} \left( \frac{h_E}{\alpha_2^2} + \alpha_{1,E} \sqrt{Re} \right) \| J_{E,h} \|_{0,E}^2. \quad (3.26)
\end{aligned}$$

Now we collect the results of all the previous subsections. In the following theorem we introduce a strictly positive parameter  $\alpha_3$  that we will exactly define in the sequel.

**Theorem 2** *There exists a constant  $C^\dagger$  such that, for each  $\alpha_3 > 0$ , the global upper bound holds*

$$\begin{aligned}
& \alpha_3 \| \mathcal{Y} \|_{\Omega} + \| \Psi \|_0 \leq C^\dagger (\alpha_3 + \alpha_2) \left\{ \sqrt{\sum_{T \in \mathcal{T}_h} \eta_{R,T}^2} \right. \\
& \left. + \left( \frac{1}{\alpha_2} + \alpha_1 \right) (\| \Pi_T a - a \|_{\infty} |u_h|_1 + \| \Pi_T f - f \|_0) \right\}. \quad (3.27)
\end{aligned}$$

*Proof.* It follows from estimates (3.12), (3.13) and definition (3.26).

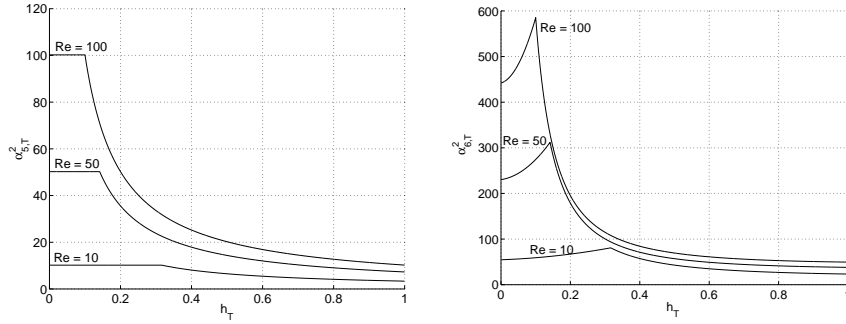
□

**Definition 4** *Let us define for each  $T \in \mathcal{T}_h$*

$$\alpha_{4,T} \stackrel{def}{=} \frac{1}{\sqrt{Re}} + \alpha_{1,T} \| a \|_{\infty, \omega_T} + h_T \sqrt{z}. \quad (3.28)$$

**Theorem 3** *There exists a constant  $C'_\downarrow$  such that the local lower bound holds*

$$\begin{aligned}
& \eta_{R,T}^2 \leq C'_\downarrow \left\{ \left\{ \left( \frac{h_T}{\alpha_2^2} + \alpha_{1,T} \sqrt{Re} \right) \frac{\alpha_{4,T}^2}{h_T} + \frac{\alpha_{1,T}^2}{h_T^2} [\alpha_2^2 \right. \right. \\
& \quad \left. \left. + \left( \frac{h_T^2}{\alpha_2^2} + \alpha_{1,T}^2 \right) \lambda^2 \| \Pi_T a \|_{\infty,T}^2] \right\} \| \mathcal{Y} \|_{\omega_T}^2 \right. \\
& + \left( \frac{h_T}{\alpha_2^2} + \alpha_{1,T} \sqrt{Re} \right) \frac{1}{h_T} \| \Psi \|_{0, \omega_T}^2 + \left( \frac{h_T}{\alpha_2^2} + \alpha_{1,T} \sqrt{Re} \right) h_T \\
& \quad \left. \times \left( \| \Pi_T a - a \|_{\infty, \omega_T}^2 |u_h|_{1, \omega_T}^2 + \| \Pi_T f - f \|_{0, \omega_T}^2 \right) \right\}. \quad (3.29)
\end{aligned}$$



**Figure 3.1.**  $\alpha_{5,T}^2$  versus  $h_T$ ,  $\|a\|_{\infty,T} = 1$ ,  $z = 1$ ,  $\lambda = 0$  **Figure 3.2.**  $\alpha_{6,T}^2$  versus  $h_T$ ,  $\|a\|_{\infty,T} = 1$ ,  $z = 1$ ,  $\lambda = 0$

*Proof.* We use definition (3.28) and equations (3.18), (3.23), (3.25). We combine these equations and we note that

$$\left( \frac{h_T^2}{\alpha_2^2} + \alpha_{1,T}^2 \right) \frac{1}{h_T^2} \leq \left( \frac{h_E}{\alpha_2^2} + \alpha_{1,E} \sqrt{Re} \right) \frac{1}{h_E}, \quad \forall E \in \mathcal{E}(T), \quad \forall T \in \mathcal{T}_h;$$

via the regularity assumption  $h_E \asymp h_T$ , we get (3.29).  $\square$

*Remark 2* Here we have not considered the modified *edge bubble functions* used in [19] and [20] because these functions give no advantage due to the presence of the pressure term in the momentum equation (2.1). In fact, the modified *edge bubble functions* depend on a parameter  $\theta_T \leq 1$  that can be chosen to balance the different contributions of the velocity terms to the lower bound of the error estimator. But, to balance the contribution of the pressure term we should take  $\theta_T \geq 1$ , so the best choice results in  $\theta_T = 1$  which corresponds to the classical definition of the *edge bubble functions*.

Now we will investigate the expressions that appear in (3.29). At first, let us define  $\bar{h} \stackrel{def}{=} \frac{1}{\sqrt{Re\sqrt{z}}}$ .

**Definition 5** For each triangle  $T \in \mathcal{T}_h$ , let us define

$$\alpha_{5,T}^2 \stackrel{def}{=} \left( \frac{h_T}{\alpha_2^2} + \alpha_{1,T} \sqrt{Re} \right) \frac{1}{h_T}. \quad (3.30)$$

The parameter  $\alpha_{5,T}^2$  as a function of  $h_T \leq 1$  has the constant value  $\bar{\alpha}_5^2 \stackrel{def}{=} \frac{1}{\alpha_2^2} + Re$  on the interval  $(0, \bar{h}]$  and then it strictly decreases (see Figure 3.1); thus,  $\bar{\alpha}_5^2$  is the absolute maximum of  $\alpha_{5,T}^2$ . We also define the following quantity on the considered triangulation  $\mathcal{T}_h$

$$\alpha_5^2 \stackrel{def}{=} \max_{T \in \mathcal{T}_h} \alpha_{5,T}^2. \quad (3.31)$$

Setting

$$\check{h} \stackrel{def}{=} \min_{T \in \mathcal{T}_h} h_T, \quad (3.32)$$

we have  $\alpha_5^2 = \bar{\alpha}_5^2$  if  $\check{h} \leq \bar{h}$ , whereas  $\alpha_5^2 < \bar{\alpha}_5^2$  if  $\check{h} > \bar{h}$ .

**Definition 6** *Let us set*

$$\alpha_{6,T}^2 \stackrel{def}{=} \alpha_{5,T}^2 \alpha_{4,T}^2 + \frac{\alpha_{1,T}^2}{h_T^2} \left[ \alpha_2^2 + \lambda^2 \left( \frac{h_T^2}{\alpha_2^2} + \alpha_{1,T}^2 \right) \| \Pi_T a \|_{\infty,T}^2 \right]. \quad (3.33)$$

The parameter  $\alpha_{6,T}^2$  as a function of  $h_T$  takes its maximum value  $\bar{\alpha}_6^2$  for  $h_T = \bar{h}$  (see Figure 3.2). We define

$$\alpha_6^2 \stackrel{def}{=} \begin{cases} \max_{T \in \mathcal{T}_h} \alpha_{6,T}^2, & \text{if } h < \bar{h}, \\ \bar{\alpha}_6^2, & \text{if } \check{h} \leq \bar{h} \leq h, \\ \max_{T \in \mathcal{T}_h} \alpha_{6,T}^2, & \text{if } \bar{h} < \check{h}. \end{cases} \quad (3.34)$$

The following Corollary is based on Theorem 3 and the previous definitions.

**Corollary 1** *There exists a constant  $C_\downarrow$  such that the local lower bound*

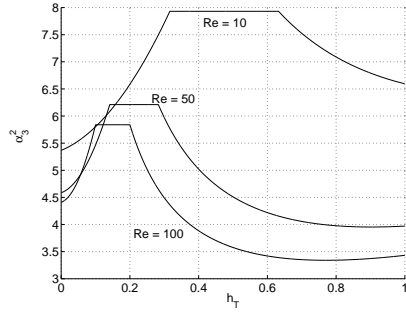
$$\eta_{R,T}^2 \leq C_\downarrow^2 \left\{ \alpha_{6,T}^2 \| \mathcal{R} \|_{\omega_T}^2 + \alpha_{5,T}^2 \| \Psi \|_{0,\omega_T}^2 + \alpha_{5,T}^2 h_T^2 \| \Pi_T a - a \|_{\infty,\omega_T}^2 | u_h |_{1,\omega_T}^2 + \alpha_{5,T}^2 h_T^2 \| \Pi_T f - f \|_{0,\omega_T}^2 \right\} \quad (3.35)$$

and the global lower bound

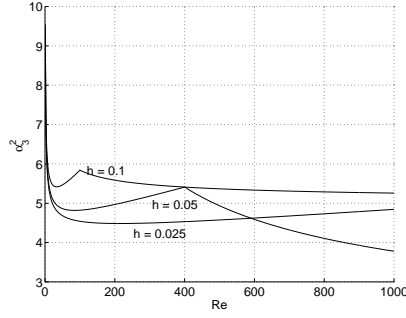
$$\frac{1}{\alpha_5} \sqrt{\sum_{T \in \mathcal{T}_h} \eta_{R,T}^2} \leq C_\downarrow \left( \frac{\alpha_6}{\alpha_5} \| \mathcal{R} \|_{\Omega} + \| \Psi \|_0 \right) + C_\downarrow h \left\{ \sum_{T \in \mathcal{T}_h} \| \Pi_T a - a \|_{\infty,\omega_T}^2 | u_h |_{1,\omega_T}^2 + \sum_{T \in \mathcal{T}_h} \| \Pi_T f - f \|_{0,\omega_T}^2 \right\}^{\frac{1}{2}} \quad (3.36)$$

hold true.

Now estimate (3.36) suggests the following choice for the, up to now, generic constant  $\alpha_3$  which appears in (3.27).



**Figure 3.3.**  $\alpha_3^2$  versus  $h$ ,  $\|a\|_{\infty,T} = 1$ ,  $z = 1$ ,  $\check{h} = \frac{h}{2}$ ,  $\lambda = 0$



**Figure 3.4.**  $\alpha_3^2$  versus  $Re$ ,  $\|a\|_{\infty,T} = 1$ ,  $z = 1$ ,  $\check{h} = \frac{h}{2}$ ,  $\lambda = 0$

**Definition 7** *Let us define*

$$\alpha_3^2 \stackrel{def}{=} \begin{cases} \left( \frac{1}{\sqrt{Re}} + \|a\|_{\infty,\Omega} \sqrt{Re} h + h \sqrt{z} \right)^2 + \frac{Re \alpha_2^4}{1 + Re \alpha_2^2} \\ \quad + \lambda^2 \| \Pi_T a \|_{\infty,\Omega}^2 Re h^2, & \text{if } h < \check{h}, \\ \left( \frac{1}{\sqrt{Re}} + \|a\|_{\infty,\Omega} \sqrt{Re} \check{h} + \check{h} \sqrt{z} \right)^2 + \frac{Re \alpha_2^4}{1 + Re \alpha_2^2} \\ \quad + \lambda^2 \| \Pi_T a \|_{\infty,\Omega}^2 Re \check{h}^2, & \text{if } \check{h} \leq h \leq h, \\ \left( \frac{1}{\sqrt{Re}} + \frac{\|a\|_{\infty,\Omega}}{\sqrt{z}} + \check{h} \sqrt{z} \right)^2 + \frac{\alpha_2^4}{z \check{h}^2 + \alpha_2^2 \sqrt{z} Re \check{h}} \\ \quad + \lambda^2 \| \Pi_T a \|_{\infty,\Omega}^2 \frac{z \check{h}^2 + \alpha_2^2}{z (z \check{h}^2 + \alpha_2^2 \sqrt{z} Re \check{h})}, & \text{if } \check{h} > h. \end{cases} \quad (3.37)$$

The parameter  $\alpha_3^2$  is defined such that the inequality  $\frac{\alpha_3^2}{\alpha_5^2} \leq \alpha_3^2$  holds true. The behaviour of  $\alpha_3^2$  as a function of  $h_T$  and  $Re$  is shown in Figures 3.3 and 3.4.

Corollary 1 and Definition 3.37 yield the following corollary.

**Corollary 2** *There exists a constant  $C_{\downarrow}$  such that the following global lower bound holds*

$$\frac{1}{\alpha_5} \sqrt{\sum_{T \in \mathcal{T}_h} \eta_{R,T}^2} \leq C_{\downarrow} (\alpha_3 \| \mathcal{Y} \|_{\Omega} + \| \Psi \|_0) \\ + C_{\downarrow} h \left\{ \sum_{T \in \mathcal{T}_h} \| \Pi_T a - a \|_{\infty,\omega_T}^2 |u_h|_{1,\omega_T}^2 + \sum_{T \in \mathcal{T}_h} \| \Pi_T f - f \|_{0,\omega_T}^2 \right\}^{\frac{1}{2}}. \quad (3.38)$$

#### 4 Equivalence between the true error and the error estimator

Now we define the true error for our problem

$$t.e. \stackrel{def}{=} \alpha_3 \|\mathcal{Y}\|_{\Omega} + \|\Psi\|_0 \quad (4.1)$$

and we define the global error estimator

$$\eta_{\Omega} \stackrel{def}{=} \sqrt{\sum_{T \in \mathcal{T}_h} \eta_{R,T}^2}. \quad (4.2)$$

For a practical and simple use of the error estimator in the construction of a sequence of adapted Delaunay triangulations, we assume that the data  $f$ ,  $a$  are interpolated by polynomials  $\Pi_T f$ ,  $\Pi_T a$  of degree  $n_1, n_2 \geq 1$  respectively, satisfying the following estimates:

$$\|\Pi_T f - f\|_{0,\omega_T} \lesssim h_T^{n_1+1} |f|_{n_1+1,\omega_T}, \quad (4.3)$$

$$\|\Pi_T a - a\|_{0,\infty,\omega_T} \lesssim h_T^{n_2+1} |a|_{n_2+1,\infty,\omega_T}. \quad (4.4)$$

We assume that  $n_1, n_2$  are large enough and  $h_T, \forall T \in \mathcal{T}_h$ , is small enough so that the errors due to the approximation appearing in inequalities (3.27), (3.38) can be made negligible with respect to the global error estimator  $\eta_{\Omega}$ . For this topics we refer to [1], [12]. Then, inequalities (3.27), (3.38) imply the following proposition.

**Proposition 7** *Under the above assumption on the data approximation, there exist two constants  $\underline{c}$  and  $\overline{C}$ , dependent upon the constants  $C^{\uparrow}$  and  $C_{\downarrow}$ , such that the following bounds for the true error in terms of the global error estimator hold*

$$\underline{c} \frac{1}{\alpha_5} \eta_{\Omega} \leq \alpha_3 \|\mathcal{Y}\|_{\Omega} + \|\Psi\|_0 \leq \overline{C} (\alpha_3 + \alpha_2) \eta_{\Omega}. \quad (4.5)$$

The effectivity index [1]

$$e.i. \stackrel{def}{=} \frac{\eta_{\Omega}}{\alpha_3 \|\mathcal{Y}\|_{\Omega} + \|\Psi\|_0} \quad (4.6)$$

plays a fundamental role in the study of the equivalence relation between the error estimator and the true error. We have the following bounds for the inverse of the effectivity index

$$\underline{c} \frac{1}{\alpha_5} \leq \frac{1}{e.i.} \leq \overline{C} (\alpha_3 + \alpha_2). \quad (4.7)$$

In the optimal situation the two bounds in (4.7) should be independent of any mesh-size.

## 5 Sensitiveness to the problem parameters

Due to the complexity of the definitions of the coefficients  $\alpha_2$ ,  $\alpha_3$  and  $\alpha_5$ , we study a case of particular interest and we make all the considerations for it. First of all we fix  $z \approx 1$ ,  $Re \geq 1$  and we recall Remark 1. Moreover we assume  $\lambda = 0$  ( $\delta_T = 0$ ). We are interested in observing the behaviour of the coefficients  $\alpha_2$ ,  $\alpha_3$  and  $\alpha_5$  when  $Re$  becomes very large and  $\check{h}$  becomes small. Under the previous hypotheses, it is easy to get:  $\alpha_2 = \mathcal{O}(1)$ ,  $\alpha_3 = \mathcal{O}(1)$ ,  $\bar{\alpha}_5 = \mathcal{O}(\sqrt{Re})$ ,  $\alpha_5 = \mathcal{O}\left(\sqrt[4]{Re} \min\left\{\sqrt[4]{Re}, \frac{1}{\sqrt{\check{h}}}\right\}\right)$ . Hence, the double inequality (4.7) becomes

$$\underline{c} \frac{1}{\sqrt[4]{Re}} \max\left\{\frac{1}{\sqrt[4]{Re}}, \sqrt{\check{h}}\right\} \lesssim \frac{1}{e.i.} \lesssim \bar{C}, \quad (5.1)$$

showing a moderate loss of robustness of our estimates when  $Re$  becomes very large.

The case with  $z = 0$  and  $Re \gg 1$  was considered in [3]. The corresponding result in the current setting is

$$\underline{c} \frac{1}{\sqrt{Re}} \lesssim \frac{1}{e.i.} \lesssim \bar{C} \sqrt{Re}. \quad (5.2)$$

We note the improvement of robustness due to the presence of the zero-order term.

*Remark 3* The lower bound in (5.1) involves  $\check{h}$ , so it is not independent of the mesh-size. This is a consequence of our definitions (3.31), (3.34) and (3.37) for the quantities  $\alpha_5^2$ ,  $\alpha_6^2$  and  $\alpha_3^2$ . Another possibility is to set  $\alpha_5^2 = \bar{\alpha}_5^2$ ,  $\alpha_6^2 = \bar{\alpha}_6^2$  (i.e., take the largest values independently of any mesh-size) and  $\alpha_3^2 = \frac{\bar{\alpha}_6^2}{\bar{\alpha}_5^2}$ . In this case we have

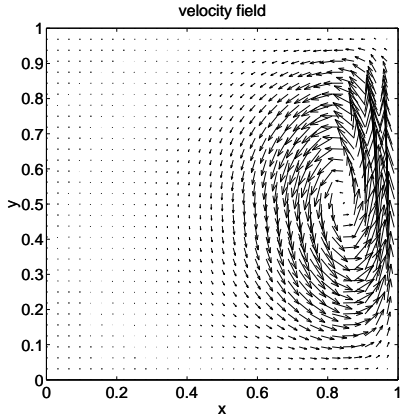
$$\underline{c} \frac{1}{\sqrt{Re}} \lesssim \frac{1}{e.i.} \lesssim \bar{C}. \quad (5.3)$$

This bound is independent of the mesh-size, but is less sharp than (5.1) for high Reynolds numbers.

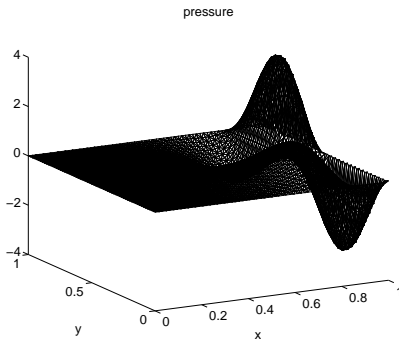
In the next subsections we want to perform some numerical investigations on the bounds of the estimates (4.5), (4.7).

### 5.1 The test problem

In order to test our error estimator we consider problem (2.1), (2.2) in the unit box  $\Omega \stackrel{def}{=} (0, 1)^2$  with homogeneous boundary conditions.



**Figure 5.1.** Exact solution: velocity field



**Figure 5.2.** Exact solution:  $p$

We define the vector field  $a = [a_1, a_2]$  as follows:

$$a_1(x, y) \stackrel{\text{def}}{=} \left(1 - \cos\left(\frac{2\pi(e^{R_1 x} - 1)}{e^{R_1} - 1}\right)\right) \sin\left(\frac{2\pi(e^{R_2 y} - 1)}{e^{R_2} - 1}\right) \frac{R_2}{2\pi} \frac{e^{R_2 y}}{(e^{R_2} - 1)},$$

$$a_2(x, y) \stackrel{\text{def}}{=} -\sin\left(\frac{2\pi(e^{R_1 x} - 1)}{e^{R_1} - 1}\right) \left(1 - \cos\left(\frac{2\pi(e^{R_2 y} - 1)}{e^{R_2} - 1}\right)\right) \frac{R_1}{2\pi} \frac{e^{R_1 x}}{(e^{R_1} - 1)}$$

where  $R_1, R_2$  are two strictly positive real parameters. With a suitable choice of  $f = [f_1, f_2]$ , the solution  $[u, p]$  of the problem is

$$u_1(x, y) = a_1(x, y),$$

$$u_2(x, y) = a_2(x, y),$$

$$p(x, y) = R_1 R_2 \sin\left(\frac{2\pi(e^{R_1 x} - 1)}{e^{R_1} - 1}\right) \sin\left(\frac{2\pi(e^{R_2 y} - 1)}{e^{R_2} - 1}\right) \times \frac{e^{R_1 x} e^{R_2 y}}{(e^{R_1} - 1)(e^{R_2} - 1)}.$$

The velocity field of this solution is similar to a counterclockwise vortex in a unit-box (see Figures 5.1, 5.2). Playing with the parameters  $R_1$  and  $R_2$  we can move the centre of this vortex that has coordinates  $x_0 = \frac{1}{R_1} \log\left(\frac{e^{R_1} + 1}{2}\right)$  and  $y_0 = \frac{1}{R_2} \log\left(\frac{e^{R_2} + 1}{2}\right)$ . Increasing  $R_1$ , the centre goes rapidly towards the right-hand vertical side, whereas increasing  $R_2$  it approaches the top edge.

Every numerical result that we shall present is obtained using continuous linear finite elements for both velocity and pressure. Moreover, every integral needed to set up the linear system is computed

assuming  $n_1 = n_2 = 3$  in (4.3), (4.4); this is achieved by computing the integrals with suitable quadrature formulas on each triangle. A quadrature formula of order 5 on each element is used for computing the norms in the true error. The parameter  $\lambda$  appearing in the stabilizing parameter  $\delta_T$  in (2.10) is set to 0.

### 5.2 Numerical results on uniform triangulations

We study how the effectivity index *e.i.*, varies with respect to the mesh-size and the Reynolds number on a uniform grid. In our test problem we consider the forcing function that corresponds to the solution which has the centre of the vortex on the horizontal line  $y_0 = 0.5125$  ( $R_2 = 0.1$ ) and the distance from the right-hand vertical wall equal to  $\frac{1}{\sqrt[4]{Re}}$ . We report the behaviour of  $\alpha_4 + \alpha_3$ ,  $\frac{1}{e.i.}$ ,  $\frac{1}{\alpha_6}$  on uniform grids with respect to  $\tilde{h} = h$  in Figures 5.3, 5.5, 5.7 and with respect to  $Re$  in Figures 5.4, 5.6, 5.8. We see that the dependence of  $\frac{1}{e.i.}$  on the Reynolds number is not far from  $\frac{1}{\alpha_6}$  as expected for the lower bound in (4.7). Figures 5.9, 5.10 show the direct comparison of the upper bound  $\alpha_3 + \alpha_2$ , of the lower bound  $\frac{1}{\alpha_5}$  with respect to  $\frac{1}{e.i.}$ , for the coarsest and the finest uniform grids we consider. The parallel behaviour of  $\frac{1}{e.i.}$  and  $\frac{1}{\alpha_5}$ , shown in these figures, confirm our opinion that the asymptotic behaviour of  $\frac{1}{e.i.}$ , for  $Re$  becoming very large, is close to the one predicted by the lower bound of (4.7). We note that our estimates are not robust because the coefficients depend on the Reynolds number, but we can say that they are sharp because for our test problem the dependence of  $\frac{1}{e.i.}$  upon  $Re$  is very close to the dependence of  $\frac{1}{\alpha_5}$  upon the Reynolds number.

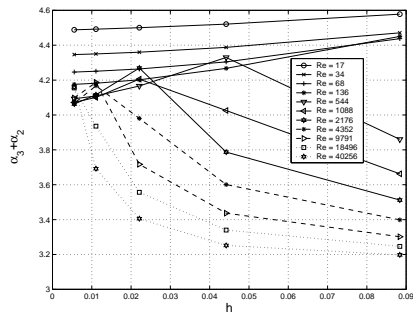
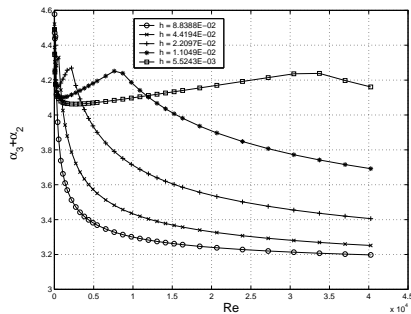
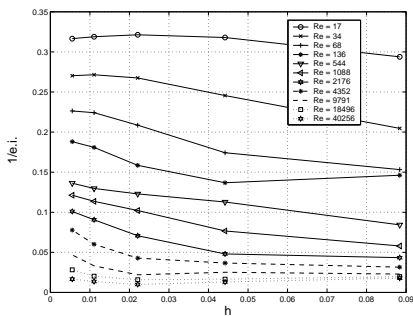
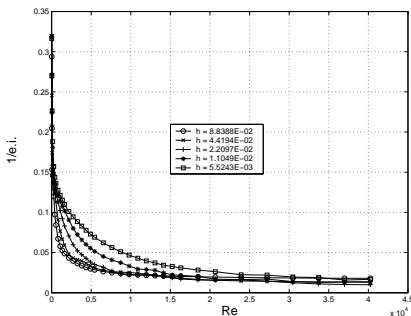
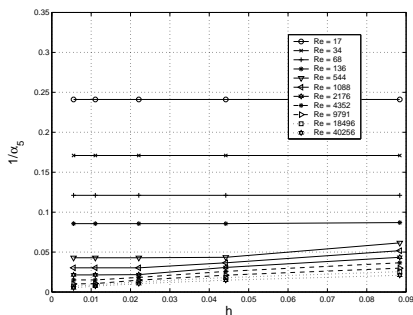
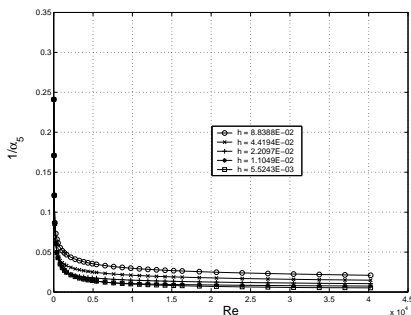
## 6 Comparisons with error estimators for reaction-convection-diffusion problems

We like to apply the principles of our analysis to the reaction-convection-diffusion equation and compare the results so obtained to the analogous ones derived in [20]. Here, we specialize the analysis of Subsection 3.3 to the following problem

$$-\frac{1}{Pe} \Delta u + a \cdot \nabla u + z u = f, \quad \text{in } \Omega, \quad (6.1)$$

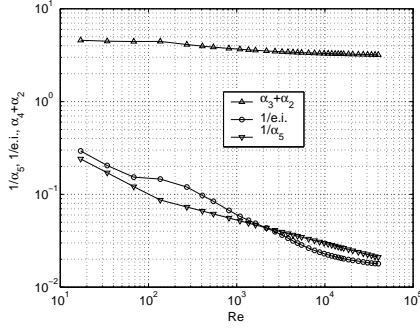
$$u = 0 \quad \text{on } \partial\Omega, \quad (6.2)$$

where  $Pe \gg 1$  is the Péclet number;  $z \in L^\infty(\Omega)$ ;  $a \in [H^1(\Omega)]^2 \cap [L^\infty(\Omega)]^2$  with  $\nabla \cdot a = 0$  in  $\bar{\Omega}$ ;  $\|a\|_\infty = \mathcal{O}(1)$ ;  $\|z\|_\infty = \mathcal{O}(1)$ ;  $f \in$

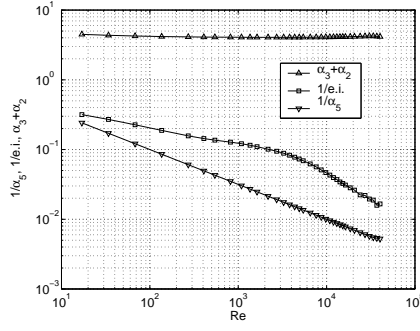
Figure 5.3.  $\alpha_3 + \alpha_2$  versus  $h$ Figure 5.4.  $\alpha_3 + \alpha_2$  versus  $Re$ Figure 5.5.  $\frac{1}{e.i.}$  versus  $h$ Figure 5.6.  $\frac{1}{e.i.}$  versus  $Re$ Figure 5.7.  $\frac{1}{\alpha_5}$  versus  $h$ Figure 5.8.  $\frac{1}{\alpha_5}$  versus  $Re$ 

$L^2(\Omega)$ . The discrete model includes a *SUPG* stabilization [9] like for the previous problems: Find  $u_h \in V_h$  such that  $\forall v_h \in V_h$

$$\begin{aligned} & \frac{1}{Pe} (\nabla u_h, \nabla v_h) + (a \cdot \nabla u_h, v_h) + (z u_h, v_h) \\ & + \sum_{T \in \mathcal{T}_h} \tau_T \left( -\frac{1}{Pe} \Delta u_h + a \cdot \nabla u_h + z u_h, a \cdot \nabla v_h \right)_T \end{aligned}$$



**Figure 5.9.**  $\alpha_3 + \alpha_2$ ,  $\frac{1}{e.i.}$ ,  $\frac{1}{\alpha_5}$  versus  $Re$ ,  $h = 8.8388E - 02$



**Figure 5.10.**  $\alpha_3 + \alpha_2$ ,  $\frac{1}{e.i.}$ ,  $\frac{1}{\alpha_5}$  versus  $Re$ ,  $h = 5.5243E - 03$

$$= (f, v_h) + \sum_{T \in \mathcal{T}_h} \tau_T (f, a \cdot \nabla v_h)_T. \quad (6.3)$$

Following [9], we set  $\tau_T \stackrel{def}{=} m_k \frac{h_T^2}{4} Pe$  if  $0 \leq m_k \frac{\|a\|_{\infty, T} h_T Pe}{2} < 1$ , whereas  $\tau_T \stackrel{def}{=} \frac{h_T}{2\|a\|_{\infty, T}}$  if  $m_k \frac{\|a\|_{\infty, T} h_T Pe}{2} \geq 1$  where  $m_k$  is defined in Subsection 2.2. For the sake of simplicity, we do not consider any approximation of the convective velocity  $a$  and of the function  $z$ . Moreover the function  $f$  is not approximated for solving the problem. We will consider an approximation of  $f$  only in the definition of the *element residual bubble function*  $w_T$ . This is to make comparisons with [20] as easy and direct as possible.

Let us define our energy norm for the solution  $u$  on  $\omega \subseteq \Omega$  in the following manner:

$$\|u\|_{\omega}^2 \stackrel{def}{=} \frac{1}{Pe} |u|_{1, \omega}^2 + \|u\|_{0, \omega}^2. \quad (6.4)$$

*Remark 4* Definition (6.4) does not include any dependence on  $z$ , following [20] and differently from (2.6); this is justified by the assumption  $\|z\|_{\infty} = \mathcal{O}(1)$ .

**Definition 8** *Let us set*

$$\alpha_{1, S} \stackrel{def}{=} \min \left\{ \sqrt{Pe} h_S, 1 \right\}, \quad S = T \in \mathcal{T}_h \text{ or } S = E \in \mathcal{E}_h.$$

We give the definitions of the *equation-residual*  $R_T(u_h)$  and of the *stress-jump*  $J_E(u_h)$  for this reaction-convection-diffusion problem.

**Definition 9**

$$R_{T, h} \stackrel{def}{=} - \frac{1}{Pe} \Delta u_h + a \cdot \nabla u_h + z u_h - \Pi_T f \Big|_T,$$

$$J_{E,h} \stackrel{def}{=} \frac{1}{Pe} \left[ \left[ \frac{\partial u_h}{\partial \hat{n}_E} \right] \right]_E.$$

**Definition 10** *Let us introduce the error indicator*

$$\eta_{R,T}^2 \stackrel{def}{=} \alpha_{1,T}^2 \|R_{T,h}\|_{0,T}^2 + \frac{1}{2} \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,\Omega}} \alpha_{1,E} \sqrt{Pe} \|J_{E,h}\|_{0,E}^2. \quad (6.5)$$

### 6.1 Global upper bound

Following the guidelines of Section 3.2, one gets the same upper bound for the error as in [20]:

$$\|u_h - u\|_{\Omega} \lesssim \sqrt{\sum_{T \in \mathcal{T}_h} \eta_{R,T}^2} + \left\{ \sum_{T \in \mathcal{T}_h} \alpha_{1,T}^2 \|\Pi_T f - f\|_0^2 \right\}^{\frac{1}{2}}. \quad (6.6)$$

### 6.2 Local lower bound

Now following the same analysis of Subsection 3.3 applied to this problem we find how  $\eta_{R,T}$  bounds the error from below.

**Definition 11** *Let us define for each  $T \in \mathcal{T}_h$*

$$\alpha_{4,T} \stackrel{def}{=} \frac{1}{\sqrt{Pe}} + \alpha_{1,T} \|a\|_{\infty,\omega_T} + h_T \|z\|_{\infty,\omega_T}. \quad (6.7)$$

Then we have the following proposition:

**Proposition 8** *The following local lower bound holds*

$$\eta_{R,T}^2 \lesssim \sqrt{Pe} \frac{\alpha_{1,T}}{h_T} \alpha_{4,T}^2 \|u_h - u\|_{\omega_T}^2 + \alpha_{1,T} h_T \sqrt{Pe} \|\Pi_T f - f\|_{0,\omega_T}^2. \quad (6.8)$$

We define

$$\mathcal{C}_{1,T}^2 \stackrel{def}{=} \sqrt{Pe} \frac{\alpha_{1,T}}{h_T} \alpha_{4,T}^2 \quad (6.9)$$

and we write the lower bound (6.8) as

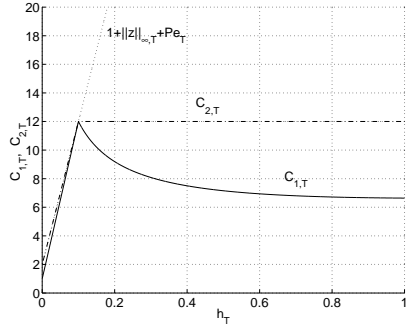
$$\eta_{R,T}^2 \lesssim \mathcal{C}_{1,T}^2 \|u_h - u\|_{\omega_T}^2 + \alpha_{1,T} h_T \sqrt{Pe} \|\Pi_T f - f\|_{0,\omega_T}^2. \quad (6.10)$$

We compare this inequality with the equivalent one given in [20]

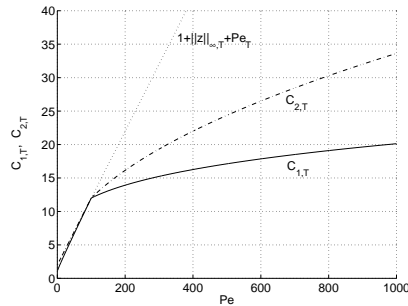
$$\eta_{R,T}^2 \lesssim \mathcal{C}_{2,T}^2 \|u_h - u\|_{\omega_T}^2 + \alpha_{1,T}^2 \|\Pi_T f - f\|_{0,\omega_T}^2, \quad (6.11)$$

where

$$\mathcal{C}_{2,T} \stackrel{def}{=} 1 + \alpha_{1,T} \|a\|_{\infty,\omega_T} \sqrt{Pe} + \|z\|_{\infty,\omega_T}. \quad (6.12)$$



**Figure 6.1.**  $C_{1,T}$  and  $C_{2,T}$  versus  $h_T$ ,  $Pe = 100$



**Figure 6.2.**  $C_{1,T}$  and  $C_{2,T}$  versus  $Pe$ ,  $h_T = 0.1$

*Remark 5* Inequality (6.11) is obtained taking  $\theta_T \stackrel{def}{=} \min \left\{ \frac{1}{\sqrt{Pe} h_T}, 1 \right\}$  in the definition of the modified *edge bubble function* [20]. This choice is done with the target to make the contribution of convection to the loss of robustness ( $C_{2,T}$ ) as close to 1 as possible. If one does not apply integration by parts in the proof of inequality (3.17) but simply takes

$$|(a \cdot \nabla(u_h - u), w_T)| \leq \|a\|_{\infty,T} \sqrt{Pe} \|u_h - u\|_T \|w_T\|_{0,T}$$

(and one proceeds similarly in proving inequalities (3.22)), then one needs the modified cut-off bubble functions to get (6.11).

We consider the comparison between  $C_{1,T}$  and  $C_{2,T}$  very interesting for analyzing the robustness of our estimates. The difference between the two factors multiplying the term  $\| \Pi_T f - f \|_{0,\omega_T}^2$  in the two previous equations is less interesting; indeed, we assume to choose the approximation  $\Pi_T f$  such that the data approximation terms are negligible with respect to the error indicator.

It is easy to verify that

$$C_{1,T} \leq 1 + \|z\|_{\infty,T} + Pe_T \quad \text{and} \quad C_{2,T} \leq 1 + \|z\|_{\infty,T} + Pe_T,$$

where  $Pe_T \stackrel{def}{=} \|a\|_{\infty,T} h_T Pe$  is the local mesh-Péclet number.

Figures 6.1, 6.2 allow us to compare  $C_{1,T}$ ,  $C_{2,T}$  and  $1 + \|z\|_{\infty,T} + Pe_T$ . We observe that our analysis leads to an estimate as sharp as the one given in [20] even if we do not take advantage of the modified cut-off functions. Furthermore, we note a slight improvement in the numerical values of the coefficients.

Going back to the Oseen problem, we conclude that the estimates derived in Subsections 3.2, 3.3 for the chosen energy-like norm of the error in the velocity are qualitatively as sharp as the ones given in [20] for the scalar reaction, convection and diffusion equation.

## 7 Divergence-free projection

In order to discuss the sharpness of our estimate (3.18) with respect to the error in pressure and the effect of the incompressibility constraint alone, we propose to consider the following model problem, obtained from our Oseen model by setting  $Re = \infty$ ,  $a = 0$ ,  $z = 1$  and enforcing admissible boundary conditions:

$$u + \nabla p = f, \quad \text{in } \Omega, \quad (7.1)$$

$$\nabla \cdot u = 0, \quad \text{in } \Omega, \quad (7.2)$$

$$u \cdot \hat{n} = 0, \quad \text{on } \partial\Omega. \quad (7.3)$$

The problem can be written in two different variational formulations. To this end, we introduce the space  $H_0(\text{div};\Omega) = \{v \in [L^2(\Omega)]^2 : \nabla \cdot v \in L^2(\Omega) \text{ and } v \cdot \hat{n} = 0 \text{ on } \partial\Omega\}$  (see, e.g. [4], [10]), equipped with the norm  $\|v\|_{\text{div}} = \left(\|v\|_0^2 + \|\nabla \cdot v\|_0^2\right)^{\frac{1}{2}}$ .

The first variational formulation of (7.1)-(7.3) we consider is the closest one to the formulation used for the previous problems:

*Find  $[u, p] \in H_0(\text{div};\Omega) \times L_0^2(\Omega)$  such that*

$$(u, v) - (p, \nabla \cdot v) = (f, v), \quad \forall v \in H_0(\text{div};\Omega), \quad (7.4)$$

$$(q, \nabla \cdot u) = 0, \quad \forall q \in L_0^2(\Omega). \quad (7.5)$$

Note that  $u$  is precisely the orthogonal projection of  $f$  upon the closed subspace of  $H_0(\text{div};\Omega)$  of the divergence-free vector fields. As usual,  $p$  can be interpreted as the Lagrange multiplier associated with the divergence-free constraint. The variational formulation (7.4), (7.5) is also related to the mixed formulation of the Poisson problem for  $p$  (see, e.g., [4], Chapter IV).

Problem (7.1)-(7.3) can also be formulated as follows:

*Find  $[u, p] \in [L^2(\Omega)]^2 \times [H^1(\Omega) \cap L_0^2(\Omega)]$  such that*

$$(u, v) + (\nabla p, v) = (f, v), \quad \forall v \in [L^2(\Omega)]^2, \quad (7.6)$$

$$(\nabla q, u) = 0, \quad \forall q \in H^1(\Omega) \cap L_0^2(\Omega). \quad (7.7)$$

Note that, here, the boundary condition (7.3) is enforced as a natural boundary condition, implicitly in (7.7) after integration by parts of (7.2).

### 7.1 Well-posedness of the continuous problem

Well-posedness of problem (7.4), (7.5) follows from classical conditions on saddle-point problems. Precisely, the bilinear form  $a(u, v) = (u, v)$  is trivially coercive, with respect to the  $H_0(\text{div}; \Omega)$ -norm, on the subspace  $K \stackrel{\text{def}}{=} \{v \in H_0(\text{div}; \Omega) : (q, \nabla \cdot v) = 0, \forall q \in L^2(\Omega)\}$ .

Here, we recall a Poincaré-Friedrichs inequality for zero mean value functions:

$$\|q\|_0 \lesssim \|\nabla q\|_0, \quad \forall q \in H^1(\Omega) \text{ such that } \int_{\Omega} q \, d\Omega = 0. \quad (7.8)$$

Moreover, we have the following lemma [10]:

**Lemma 9** *On the space  $H_0(\text{div}; \Omega) \times L_0^2(\Omega)$ , the bilinear form  $b(q, v) = -(q, \nabla \cdot v)$  satisfies the following inf-sup condition*

$$\sup_{v \in H_0(\text{div}; \Omega) \setminus \{0\}} \frac{(q, \nabla \cdot v)}{\|v\|_{\text{div}}} \geq \frac{1}{\beta} \|q\|_0, \quad \forall q \in L_0^2(\Omega). \quad (7.9)$$

As a consequence, the solution of (7.4), (7.5) satisfies the estimate

$$\|u\|_{\text{div}} + \|p\|_0 \lesssim \|f\|_0. \quad (7.10)$$

*Remark 6* From equation (7.1), we get  $\nabla p = f - u \in L^2(\Omega)$ , whence  $p \in H^1(\Omega)$  with  $\|p\|_1 \lesssim \|f\|_0$ .

We can easily get the well-posedness of problem (7.6), (7.7). In fact, the bilinear form  $a(u, v) = (u, v)$  is trivially coercive on  $L^2(\Omega)$ . Moreover the form  $b(q, v) = (\nabla q, v)$  trivially satisfies an *inf-sup* condition on  $[L^2(\Omega)]^2 \times [H^1(\Omega) \cap L_0^2(\Omega)]$ , indeed

$$\|\nabla q\|_0 = \sup_{v \in [L^2(\Omega)]^2 \setminus \{0\}} \frac{(\nabla q, v)}{\|v\|_0}. \quad (7.11)$$

Finally, the solution of (7.6), (7.7) satisfies the estimate

$$\|u\|_0 + \|\nabla p\|_0 \lesssim \|f\|_0. \quad (7.12)$$

### 7.2 Discretization and a priori estimates

Now we introduce a finite dimensional approximation of the variational problem (7.4), (7.5). Let  $V_h \subset H_0(\text{div}; \Omega)$  and  $Q_h \subset L_0^2(\Omega)$  be finite dimensional subspaces of continuous piecewise polynomial

functions on a triangulation  $\mathcal{T}_h$ . We consider the following stabilized problem: *Find*  $[u_h, p_h] \in V_h \times Q_h$  such that  $\forall [v_h, q_h] \in V_h \times Q_h$

$$(u_h, v_h) - (p_h, \nabla \cdot v_h) + \sum_{T \in \mathcal{T}_h} \delta_T (\nabla \cdot u_h, \nabla \cdot v_h)_T = (\Pi_T f, v), \quad (7.13)$$

$$(q_h, \nabla \cdot u_h) + \sum_{T \in \mathcal{T}_h} \tau_T (u_h + \nabla p_h - \Pi_T f, \nabla q_h)_T = 0, \quad (7.14)$$

where  $\delta_T \geq 0$  and  $\tau_T > 0$  are stabilization parameters whose definition will be discussed below.

*Remark 7* The present discrete formulation differs from the discrete formulation given in [5]. Therein the authors assume to use a continuous subspace  $V_h \subset H_0(\text{div}; \Omega)$ , but such that the couple  $V_h, Q_h$  of the discrete subspaces satisfy a discrete *inf-sup* condition. Doing so they do not need the terms multiplied by  $\tau_T$  to circumvent the Babuška-Brezzi condition. Moreover, they choose the parameter  $\delta_T = 1$  to get the needed coercivity of the bilinear form  $a_h(u_h, v_h) = (u_h, v_h) + \sum_{T \in \mathcal{T}_h} \delta_T (\nabla \cdot u_h, \nabla \cdot v_h)_T$  in the space  $V_h$ . Here we need the terms multiplied by  $\tau_T$  to circumvent the discrete *inf-sup* condition.

*Remark 8* We will use continuous finite element spaces  $V_h, Q_h$ , so we have  $V_h \subset [L^2(\Omega)]^2$ ,  $Q_h \subset [H^1(\Omega) \cap L_0^2(\Omega)]$  as well and thanks to the boundary condition (7.3) the stabilized discretization (7.13), (7.14) is also a discrete formulation of problem (7.6), (7.7).

*7.2.1 Stability of the discrete stabilized problem* In order to prove the well-posedness of the stabilized discrete problem, we choose  $v_h = u_h$  in (7.13),  $q_h = p_h$  in (7.14) and we sum the two equations, then we apply the Cauchy-Schwarz and Young inequalities:

$$\begin{aligned} & \|u_h\|_0^2 + \sum_{T \in \mathcal{T}_h} \delta_T \|\nabla \cdot u_h\|_{0,T}^2 + \sum_{T \in \mathcal{T}_h} \tau_T \|\nabla p_h\|_{0,T}^2 = (\Pi_T f, u_h) \\ & - \sum_{T \in \mathcal{T}_h} \tau_T (u_h, \nabla p_h)_T + \sum_{T \in \mathcal{T}_h} \tau_T (\Pi_T f, \nabla p_h)_T \leq \frac{1}{2k_1} \|\Pi_T f\|_0^2 \\ & + \frac{k_1}{2} \|u_h\|_0^2 + \sum_{T \in \mathcal{T}_h} \frac{1}{2k_2} \tau_T \|u_h\|_{0,T}^2 + \sum_{T \in \mathcal{T}_h} \frac{k_2}{2} \tau_T \|\nabla p_h\|_{0,T}^2 \\ & + \sum_{T \in \mathcal{T}_h} \frac{1}{2k_3} \tau_T \|\Pi_T f\|_{0,T}^2 + \sum_{T \in \mathcal{T}_h} \frac{k_3}{2} \tau_T \|\nabla p_h\|_{0,T}^2. \quad (7.15) \end{aligned}$$

Choosing  $k_1 = k_2 = k_3 = \frac{2}{3}$  and assuming  $\max_{T \in \mathcal{T}_h} \tau_T \leq \frac{4}{9}$ , we easily get

$$\|u_h\|_0^2 + \sum_{T \in \mathcal{T}_h} \delta_T \|\nabla \cdot u_h\|_{0,T}^2 + \sum_{T \in \mathcal{T}_h} \tau_T \|\nabla p_h\|_{0,T}^2 \lesssim \|II_T f\|_0^2. \quad (7.16)$$

*Remark 9* We note that the stability condition of the discrete formulation does not fix any upper bound for  $\tau_T$  dependent on the local mesh-size  $h_T$  differently from [8], [9]; here we have only an upper constraint of the type  $\max_{T \in \mathcal{T}_h} \tau_T = \mathcal{O}(1)$ .

**Assumption 4** From now on we set  $\tau_T \stackrel{\text{def}}{=} \tau$  and  $\delta_T \stackrel{\text{def}}{=} \delta$ ,  $\forall T \in \mathcal{T}_h$ .

**Proposition 9** Assuming  $\tau \leq \frac{4}{9}$  the following uniform stability estimate for the discrete formulation (7.13), (7.14) holds true

$$\|u_h\|_0^2 + \delta \|\nabla \cdot u_h\|_0^2 + \tau \|\nabla p_h\|_0^2 \lesssim \|II_T f\|_0^2. \quad (7.17)$$

Finally, if we consider e.g.  $\tau = \frac{4}{9}$  and either  $\delta = 0$  or  $\delta = 1$ , we write explicitly the uniform stability estimates  $\|u_h\|_0 + \|\nabla p_h\|_0 \lesssim \|II_T f\|_0$  or  $\|u_h\|_{\text{div}} + \|\nabla p_h\|_0 \lesssim \|II_T f\|_0$ , respectively. Then, by standard arguments, *a priori* error estimates in the norms in which stability is stated can be obtained.

### 7.3 *A posteriori estimates*

Here we want to investigate the robustness of an error estimator very close to the error estimators used before, when we are using different norms for the true error. For this reason, sometimes, we switch between the two variational formulations of the continuous problem (see Remark 8). Again, we deal separately with the velocity error  $\mathcal{R}$  and the pressure error  $\Psi$ .

Let us define  $R_{T,h} \stackrel{\text{def}}{=} u_h + \nabla p_h - II_T f|_T$ .

**Proposition 10** Under the assumptions of the continuous problems (7.4)-(7.5), (7.6)-(7.7) and the discrete problem (7.13)-(7.14), there exist a positive constant  $C_{\mathcal{R}}$  such that, for each positive constant  $k_1$ , we have

$$\|\mathcal{R}\|_0 \leq C_{\mathcal{R}} \left\{ \frac{1}{\sqrt{k_1}} \|\nabla \Psi\|_0 + \sqrt{k_1} \|\nabla \cdot u_h\|_0 + \sqrt{\sum_{T \in \mathcal{T}_h} \|R_{T,h}\|_{0,T}^2 + \|II_T f - f\|_0^2} \right\}. \quad (7.18)$$

*Proof.* From the continuous equation (7.4) we get

$$(\mathcal{Y}, v) - (\Psi, \nabla \cdot v) = (u_h, v) - (p_h, \nabla \cdot v) - (f, v), \quad \forall v \in V. \quad (7.19)$$

We take  $v = \mathcal{Y}$  as a test function and we proceed similarly to the proof of Proposition 1, using the fact that  $p_h$  and  $u_h$  are continuous functions in  $\Omega$ ,  $[\hat{n} \cdot \mathcal{Y}]_E = 0$ ,  $\forall E \in \mathcal{E}_{h,\Omega}$  and  $\hat{n} \cdot \mathcal{Y} = 0$  on  $\partial\Omega$  to conclude that  $\sum_{T \in \mathcal{T}_h} (\hat{n} p_h, \mathcal{Y})_{\partial T} = 0$ . Moreover, we use Cauchy-Schwarz's and Young's inequalities. Then, applying (7.8) we get (7.18).  $\square$

**Proposition 11** *Under the assumptions of the continuous problem (7.6), (7.7) and the discrete problem (7.13), (7.14) we have*

$$\|\nabla \Psi\|_0 \leq \|\mathcal{Y}\|_0 + \sqrt{\sum_{T \in \mathcal{T}_h} \|R_{T,h}\|_0^2} + \|I_T f - f\|_0. \quad (7.20)$$

*Proof.* From the *inf-sup* condition (7.11) and equation (7.6) we have:

$$\begin{aligned} \|\nabla \Psi\|_0 &= \sup_{v \in [L^2(\Omega)]^2 \setminus \{0\}} \frac{(\nabla \Psi, v)}{\|v\|_0} \\ &\leq \|\mathcal{Y}\|_0 + \sup_{v \in [L^2(\Omega)]^2 \setminus \{0\}} \frac{1}{\|v\|_0} \left\{ \sum_{T \in \mathcal{T}_h} (R_{T,h}, v)_T + (I_T f - f, v) \right\}. \end{aligned}$$

Then, we get (7.20).  $\square$

**Definition 12** *Let us define the residual-based a posteriori error estimator on the triangle  $T \in \mathcal{T}_h$ :*

$$\eta_{R,T}^2 \stackrel{def}{=} \|R_{T,h}\|_{0,T}^2 + \|\nabla \cdot u_h\|_{0,T}^2. \quad (7.21)$$

**Theorem 5** *Under the assumptions of the continuous problems (7.4)-(7.5), (7.6)-(7.7) and the discrete problem (7.13)-(7.14), there exists a constant  $C^\dagger$  depending on the smallest angle of the triangulation and independent of any mesh size such that the following global upper bound holds*

$$\|\mathcal{Y}\|_{\text{div}} + \|\nabla \Psi\|_0 \leq C^\dagger (\eta_\Omega + \|I_T f - f\|_0). \quad (7.22)$$

*Proof.* Using inequalities (7.18), (7.20) and suitably choosing the constant  $k_1$ , we get the global upper bound for the error  $\mathcal{Y}$  independent of  $\Psi$ . Then using inequality (7.20) we get the global upper bound for the error  $\Psi$ . Recalling the definition of the global error estimator (4.2) and collecting the previous results, definition (7.21) and  $\|\nabla \cdot \mathcal{Y}\|_0 = \|\nabla \cdot u_h\|_0$  we get the thesis.  $\square$

**7.3.1 Local lower bounds** As in Section 3.3, for  $T \in \mathcal{T}_h$  let us define  $w_T \stackrel{\text{def}}{=} R_{T,h} b_T$ , where  $b_T$  is the usual bubble function on the triangle.

**Proposition 12** *Under the assumptions of the continuous problem (7.1)-(7.3) and the discrete problem (7.13), (7.14) we have*

$$\|R_{T,h}\|_{0,T} \lesssim \|\mathcal{Y}\|_{0,T} + \frac{1}{h_T} \|\Psi\|_{0,T} + \|\Pi_T f - f\|_{0,T} \quad (7.23)$$

and

$$\|R_{T,h}\|_{0,T} \lesssim \|\mathcal{Y}\|_{0,T} + \|\nabla \Psi\|_{0,T} + \|\Pi_T f - f\|_{0,T}. \quad (7.24)$$

*Proof.* First we use inequalities (3.14)-(3.16), then we observe that

$$(R_{T,h}, w_T)_T = (\mathcal{Y}, w_T)_T - (\Psi, \nabla \cdot w_T)_T - (\Pi_T f - f, w_T)_T. \quad (7.25)$$

Then, from (3.16) we obtain (7.23). Applying an integration by parts of the term  $(\Psi, \nabla \cdot w_T)$  in (7.25), we get (7.24).  $\square$

Moreover we can show how the residual of the continuity equation bounds from below the error for the velocity on  $T$  in the  $L^2$ -norm.

We set  $w_T \stackrel{\text{def}}{=} [\nabla \cdot u_h] b_T$  and we have

$$\|\nabla \cdot u_h\|_{0,T}^2 \lesssim \|\mathcal{Y}\|_{0,T} \|\nabla w_T\|_{0,T} \lesssim \|\mathcal{Y}\|_{0,T} \frac{1}{h_T} \|\nabla \cdot u_h\|_{0,T},$$

thus we find

$$\|\nabla \cdot u_h\|_{0,T} \lesssim \frac{1}{h_T} \|\mathcal{Y}\|_{0,T}. \quad (7.26)$$

*Remark 10* If we want to control the error measured by one of the two norms  $\|\mathcal{Y}\|_0 + \|\Psi\|_0$  or  $\|\mathcal{Y}\|_0 + \|\nabla \Psi\|_0$  we can apply the inverse inequality (7.26) on each element  $T$ .

Then, we recall definition (3.32) and we collect the previous results in the following theorem:

**Theorem 6** *There exist four constants  $C_{\downarrow,1}$ ,  $C_{\downarrow,2}$ ,  $C_{\downarrow,3}$  and  $C_{\downarrow,4}$  depending on the smallest angle of the triangulation and independent of any mesh size such that the global lower bounds holds*

$$\check{h} \eta_\Omega \leq C_{\downarrow,1} (\|\mathcal{Y}\|_0 + \|\Psi\|_0 + \check{h} \|\Pi_T f - f\|_0), \quad (7.27)$$

$$\check{h} \eta_\Omega \leq C_{\downarrow,2} (\|\mathcal{Y}\|_{\text{div}} + \|\Psi\|_0 + \check{h} \|\Pi_T f - f\|_0), \quad (7.28)$$

$$\check{h} \eta_\Omega \leq C_{\downarrow,3} (\|\mathcal{Y}\|_0 + \|\nabla \Psi\|_0 + \check{h} \|\Pi_T f - f\|_0), \quad (7.29)$$

$$\eta_\Omega \leq C_{\downarrow,4} (\|\mathcal{Y}\|_{\text{div}} + \|\nabla \Psi\|_0 + \|\Pi_T f - f\|_0). \quad (7.30)$$

#### 7.4 Numerical results

In this section we use again the hypotheses of Section 4 about the approximation of the data. From Theorems 5 and 6 we easily get that measuring the true error by one of the norms  $\|\mathcal{Y}\|_0 + \|\Psi\|_0$ ,  $\|\mathcal{Y}\|_{\text{div}} + \|\Psi\|_0$  or  $\|\mathcal{Y}\|_0 + \|\nabla\Psi\|_0$  yields the bounds

$$\underline{c}\check{h} \leq \frac{1}{e.i.} \leq \overline{C} \quad (7.31)$$

for the inverse of the effectivity index.

Instead, if we define the true error as *t.e.*  $\stackrel{\text{def}}{=} \|\mathcal{Y}\|_{\text{div}} + \|\nabla\Psi\|_0$ , then we have

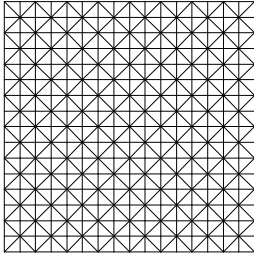
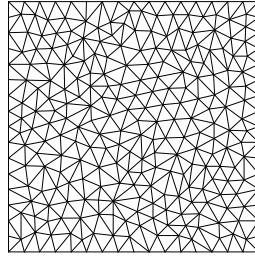
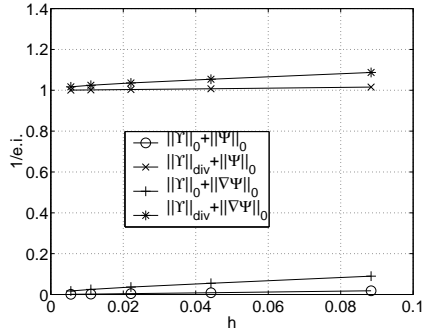
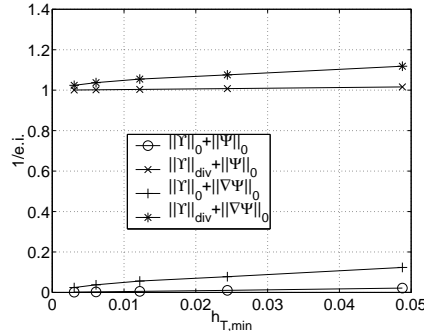
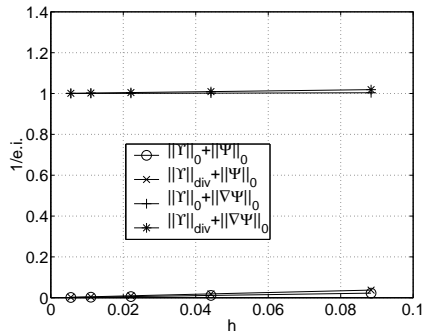
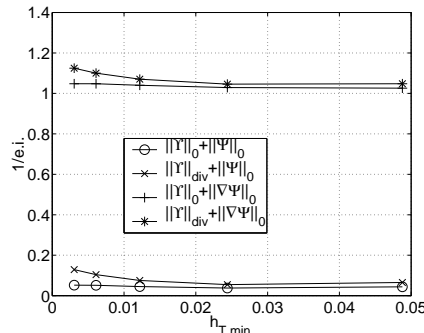
$$\underline{c} \leq \frac{1}{e.i.} \leq \overline{C}. \quad (7.32)$$

So, we have found the most appropriate norm for measuring the error in the solution of problem (7.1), (7.3), since it yields robust *a posteriori* estimates.

In order to test our error estimators, we consider problem (7.1), (7.2) in the unit square with boundary condition (7.3). We choose the forcing function  $f = [f_1, f_2]$  such that the solution  $[u, p]$  of the problem is the same as the one described in Subsection 5.1. In this case we fix  $R_1 = R_2 = 0.3$ . We consider two different grids: a structured uniform grid (Figure 7.1) and a quasi uniform unstructured grid (Figure 7.2). Finer grids are obtained splitting each triangle in four similar triangles for both grids. In Figures 7.3-7.6 we report the behaviour of  $1/e.i.$  versus  $\check{h}$  for the errors measured in the four norms considered in the previous subsection. We consider both the stabilization cases  $\delta = 0$  and  $\delta = 1$  with  $\tau = 4/9$ . The numerical results confirm our theoretical estimates. Indeed, if we measure the error in the norm  $\|\mathcal{Y}\|_{\text{div}} + \|\nabla\Psi\|_0$ , the inverse of the effectivity index is always close to 1 when  $\check{h}$  tends to 0. If we measure the error in any other norm we always have, at least, one case for which the inverse of the effectivity index tends to 0 when  $\check{h}$  tends to 0. Note that setting  $\delta = 0$  the “good” norms are those that include the term  $\|\mathcal{Y}\|_{\text{div}}$  (see Figures 7.3, 7.4). Conversely setting  $\delta = 1$  the “good” norms are those that include the term  $\|\nabla\Psi\|_0$  (see Figures 7.5, 7.6).

## References

1. I. Babuška, R. Durán, R. Rodríguez, *Analysis of the efficiency of an a posteriori error estimator for linear triangular elements*, SIAM J. Numer. Anal. 29 (1992), no. 4, 947-964.


**Figure 7.1.** Coarsest Uniform Grid,  $N = 289$ 

**Figure 7.2.** Coarsest Quasi Uniform Grid,  $N = 284$ 

**Figure 7.3.** Uniform Grid:  $\delta = 0$ ,  $\tau = 4/9$ 

**Figure 7.4.** Quasi Uniform Grid:  $\delta = 0$ ,  $\tau = 4/9$ 

**Figure 7.5.** Uniform Grid:  $\delta = 1$ ,  $\tau = 4/9$ 

**Figure 7.6.** Quasi Uniform Grid:  $\delta = 1$ ,  $\tau = 4/9$ 

2. I. Babuška, W.C. Rheinboldt, *Error estimates for adaptive finite element method*, SIAM J. Numer. Anal. 15 (1978), no. 4, 736-754.
3. S. Berrone, *Adaptive discretization of the stationary and incompressible Navier-Stokes equations by stabilized Finite Element Methods*, Comput. Methods Appl. Mech. Engrg. 190/34 (2001), 4435-4455.
4. F. Brezzi, M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York-Berlin, 1991.
5. F. Brezzi, M. Fortin, L.D. Marini, *Mixed finite element methods with continuous stresses*, M<sup>3</sup>AS 3 (1993), no. 2, 275-287.

6. P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland Publishing Company, Amsterdam, 1978.
7. P. Clément, *Approximation by finite element functions using local regularization*, R.A.I.R.O. Série Rouge Anal. Numér. 9 (1975), R-2, 77-84.
8. L.P. Franca, S.L. Frey, *Stabilized finite element methods: II. The incompressible Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg. 99 (1992), 209-233.
9. L.P. Franca, S.L. Frey, T.J.R. Hughes, *Stabilized finite element methods: I. Application to the advective-diffusion model*, Comput. Methods Appl. Mech. Engrg. 95 (1992), 253-276.
10. V. Girault, P.A. Raviart, *Finite Element Methods for Navier-Stokes Equations, Theory and Algorithms*, Springer-Verlag, Berlin Heidelberg, 1986.
11. I. Harari, T.J.R. Hughes, *What are  $C$  and  $h$ ?: Inequalities for the analysis and design of finite element methods*, Comput. Methods Appl. Mech. Engrg. 97 (1992), 157-192.
12. J. Medina, M. Picasso, J. Rappaz, *Error estimates and adaptive finite elements for nonlinear diffusion-convection problems*, Math. Models. and Math. in Appl. Sci. 6 (1996), 689-712.
13. H.G. Roos, M. Stynes, L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations, Convection-Diffusion and Flow Problems*, Springer-Verlag, Berlin Heidelberg, 1996.
14. R. Temam, *Navier-Stokes Equations, Theory and Numerical Analysis*, North-Holland Publishing Company, Amsterdam, 1979.
15. L. Tobiska, R. Verfürth, *Analysis of a Streamline Diffusion Finite Element Method for the Stokes and Navier-Stokes Equations*, SIAM J. Numer. Anal. 33 (1996), no. 1, 107-127.
16. R. Verfürth, *A posteriori error estimators for Stokes equations*, Numer. Math. 55 (1989), 309-325.
17. R. Verfürth, *A posteriori error estimation and adaptive mesh-refinement techniques*, J. Comput. Appl. Math. 50 (1994), 67-83.
18. R. Verfürth, *A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*, John Wiley & Sons, Chichester-New York, 1996.
19. R. Verfürth, *Robust a posteriori error estimators for singularly perturbed reaction-diffusion equation*, Numer. Math. 78 (1998), 479-493.
20. R. Verfürth, *A posteriori error estimators for convection-diffusion equations*, Numer. Math. 80 (1998), 641-663.